Symbolic Analysis

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What is Symbolic Analysis

- Static analysis considers all paths are feasible
- Dynamic considers one path or a number of paths
- Symbolic analysis reasons about path feasibility
  - Much more precise
  - Scalability is an issue
- A lot of security applications
  - Exploit generation
  - Vulnerability detection
  - Expose hidden behavior
  - Verification
  - Equivalence checking for analyzing obfuscated code

An Example

```
1:    x=input()
2:    if (x>0)
3:       y=…;
4:    else
5:       y=…;
6:    if (x>10)
10:      z=y
```

Basic Idea

- Explore individual paths in the program; models the conditions and the symbolic values along a path to a symbolic constraint; a path is feasible if the corresponding constraint is satisfiable (SAT)
- Similar to our per-path static analysis, a worklist is used to maintain the paths being explored
- Upon a function invocation, the current worklist is pushed to a stack and a new worklist is initialized for path exploration within the callee
- Upon a return, the symbolic value of the return variable is passed back to the caller

Another Example

```
1:    x=input()
2:    if (x>0)
3:       y=…;
4:    else
5:       y=…;
6:    t= f (x)
7:    if (t>0)
8:      z=y
```

```
10:  f (k) {
11:    if (k<=-10)
12:          return  k+10;
13:    else
14:          return k;
```

Language

Program	$p ::= m^*$
Function	$m ::= f(x)\{s\}$
Constant	$c ::= \ldots \mid -1 \mid 0 \mid 1 \mid \ldots \mid \textbf{true} \mid \textbf{false}$
Expression	$e ::= x \mid \mathrm{c} \mid e_1 \, \textbf{op} \, e_2$
Statement	$s ::= x = e \mid x = f(e) \mid x = \textbf{unknown}() \mid s_1; s_2 \mid$ $\textbf{if } (e) \textbf{ then } s_1 \textbf{ else } s_2 \mid \textbf{ret}^{\mathrm{f}} x$
Operators	$\textbf{op} ::= + \mid - \mid \ldots \mid > \mid \geq \mid = \mid \neq \mid \wedge \mid \vee \mid \ldots$

Definitions

$v \in SymValue$		$\rho \in Constraint$
$\sigma \in SymStore$	::=	$Variable \mapsto SymValue$
$\omega \in Worklist$	::=	$\mathcal{P}(Stmt \times Constraint \times SymStore)$
$\Omega \in WLStack$	::=	$\overline{Worklist}$
$\gamma \in RetConstraint$	::=	$\mathcal{P}(Constraint)$
$\Gamma \in RCStack$	::=	$\overline{RetConstraint}$

$\sigma\downarrow ::= \bigwedge_{x\in\sigma} x = \sigma(x)$, the operator turns a store to a logical conjunction.

$\langle s,\rho,\sigma\rangle \bowtie_x^f \gamma \;::= \bigcup_{\rho'\in\gamma}\{\langle s,\rho\wedge\rho',\sigma[x\mapsto rt_f]\rangle\}$, the operator propagates the constraints collected in the callee to the caller for an invocation $x=f(e)$; s is the statement right after the invocation. rt_f represents the return value of f.

select(ω) chooses the next symbolic state $\langle s,\rho,\sigma\rangle$ to explore.

follow(ρ) determines the feasibility of a path condition ρ and realizes some user specified path pruning heuristics.

Symbolic Execution Semantics

EXPRESSION RULES $\boxed{e \twoheadrightarrow v}$

$$\frac{}{c \twoheadrightarrow c} \quad \text{(E-CONST)} \qquad \frac{}{x \twoheadrightarrow x} \quad \text{(E-VAR)}$$

$$\frac{e_1 \twoheadrightarrow v_1 \quad e_2 \twoheadrightarrow v_2}{e_1 \,\mathbf{op}\, e_2 \twoheadrightarrow v_1 \,\boldsymbol{op}\, v_2} \quad \text{(E-OP)}$$

Symbolic Execution Semantics

STATEMENT RULES $\boxed{\langle s, \rho, \sigma\rangle \rightarrow \mathcal{P}(\langle s, \rho, \sigma\rangle)}$

$$\frac{e \twoheadrightarrow \rho_0}{\langle x = e; s,\ \rho, \sigma\rangle \rightarrow \{\langle s, \rho, \sigma[x \mapsto \rho_0]\rangle\}} \quad \text{(S-ASSIGN)}$$

$$\frac{e \twoheadrightarrow v \qquad \rho_1 = \rho \wedge v \qquad \rho_2 = \rho \wedge \neg v \qquad follow(\rho_1 \wedge \sigma\downarrow) = \textbf{true} \qquad follow(\rho_2 \wedge \sigma\downarrow) = \textbf{false}}{\langle \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2; s,\ \rho, \sigma\rangle \rightarrow \{\langle s_1; s, \rho_1, \sigma\rangle\}} \quad \text{(S-IF-T)}$$

$$\frac{e \twoheadrightarrow v \qquad \rho_1 = \rho \wedge v \qquad \rho_2 = \rho \wedge \neg v \qquad follow(\rho_1 \wedge \sigma\downarrow) = \textbf{\textit{true}} \qquad follow(\rho_2 \wedge \sigma\downarrow) = \textbf{\textit{true}}}{\langle \textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2; s,\ \rho, \pi, \sigma\rangle \rightarrow \{\langle s_1; s, \rho_1, \sigma\rangle, \langle s_2; s, \rho_2, \sigma\rangle\}} \quad \text{(S-IF-BOTH)}$$

$$\frac{r \text{ a fresh symbolic variable}}{\langle x = \textbf{unknown}(); s,\ \rho, \sigma\rangle \rightarrow \{\langle s, \rho, \sigma[x \mapsto r]\rangle\}} \quad \text{(S-UNINTPRT)}$$

Symbolic Execution Semantics

GLOBAL RULES $\boxed{\langle \Omega, \Gamma \rangle \Rightarrow \langle \Omega', \Gamma' \rangle}$

$$\frac{\langle s, \rho, \sigma \rangle = select(\omega) \quad \langle s, \rho, \sigma \rangle \to t}{\langle \Omega \circ \omega, \Gamma \rangle \Rightarrow \langle \Omega \circ \omega[t/\langle s, \rho, \sigma \rangle], \Gamma \rangle} \quad \text{(G-STMT)}$$

$$\frac{\begin{array}{c}\langle \mathrm{y} = f(e); s_0, \rho, \sigma \rangle = select(\omega) \quad f(x)\{s\} \text{ is a method} \\ e \twoheadrightarrow v \qquad \omega' = \{\langle s, \rho, \sigma[x \mapsto v \ ] \rangle\}\end{array}}{\langle \Omega \circ \omega, \Gamma \rangle \Rightarrow \langle \Omega \circ \omega \circ \omega', \Gamma \circ \{\} \rangle} \quad \text{(G-CALL)}$$

$$\frac{\begin{array}{c}\langle \mathbf{ret}_f \ x, \rho, \sigma \rangle = select(\omega) \quad f(y)\{s\} \text{ is a method} \\ \omega' = \omega - \langle \boldsymbol{ret}_f \ x, \rho, \sigma \rangle \\ \gamma' = \gamma \cup \{ \rho \wedge \sigma\downarrow \}\end{array}}{\langle \Omega \circ \omega, \Gamma \circ \gamma \rangle \Rightarrow \langle \Omega \circ \omega', \Gamma \circ \gamma' \rangle} \quad \text{(G-RET)}$$

$$\frac{\begin{array}{c}\langle x = f(e); s, \rho, \sigma \rangle = select(\omega) \\ \omega' = \omega[\langle s, \rho, \sigma \rangle \bowtie_x^f \gamma \ / \ \langle x = f(e); s, \rho, \sigma \rangle]\end{array}}{\langle \Omega \circ \omega \circ \{\}, \Gamma \circ \gamma \rangle \Rightarrow \langle \Omega \circ \omega', \Gamma \rangle} \quad \text{(G-POST-CALL)}$$

Technical Challenges

- How to encode a program to constraints
  - Arrays, loops, heap, strings
- How to solve constraints
  - Propositional logic and SAT/SMT solving

Propositional logic

Compiled and modified from slides by
Michael Reniers, Julia Lawall and Neil Jones

Syntax of propositional logic

$$F ::= (P) \mid (\neg F) \mid (F \vee F) \mid (F \wedge F) \mid (F \rightarrow F)$$
$$P ::= p \mid q \mid r \mid \ldots$$

- propositional atoms: $p, q, r, \ldots$ for describing declarative sentences such as:
  - All students have to follow the course Programming and Modal Logic
  - 1037 is a prime number

- connectives:

Connective	Symbol	Alternative symbols
negation	$\neg$	$\sim$
disjunction	$\vee$	$\mid$
conjunction	$\wedge$	$\&$
implication	$\rightarrow$	$\Rightarrow$, $\supset$, $\supseteq$

Sometimes also bi-implication ($\leftrightarrow$, $\Leftrightarrow$, $\equiv$) is considered as a connective.

Syntax of propositional logic

Binding priorities

$$\neg$$

$$\vee \qquad \wedge$$

$$\rightarrow \qquad (\leftrightarrow)$$

for reducing the number of brackets.

Also outermost brackets are often omitted.

Semantics of propositional logic

The meaning of a formula depends on:

- The meaning of the propositional atoms (that occur in that formula)
- The meaning of the connectives (that occur in that formula)

Semantics of propositional logic

The meaning of a formula depends on:

- The meaning of the propositional atoms (that occur in that formula)
  - a declarative sentence is either true or false
  - captured as an assignment of truth values ($\mathbb{B} = \{\mathtt{T}, \mathtt{F}\}$) to the propositional atoms: a *valuation* $\boxed{v : P \to \mathbb{B}}$
- The meaning of the connectives (that occur in that formula)
  - the meaning of an n-ary connective $\oplus$ is captured by a function $\boxed{f_\oplus : \mathbb{B}^n \to \mathbb{B}}$
  - usually such functions are specified by means of a truth table.

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \to B$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Exercise

Find the meaning of the formula $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ by constructing a truth table from the subformulas.

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p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Exercise

Find the meaning of the formula $(p \to q) \wedge (q \to r) \to (p \to r)$ by constructing a truth table from the subformulas.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Formally (this is not in the book)

$$[\![\_]\!] : F \to ((P \to \mathbb{B}) \to \mathbb{B})$$

$$\begin{array}{lcl} [\![p]\!](v) & = & v(p) \\ [\![\neg\phi]\!](v) & = & f_{\neg}([\![\phi]\!](v)) \end{array} \qquad \begin{array}{lcl} [\![\phi \wedge \psi]\!](v) & = & f_{\wedge}([\![\phi]\!](v), [\![\psi]\!](v)) \\ [\![\phi \vee \psi]\!](v) & = & f_{\vee}([\![\phi]\!](v), [\![\psi]\!](v)) \\ [\![\phi \to \psi]\!](v) & = & f_{\to}([\![\phi]\!](v), [\![\psi]\!](v)) \end{array}$$

Deciding validity and satisfiability of propositional formulas

- Validity: A formula ϕ is *valid* if for any valuations v, $[\![\phi]\!](v) = \mathtt{T}$.
- Satisfiability: A formula ϕ is *satisfiable* if there exists a valuation v such that $[\![\phi]\!](v) = \mathtt{T}$.

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Examples

$p \wedge q$	valid? satisfiable?
$p \rightarrow (q \rightarrow p)$	valid? satisfiable?
$p \wedge \neg p$	valid? satisfiable?

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Examples

$p \wedge q$	satisfiable
$p \rightarrow (q \rightarrow p)$	valid
$p \wedge \neg p$	unsatisfiable

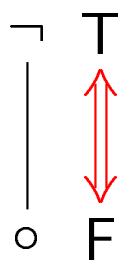
Given a propositional formula ϕ, how to check whether it is valid? satisfiable?

SAT Solver

- Finding satisfying valuations to a propositional formula.

