Distance between two hyperplanes

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Suppose we have two parallel hyperplanes $L_1 : w^T x + b_1 = 0, L_2 : w^T x + b_2 = 0$, the distance between them is $d = \frac{|b_1 - b_2|}{\|w\|}$.

proof: There must exist two points $x_1, x_2$, while $x_1$ is in $L_1$, and $x_2$ is in $L_2$. Also, a pair of $x_1, x_2$ satisfies $\|x_1 - x_2\| = |b_1 - b_2|$. let $d$ be a vector perpendicular to $L_1$ and $L_2$ and $\|d\| = d$. Since $w \perp L_1$, let $d = cw$.

Then we have

$$(x_2 - x_1 - cw)^T cw = 0 \implies w^T (x_1 - x_2) - c \|w\|^2 = 0.$$  

Then

$$c = \frac{w^T (x_2 - x_1)}{\|w\|^2}.$$  

Since $w^T (x_2 - x_1) = b_1 - b_2 \implies c = \frac{b_1 - b_2}{\|w\|^2}$. Therefore, $d = \|d\| = |c| \|w\| = \frac{|b_1 - b_2|}{\|w\|}$.