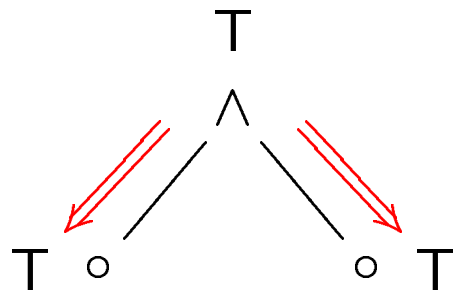
Forcing laws – negation

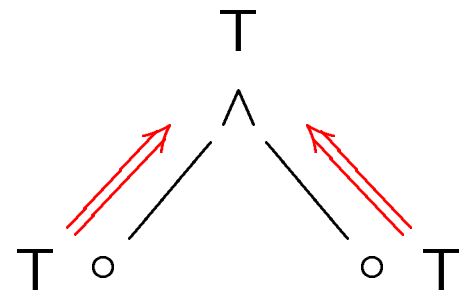
ϕ	$\neg\phi$
T	F
F	T

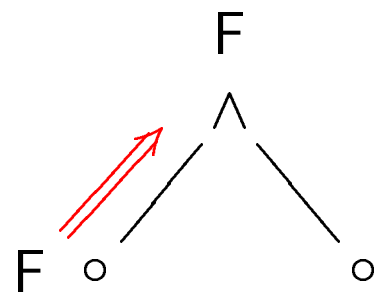


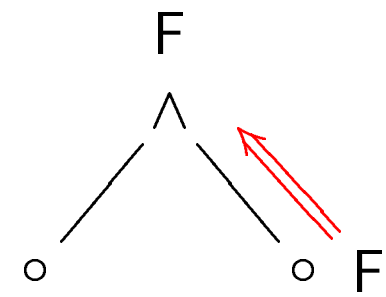
Forcing laws – conjunction

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

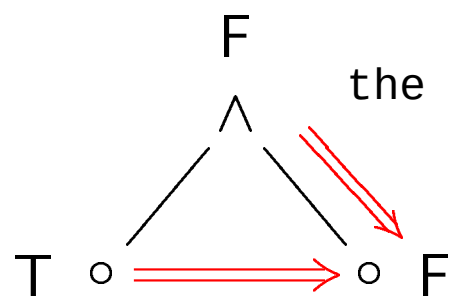




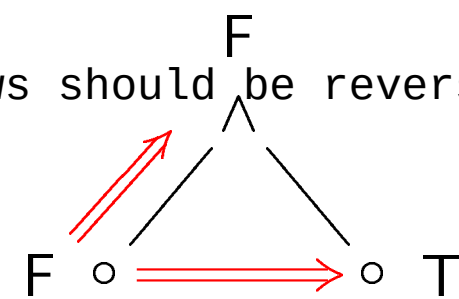




Other laws possible,
but $\neg$ and $\wedge$
are adequate



the arrows should be reversed



Using the SAT solver

1. Convert to $\neg$ and $\wedge$.

$$T(p) = p \qquad T(\neg\phi) = \neg T(\phi)$$
$$T(\phi \wedge \psi) = T(\phi) \wedge T(\psi) \qquad T(\phi \vee \psi) = \neg(\neg T(\phi) \wedge \neg T(\psi))$$
$$T(\phi \rightarrow \psi) = \neg(T(\phi) \wedge \neg T(\psi))$$

   Linear growth in formula size (no distributivity).

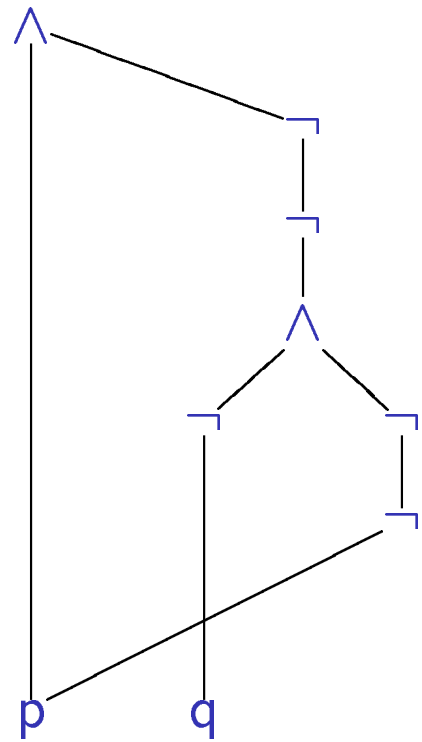
2. Translate the formula to a DAG, sharing common subterms.

3. Set the root to T and apply the forcing rules.

Satisfiable if all nodes are consistently annotated.

Example: satisfiability

Formula: $p \wedge \neg(q \vee \neg p) \equiv p \wedge \neg\neg(\neg q \wedge \neg\neg p)$



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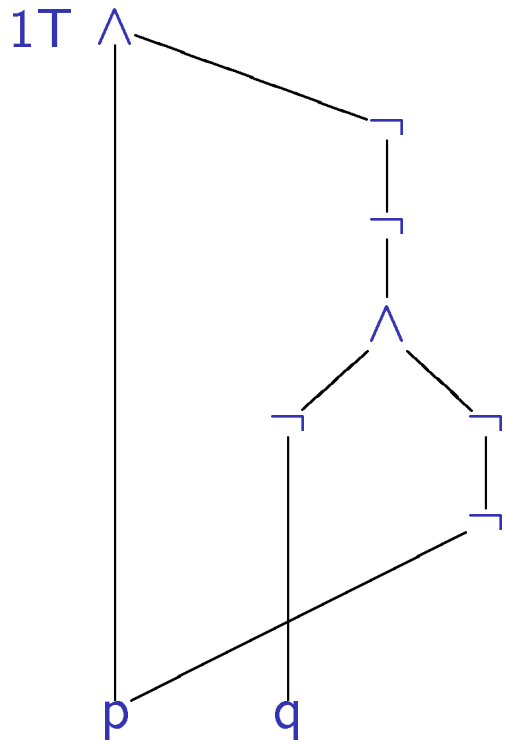

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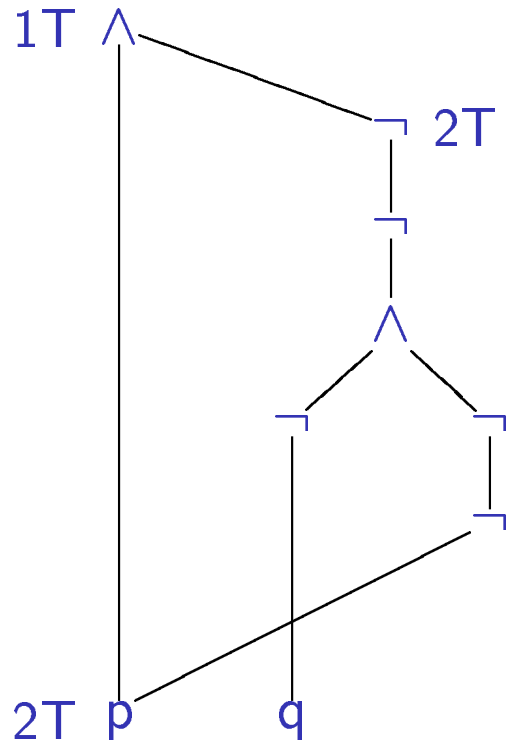

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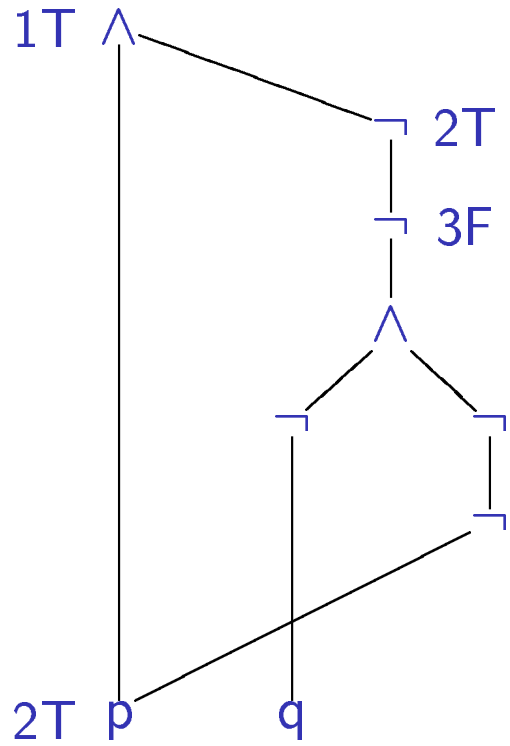

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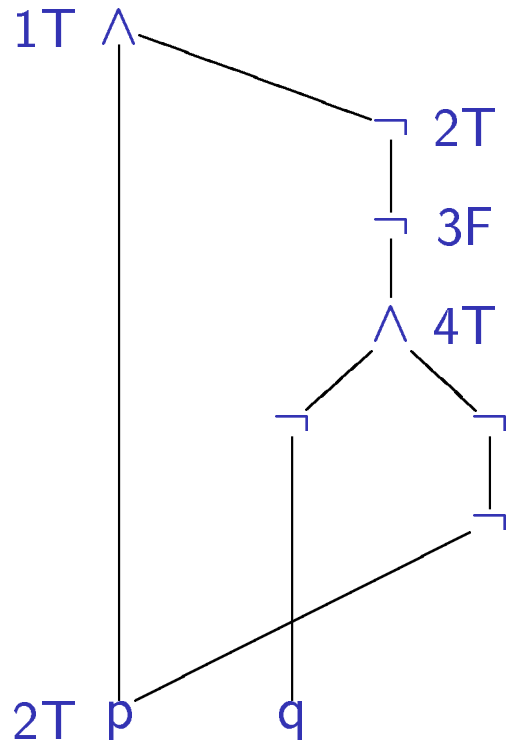

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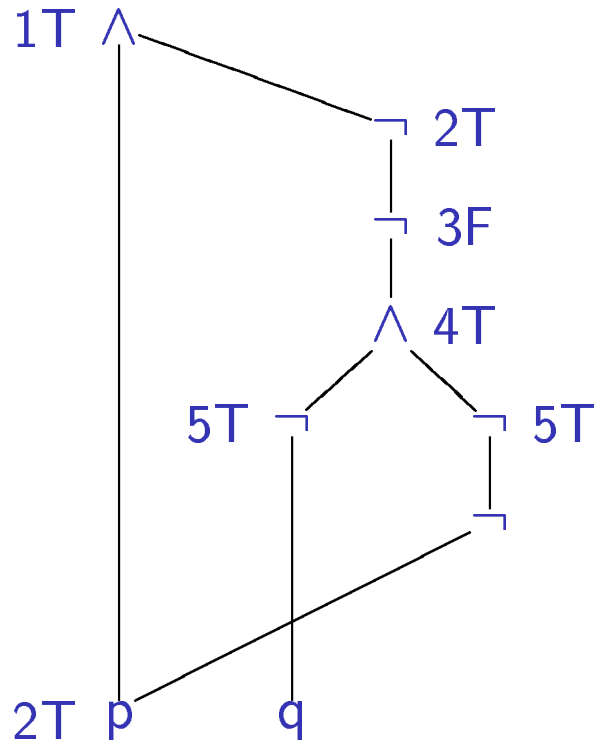

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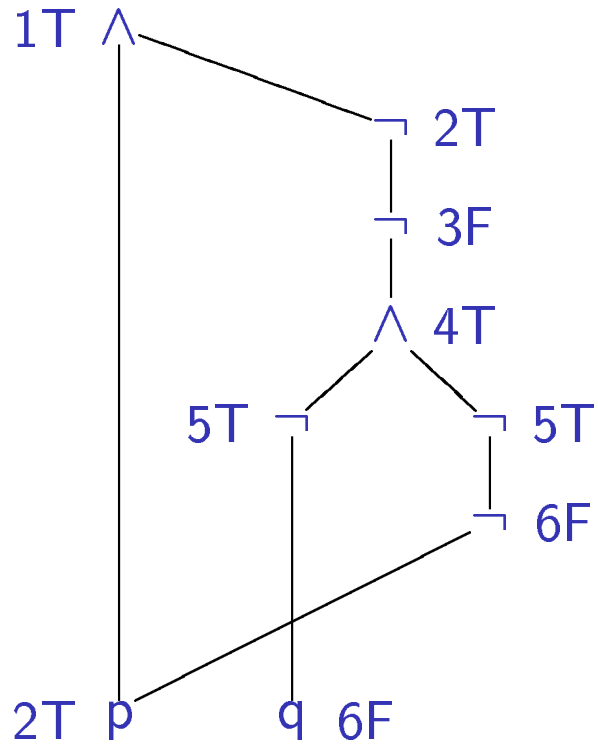


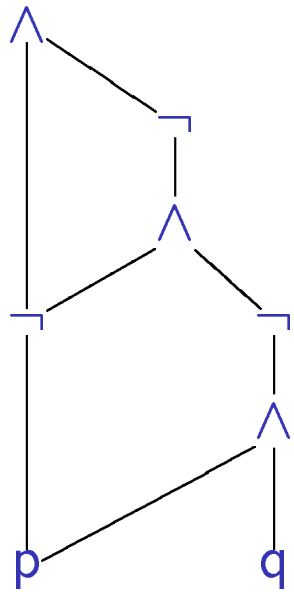
Satisfiable?

Example: validity

Formula: $(p \vee (p \wedge q)) \rightarrow p$

Valid if $\neg((p \vee (p \wedge q)) \rightarrow p)$ is not satisfiable

Translated formula: $\neg(\neg p \wedge \neg(p \wedge q)) \wedge \neg p$



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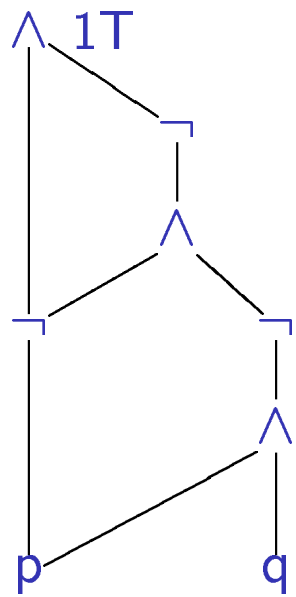

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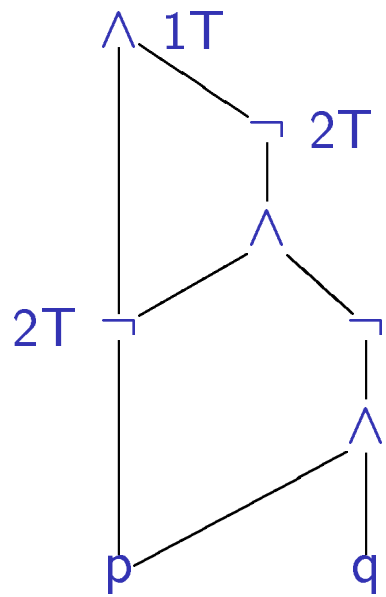

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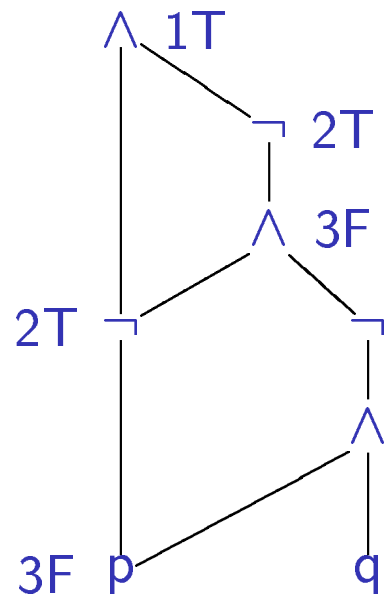

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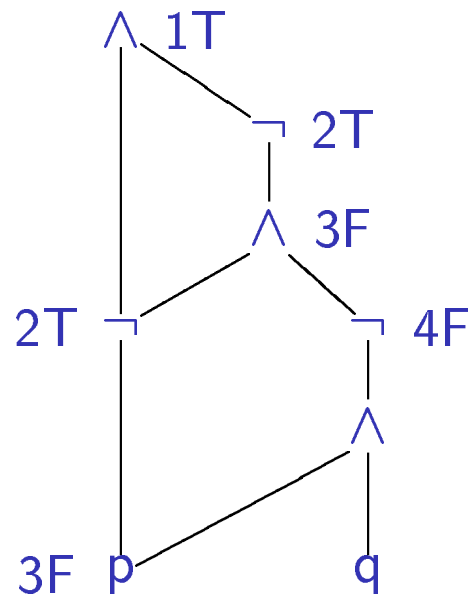

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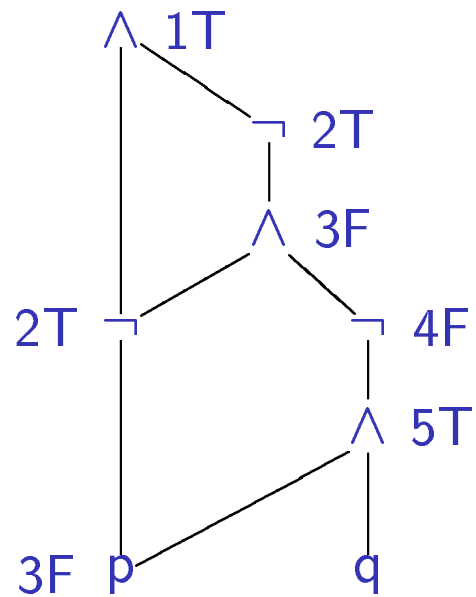

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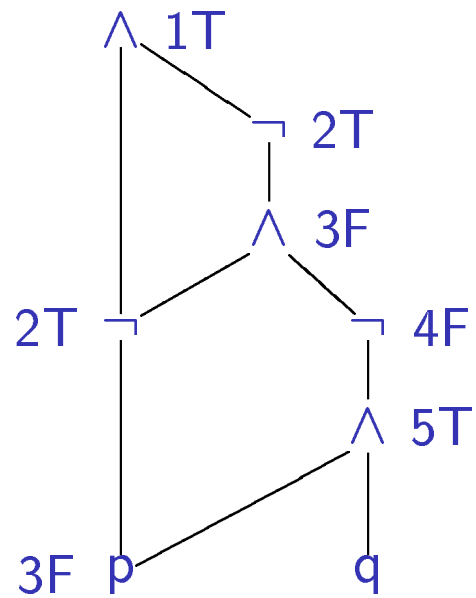
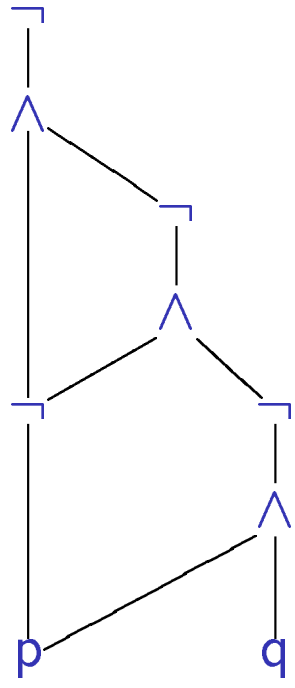


Contradiction.

Example: satisfiability

Formula: $(p \vee (p \wedge q)) \rightarrow p \equiv \neg(\neg(\neg p \wedge \neg(p \wedge q)) \wedge \neg p)$



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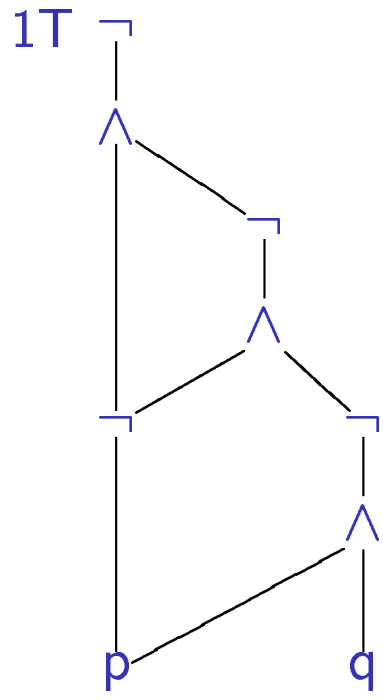
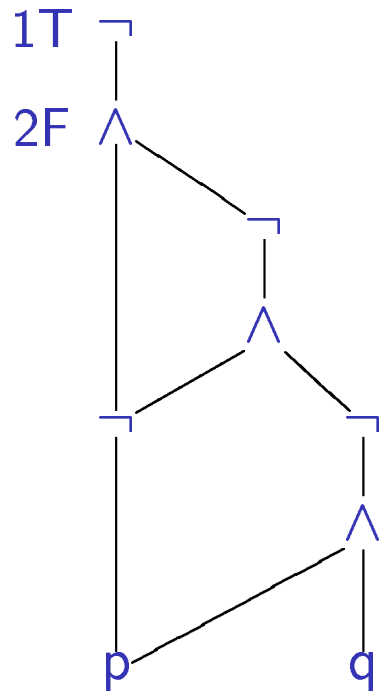


Example: satisfiability

Formula: $(p \vee (p \wedge q)) \rightarrow p \equiv \neg(\neg(\neg p \wedge \neg(p \wedge q)) \wedge \neg p)$



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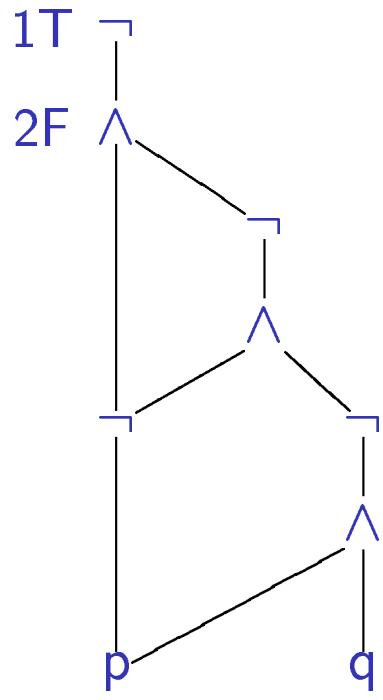
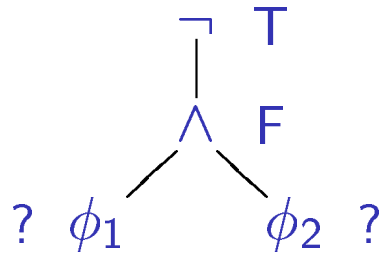


Now what?

Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.

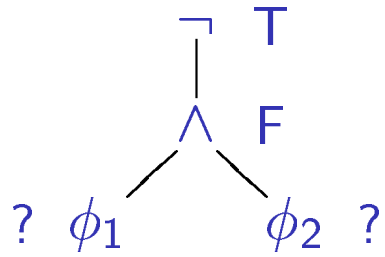


Some are valid, and thus satisfiable:

$$\top$$

Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.

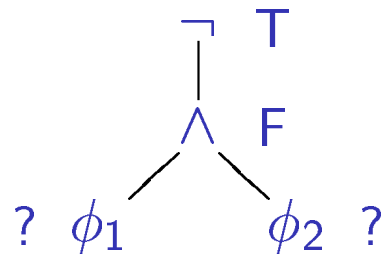


Some are valid, and thus satisfiable:

$$\top \equiv p \rightarrow p$$

Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.

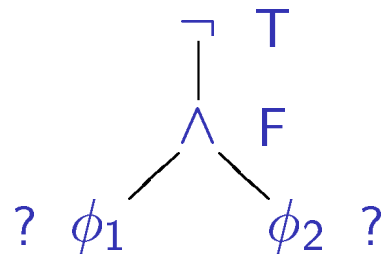


Some are valid, and thus satisfiable:

$$\top \equiv p \rightarrow p \equiv \neg(p \wedge \neg p)$$

Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.



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Some are not valid, and thus not satisfiable:

$\bot$

Limitation of the SAT solver algorithm

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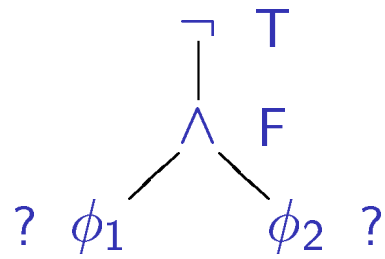

Some are valid, and thus satisfiable:

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Some are not valid, and thus not satisfiable:

$\bot \equiv \neg\top$

Limitation of the SAT solver algorithm

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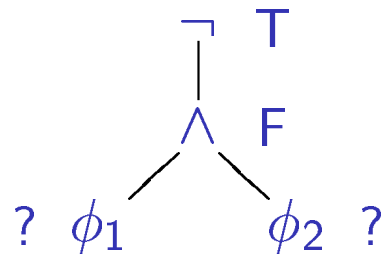

Some are valid, and thus satisfiable:

$$\top \equiv p \rightarrow p \equiv \neg(p \wedge \neg p)$$

Some are not valid, and thus not satisfiable:

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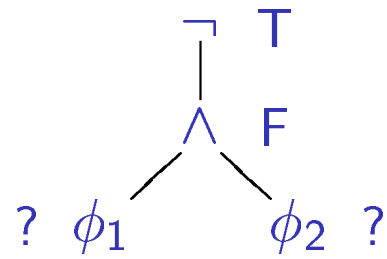
Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.



Some are valid, and thus satisfiable:

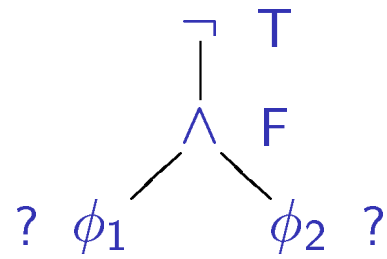
$$\top \equiv p \rightarrow p \equiv \neg(p \wedge \neg p)$$

Some are not valid, and thus not satisfiable:

$$\bot \equiv \neg\top \equiv \neg(\top \wedge \top) \equiv \neg(p \rightarrow p \wedge p \rightarrow p)$$

Limitation of the SAT solver algorithm

Fails for all formulas of the form $\neg(\phi_1 \wedge \phi_2)$.



Some are valid, and thus satisfiable:

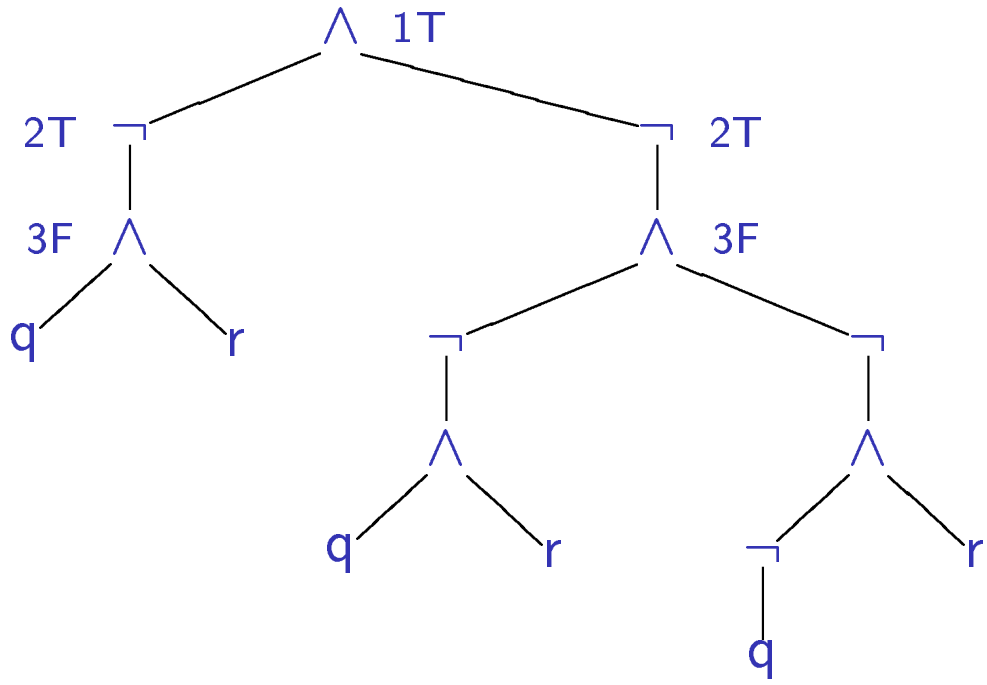
$$\top \equiv p \rightarrow p \equiv \neg(p \wedge \neg p)$$

Some are not valid, and thus not satisfiable:

$$\bot \equiv \neg\top \equiv \neg(\top\wedge\top) \equiv \neg(p \rightarrow p\wedge p \rightarrow p) \equiv \neg(\neg(p\wedge\neg p)\wedge\neg(p\wedge\neg p))$$

Extending the algorithm

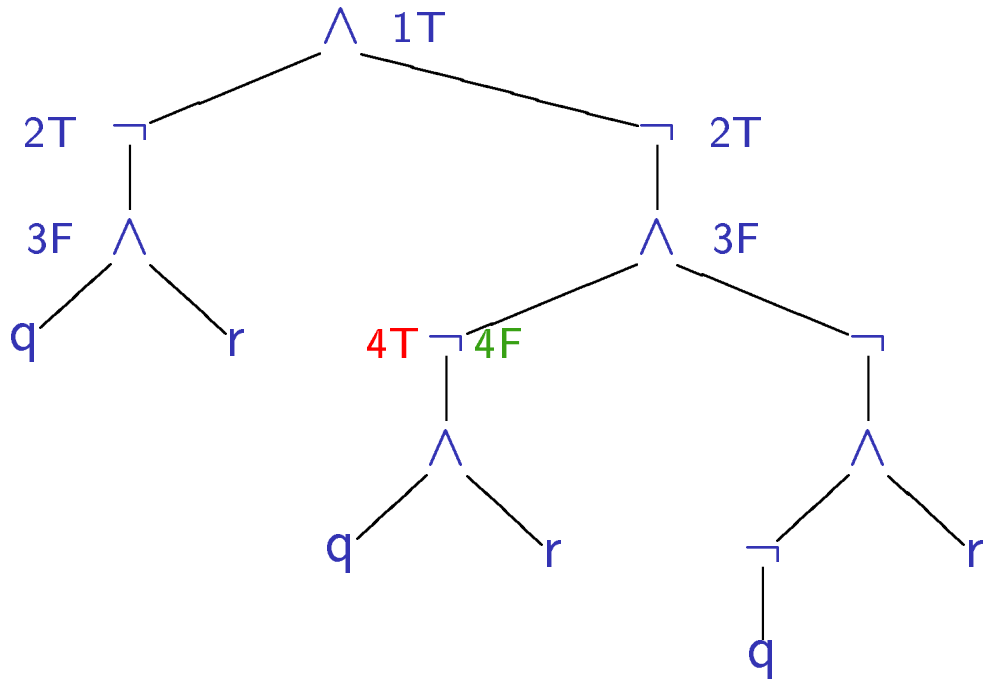
Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



Idea: pick a node and try both possibilities

Extending the algorithm

Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



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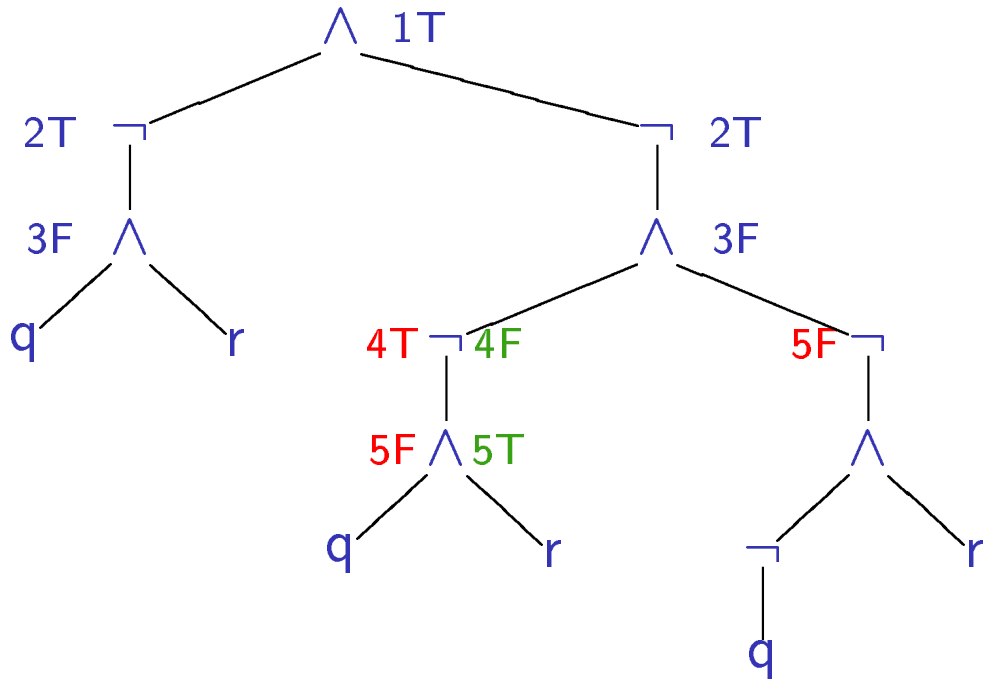

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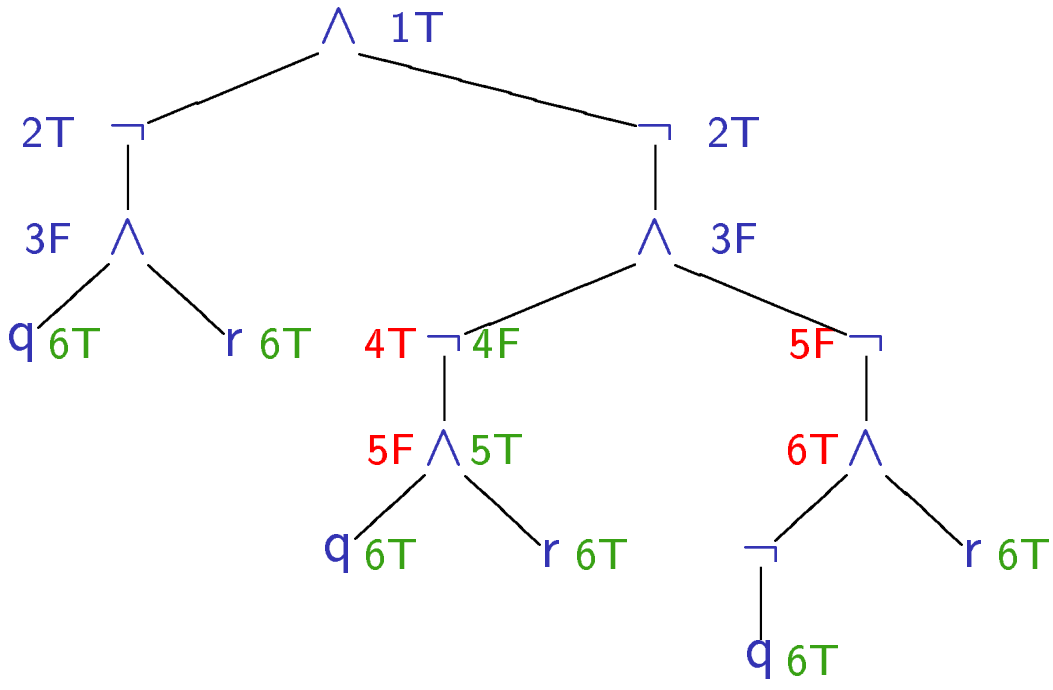

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Extending the algorithm

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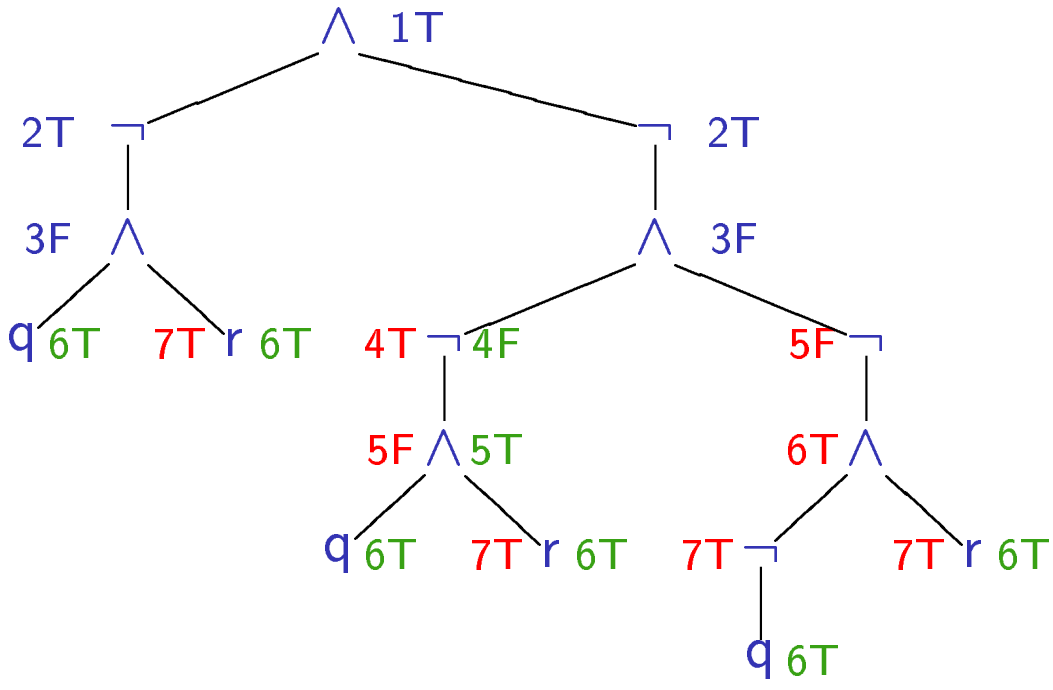

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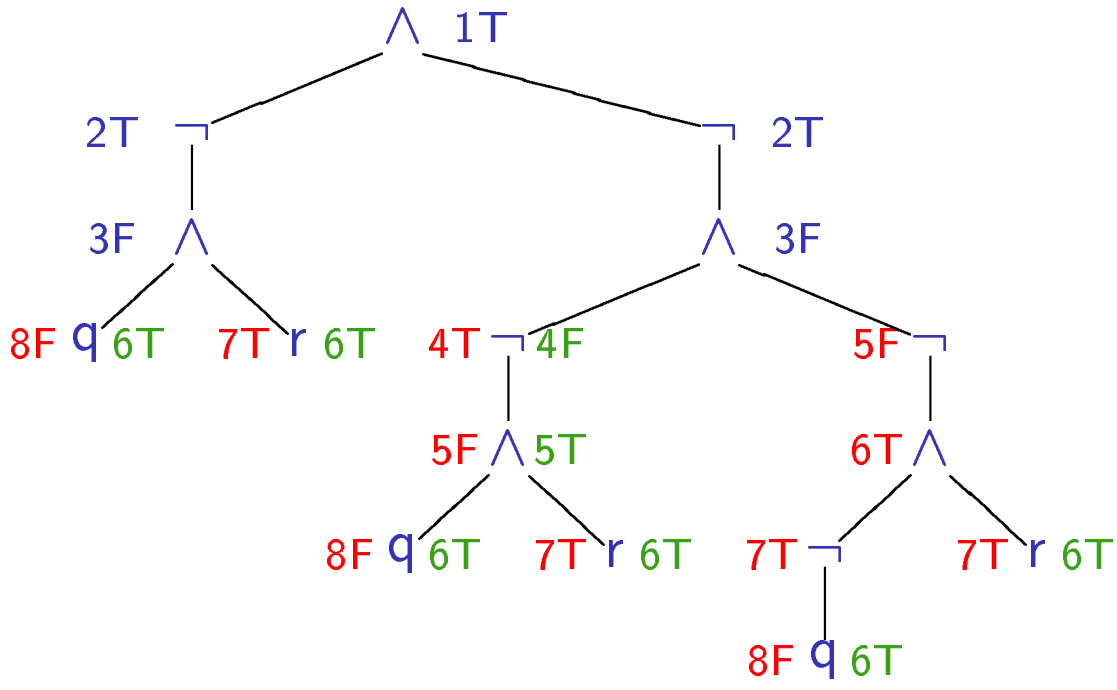
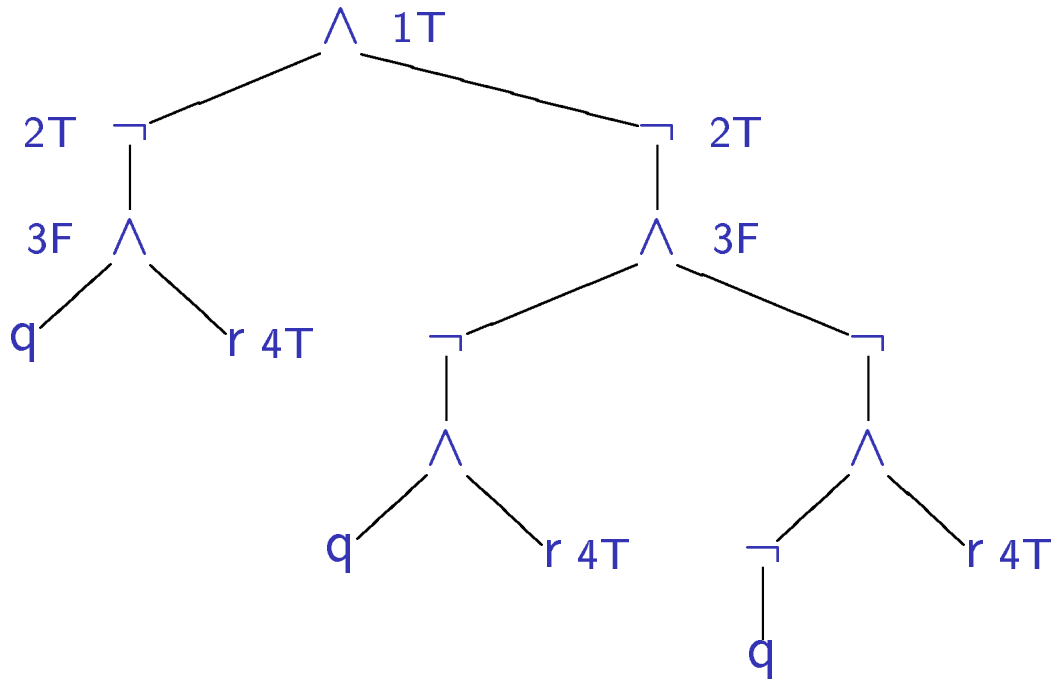
Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



r is true in both cases

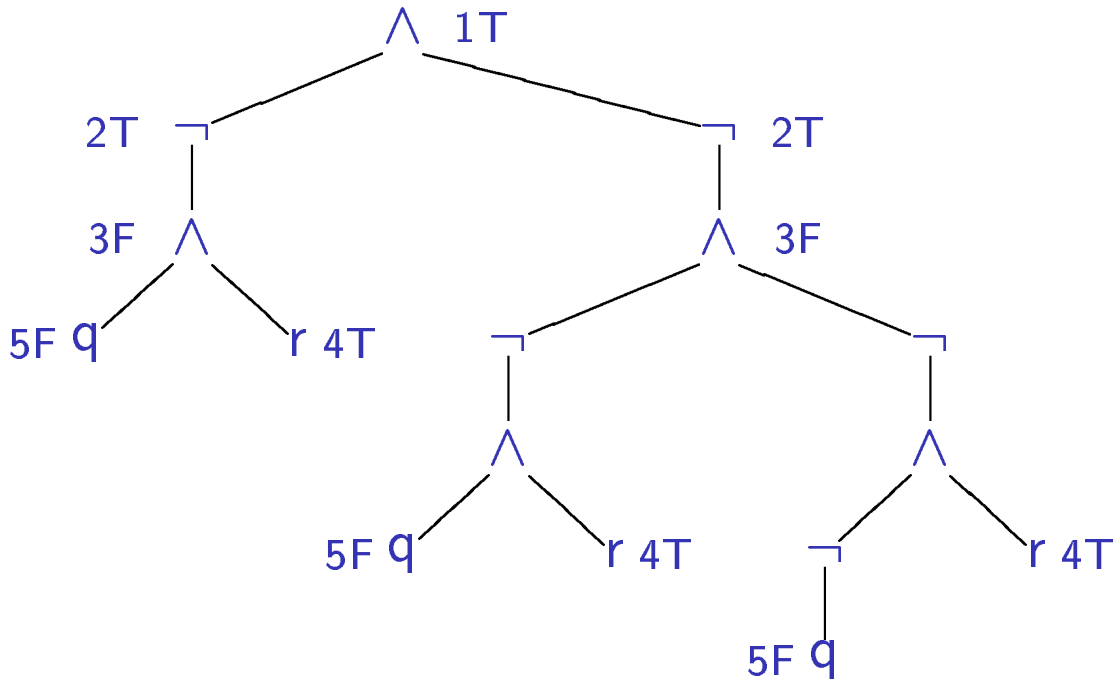
Using the value of r

Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



Using the value of r

Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



Using the value of r

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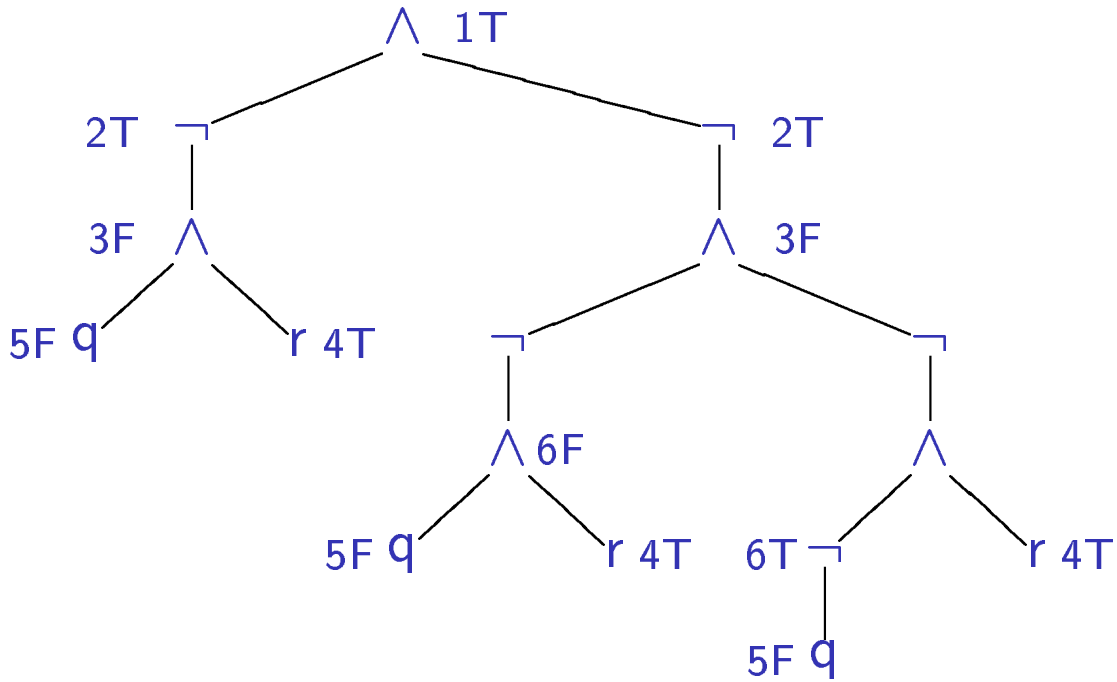
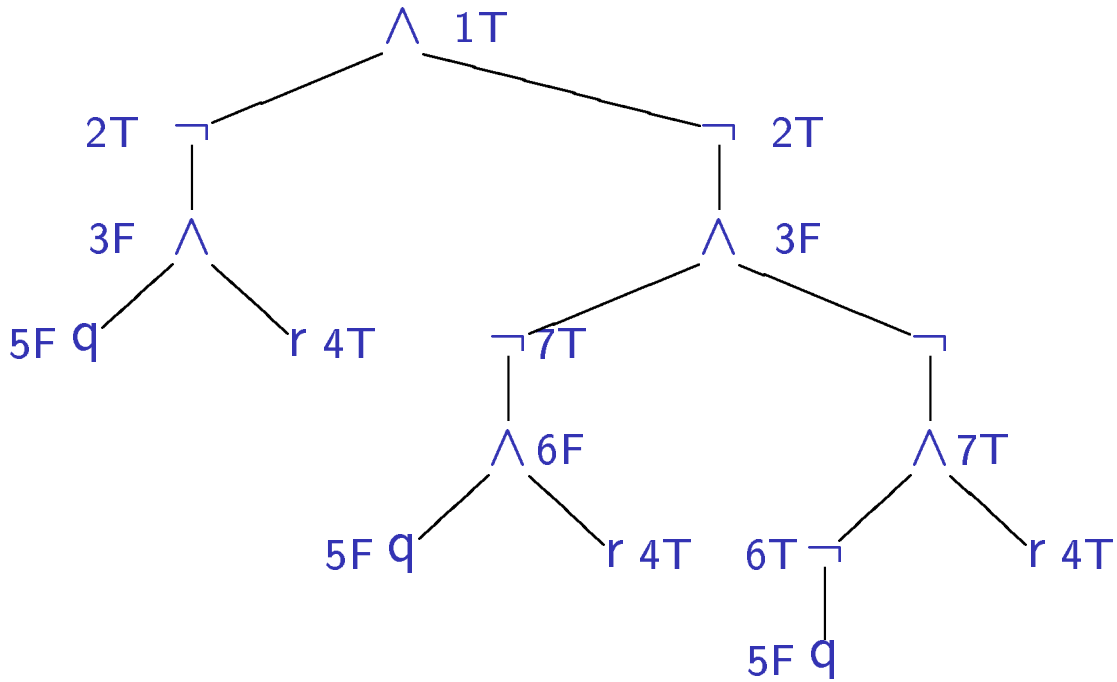


Using the value of r

Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



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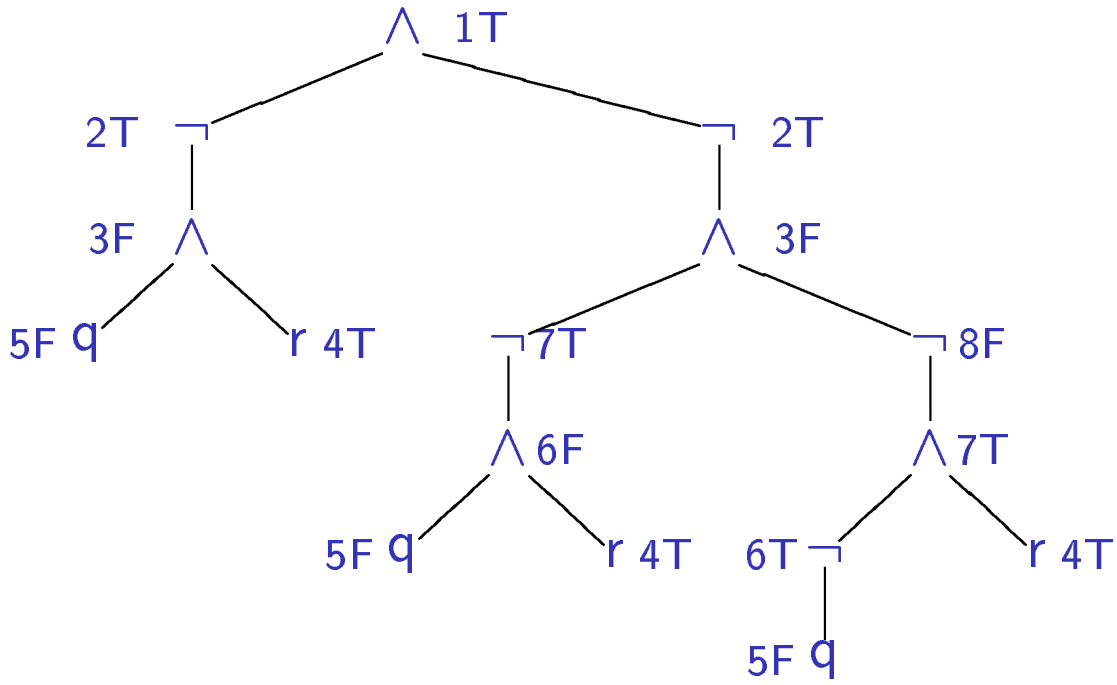


Satisfiable.

Extended algorithm

Algorithm:

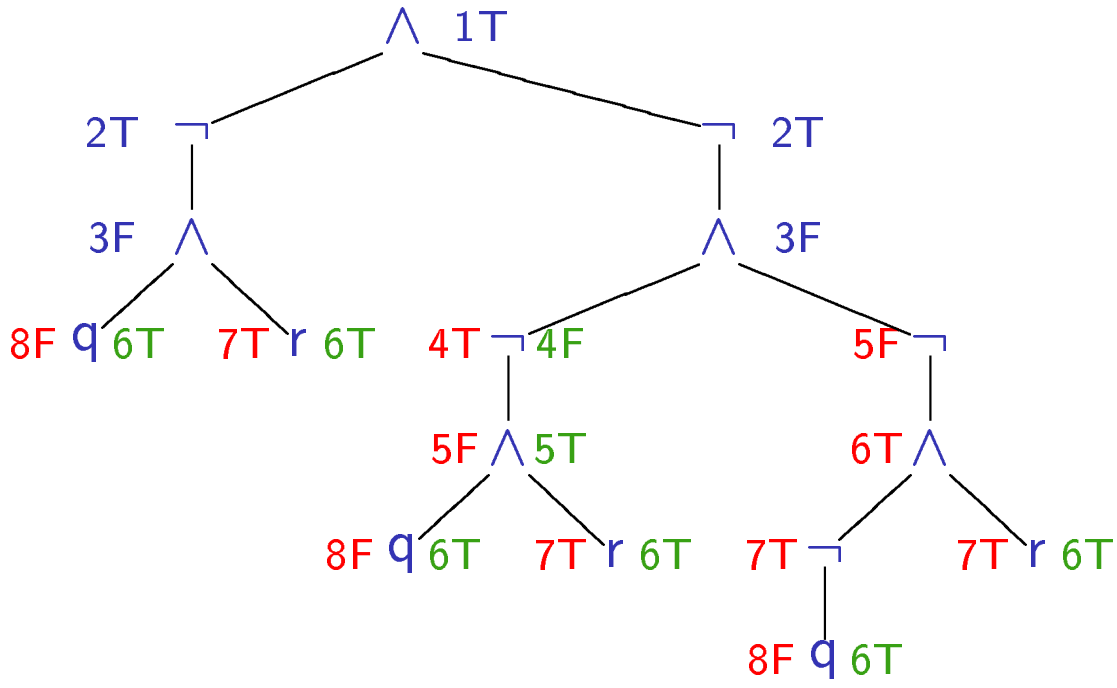
1. Pick an unmarked node and add temporary T and F marks.
2. Use the forcing rules to propagate both marks.
3. If both marks lead to a contradiction, report a contradiction.
4. If both marks lead to some node having the same value, permanently assign the node that value.
5. Erase the remaining temporary marks and continue.

Complexity $O(n^3)$:

1. Testing each unmarked node: $O(n)$
2. Testing a given unmarked node: $O(n)$
3. Repeating the whole thing when a new node is marked: $O(n)$

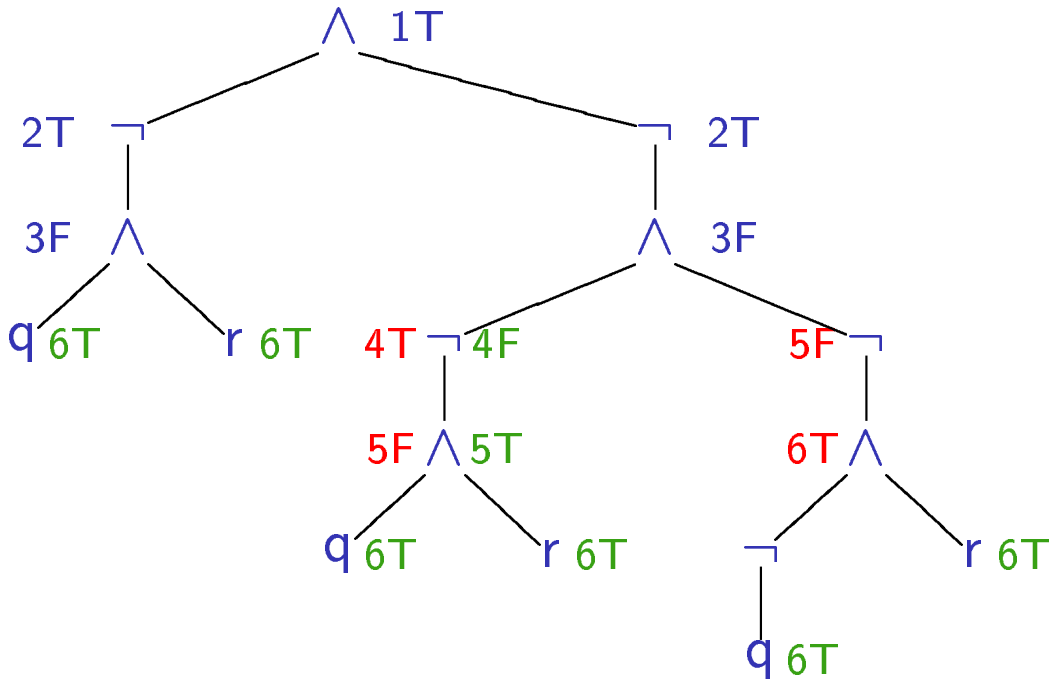
An optimization

Formula: $\neg(q \wedge r) \wedge \neg(\neg(q \wedge r) \wedge \neg(\neg q \wedge r))$



We could stop here: red values give a complete and consistent valuation.

Another optimization



- Contradiction in the leftmost subtree.
- No need to analyze q, etc.
- Permanently mark “4T4F” as T.

Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)

Perform backtracking search over values of variables

Try to satisfy each clause

M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. *Communications of the ACM, 5:394–397, 1962*

Slides from Aarti Gupta

Basic DLL Search

Performs backtracking search over variable assignments

Basic definitions

Under a given partial assignment (PA) to variables

- A variable may be
  - **assigned** (true/false literal)
  - **unassigned**.

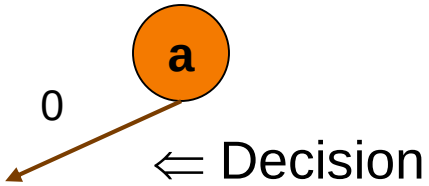
- A clause may be
  - **satisfied** (≥1 true literal)
  - **unsatisfied** (all false literals)
  - **unit** (one unassigned literal, rest false)
  - **unresolved** (otherwise)

Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)

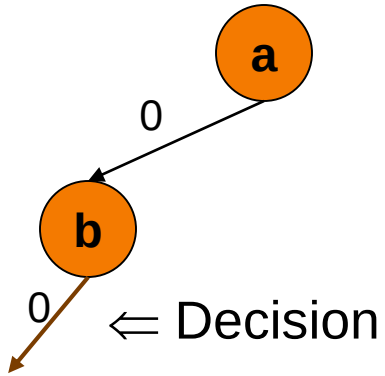
Basic DLL Search

→ **(a’ + b + c)**
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
→ **(a’ + b + c’)**
→ **(a’ + b’ + c)**



Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
→ (b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)



Basic DLL Search

(a’ + b + c)

(a + c + d)

(a + c + d’)

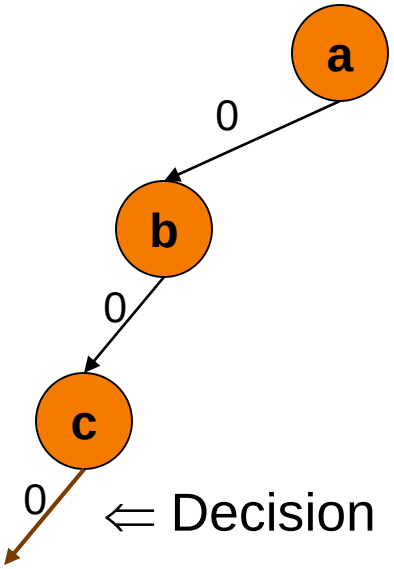
→ (a + c’ + d)

→ (a + c’ + d’)

(b’ + c’ + d)

(a’ + b + c’)

(a’ + b’ + c)



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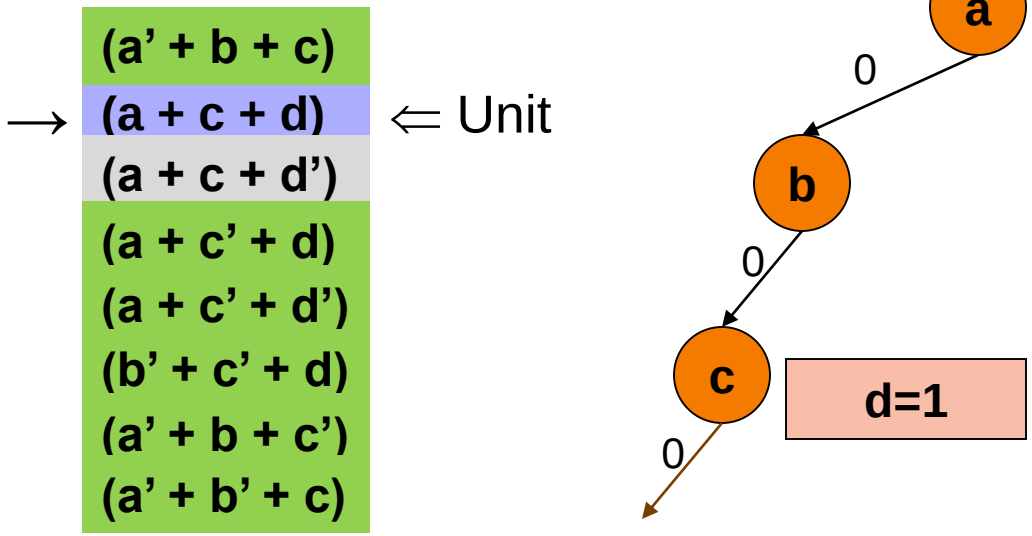
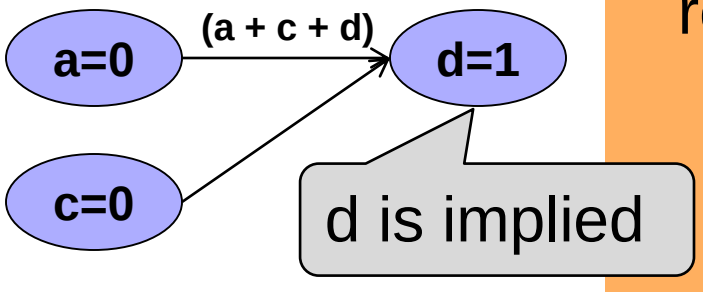


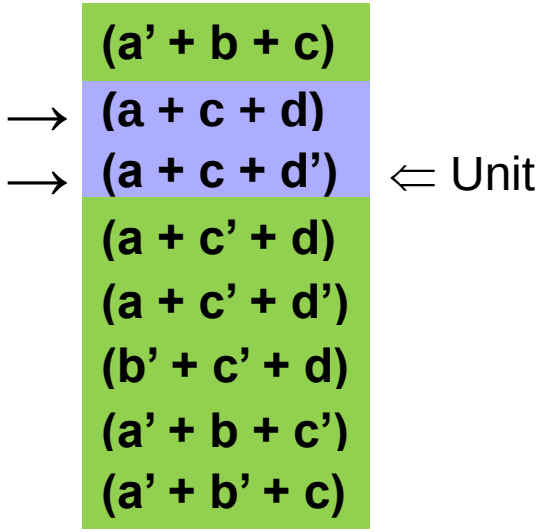
Implication Graph

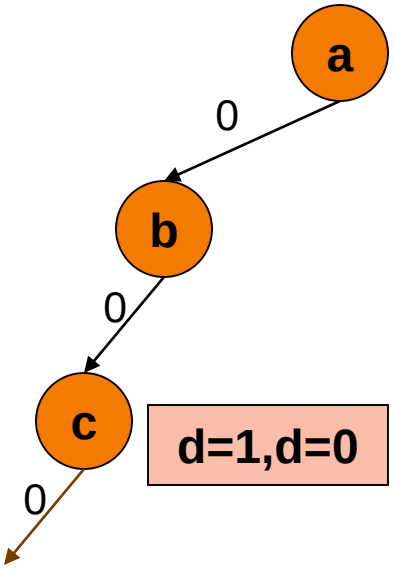


BCP: Boolean Constraint Propagation repeatedly applies *Unit Clause Rule*

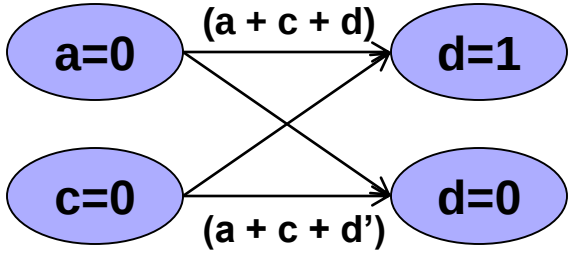
$$\frac{\text{lit} \quad \text{clause}[\text{lit}']}{\text{clause}[\bot]}$$

Basic DLL Search





Implication Graph



Basic DLL Search

(a' + b + c)

→ (a + c + d)

→ (a + c + d')

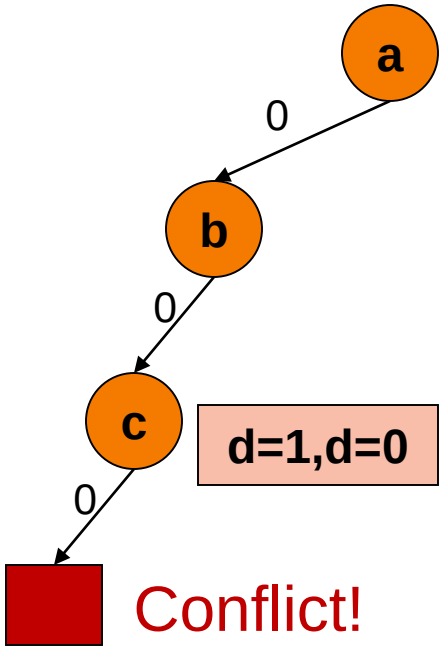
(a + c' + d)

(a + c' + d')

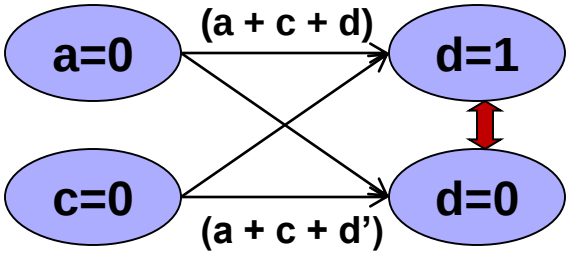
(b' + c' + d)

(a' + b + c')

(a' + b' + c)



Implication Graph



Conflict!

Basic DLL Search

(a’ + b + c)

→ (a + c + d)

→ (a + c + d’)

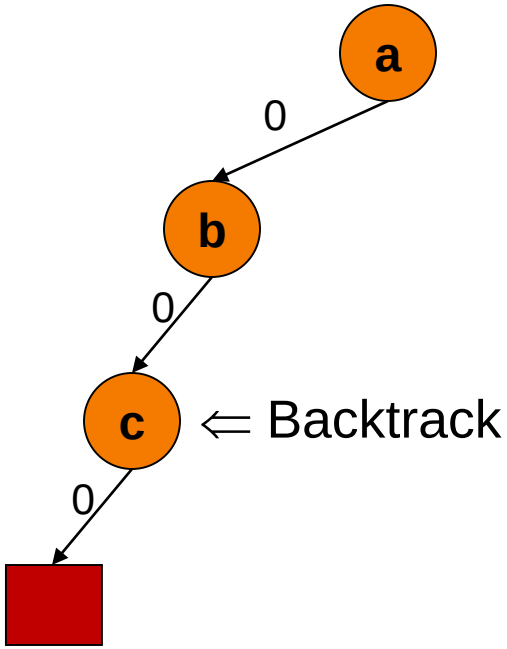
→ (a + c’ + d)

→ (a + c’ + d’)

(b’ + c’ + d)

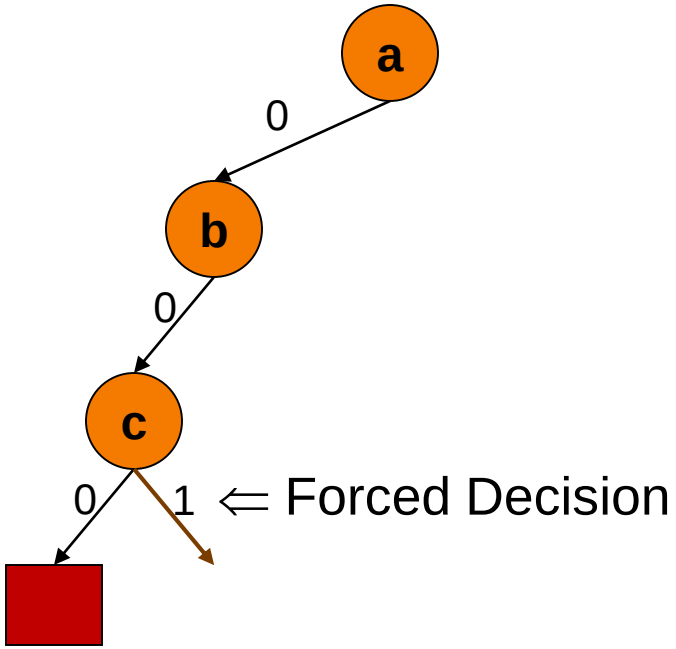
(a’ + b + c’)

(a’ + b’ + c)



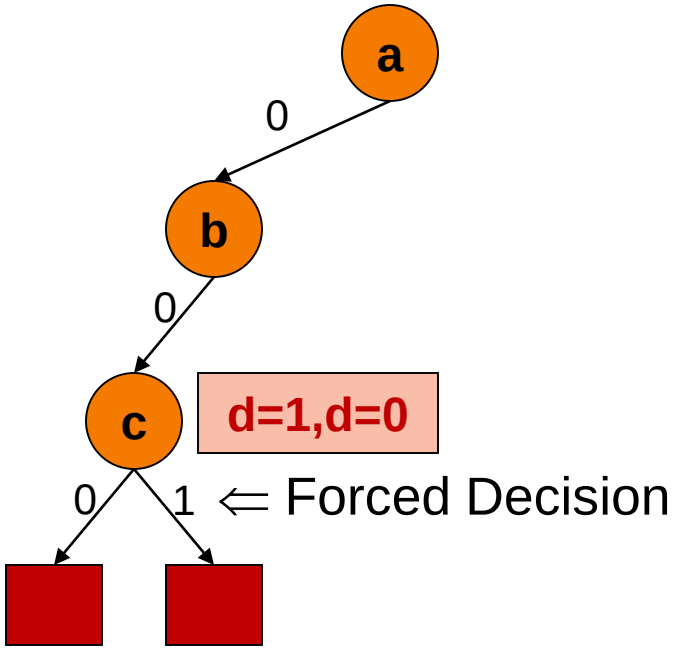
Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
→ (a + c’ + d)
→ (a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)



Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
→ (a + c’ + d)
→ (a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)



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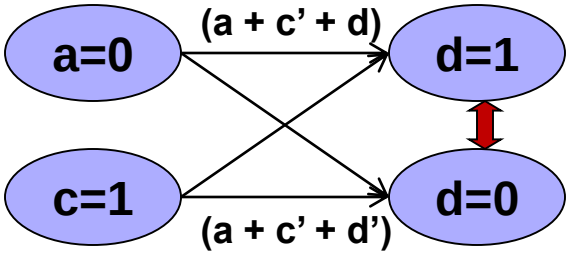


Conflict!

Basic DLL Search

(a’ + b + c)

→ (a + c + d)

→ (a + c + d’)

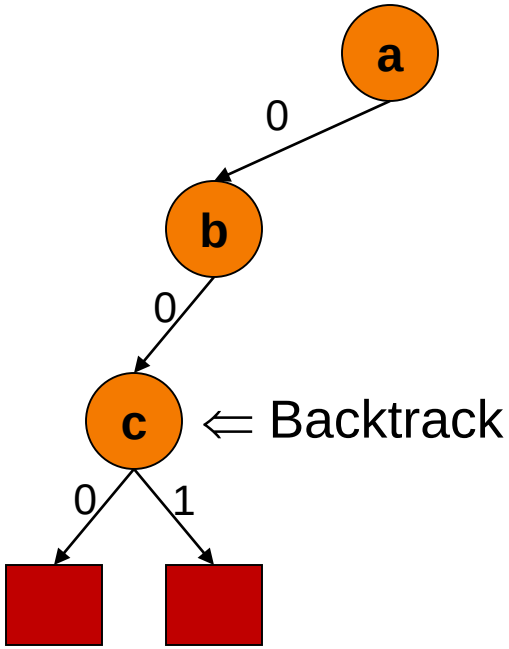
→ (a + c’ + d)

→ (a + c’ + d’)

→ (b’ + c’ + d)

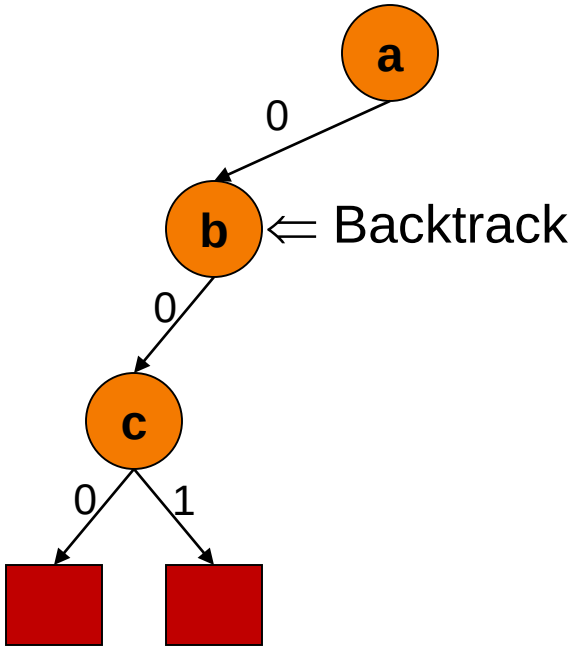
(a’ + b + c’)

(a’ + b’ + c)



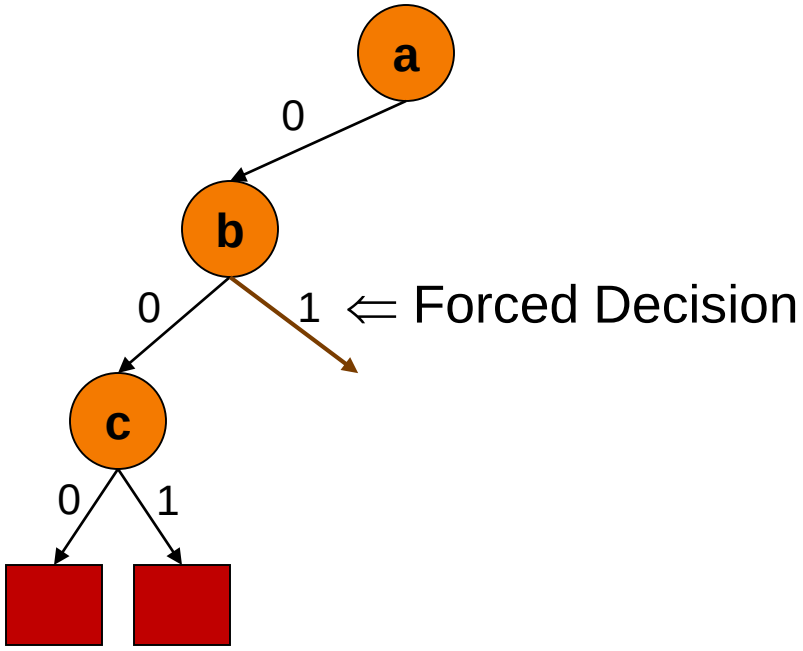
Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)



Basic DLL Search

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)



Basic DLL Search

(a’ + b + c)

→ (a + c + d)

→ (a + c + d’)

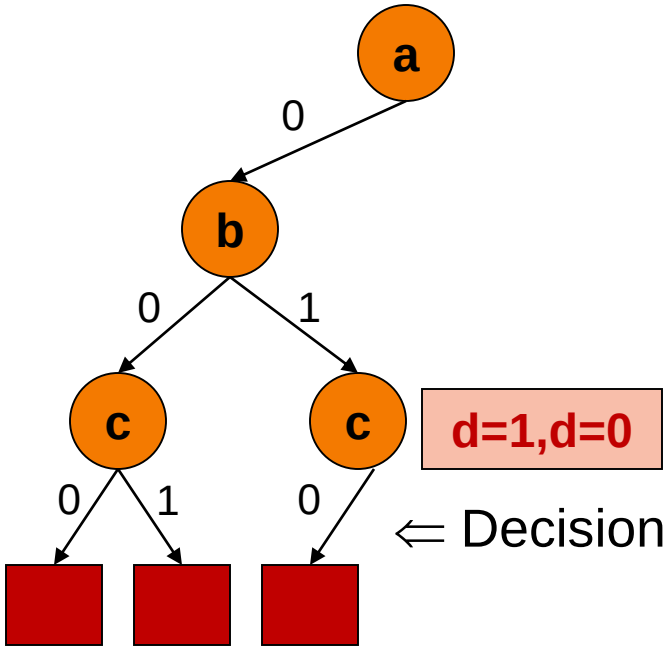
→ (a + c’ + d)

→ (a + c’ + d’)

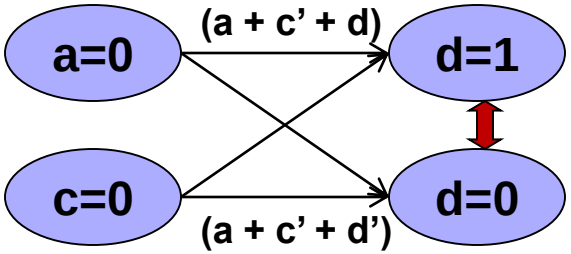
→ (b’ + c’ + d)

(a’ + b + c’)

(a’ + b’ + c)



Implication Graph



Conflict!

Basic DLL Search

(a’ + b + c)

→ (a + c + d)

→ (a + c + d’)

→ (a + c’ + d)

→ (a + c’ + d’)

→ (b’ + c’ + d)

(a’ + b + c’)

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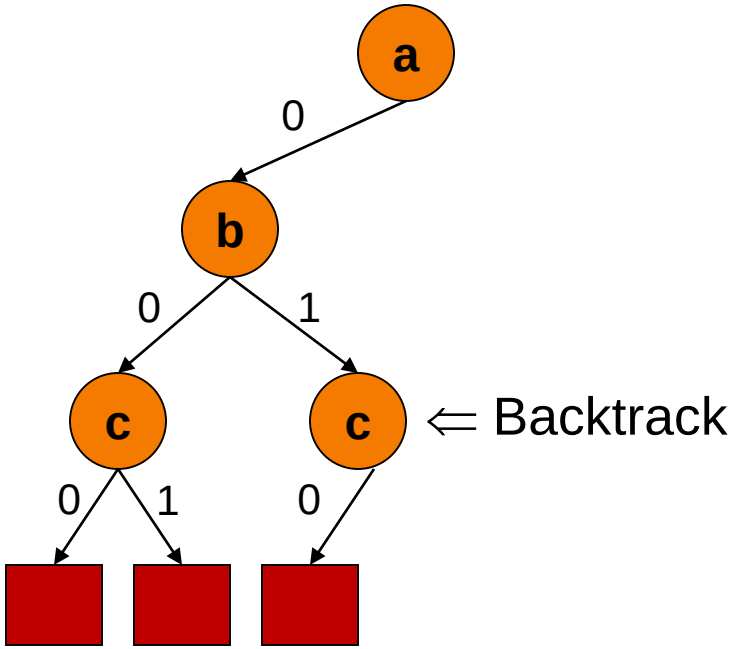

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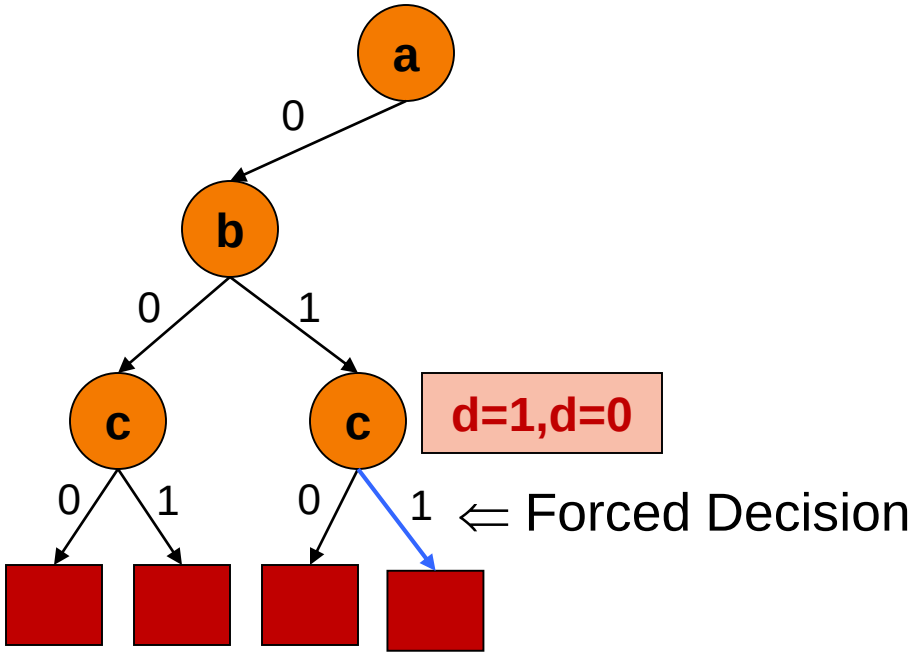

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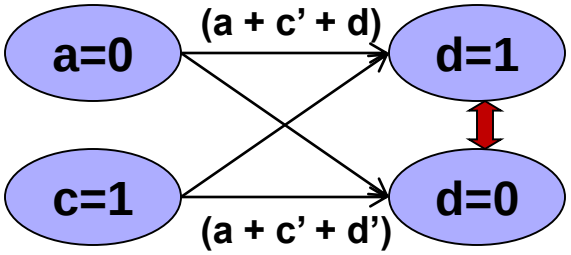

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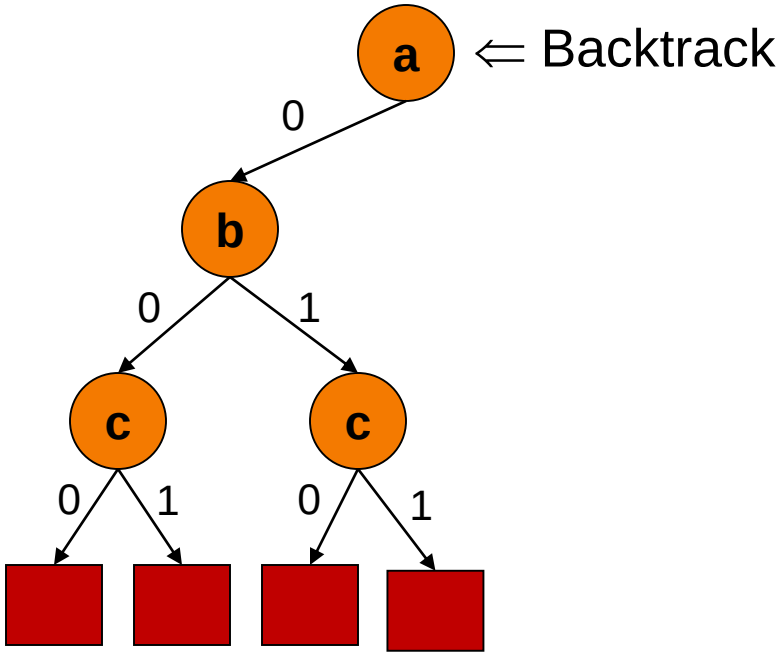

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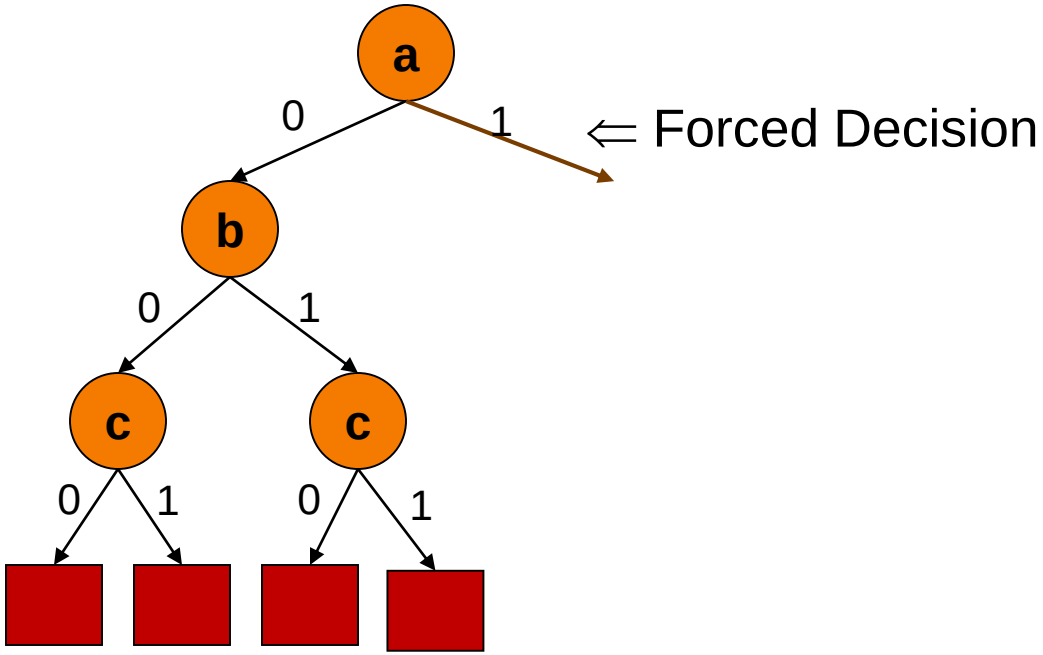

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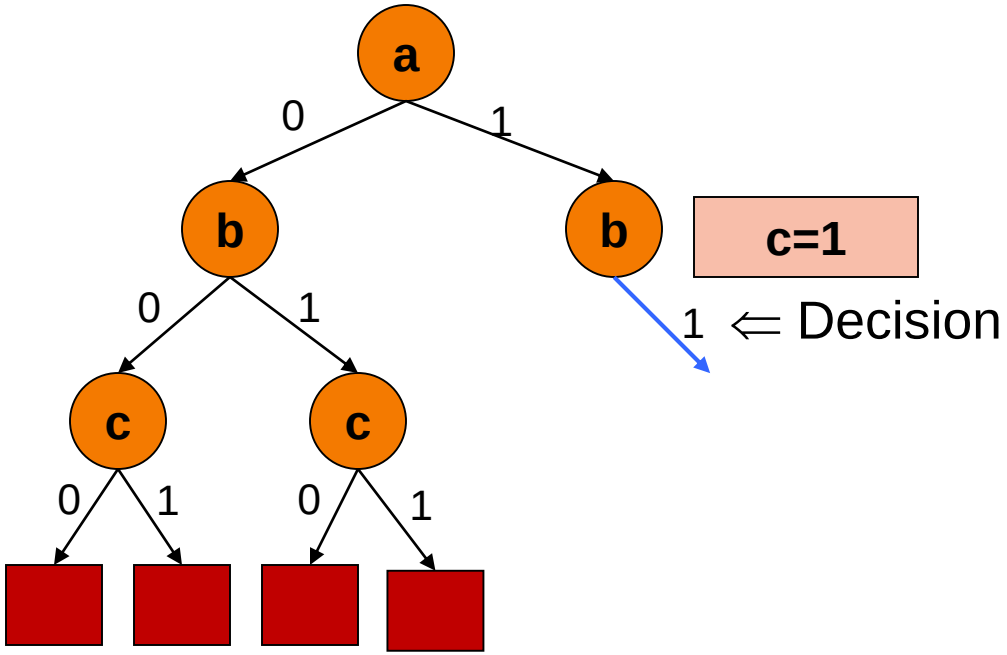

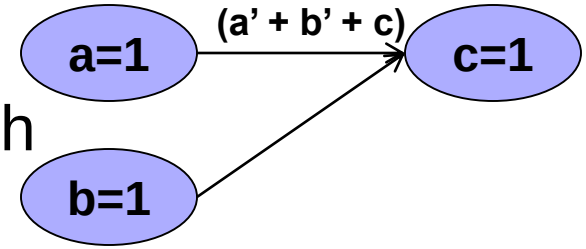

Implication Graph

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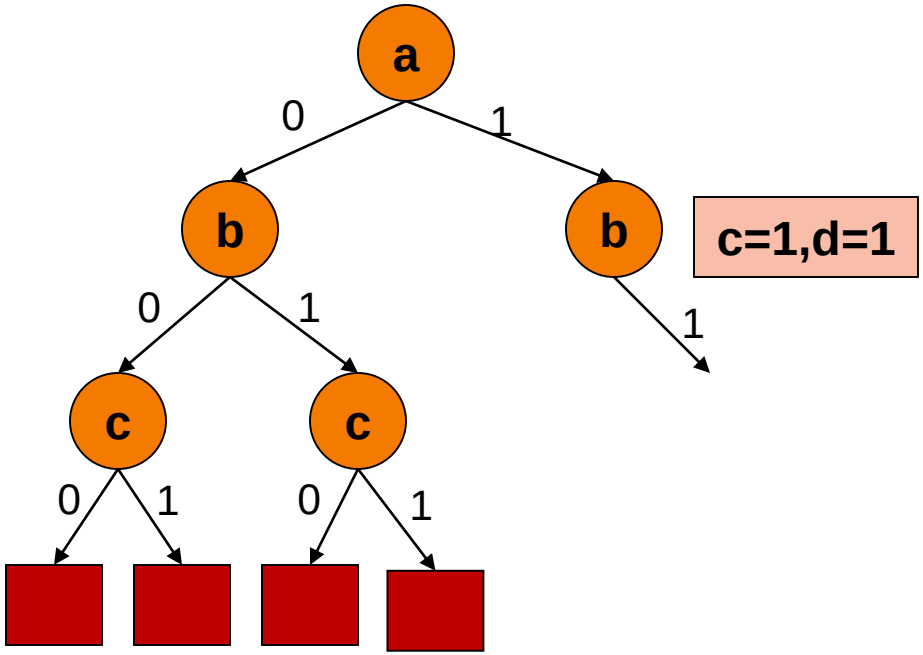
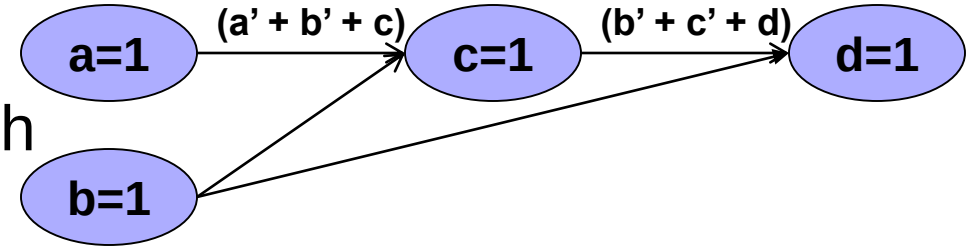


Implication Graph



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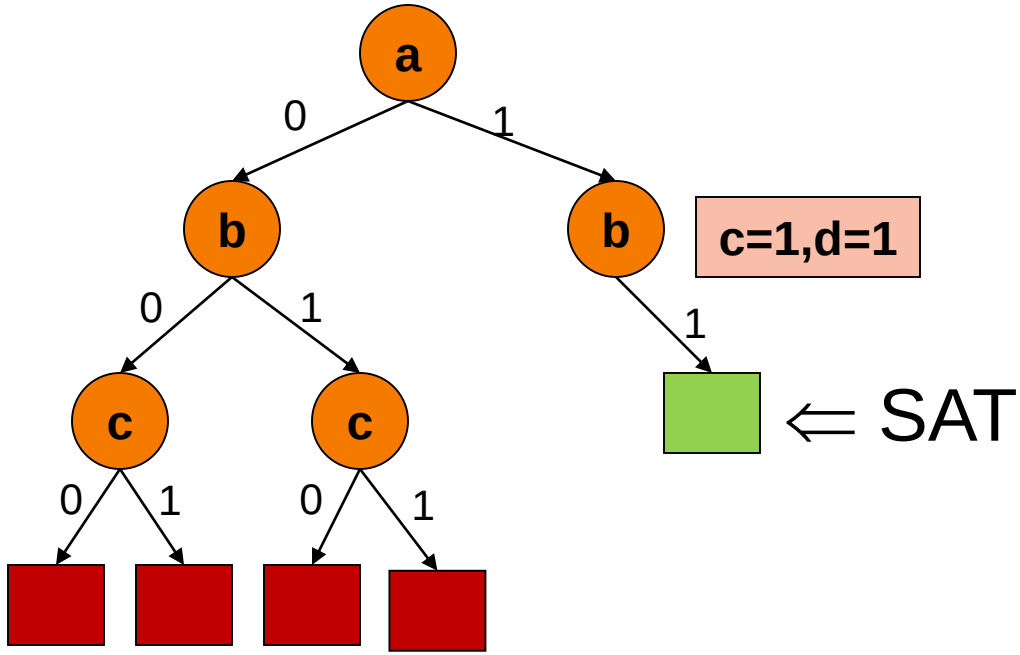
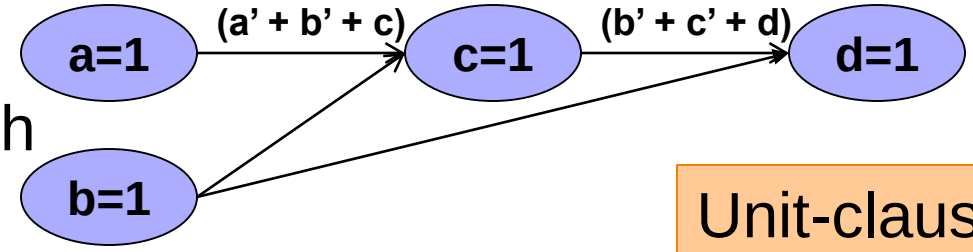
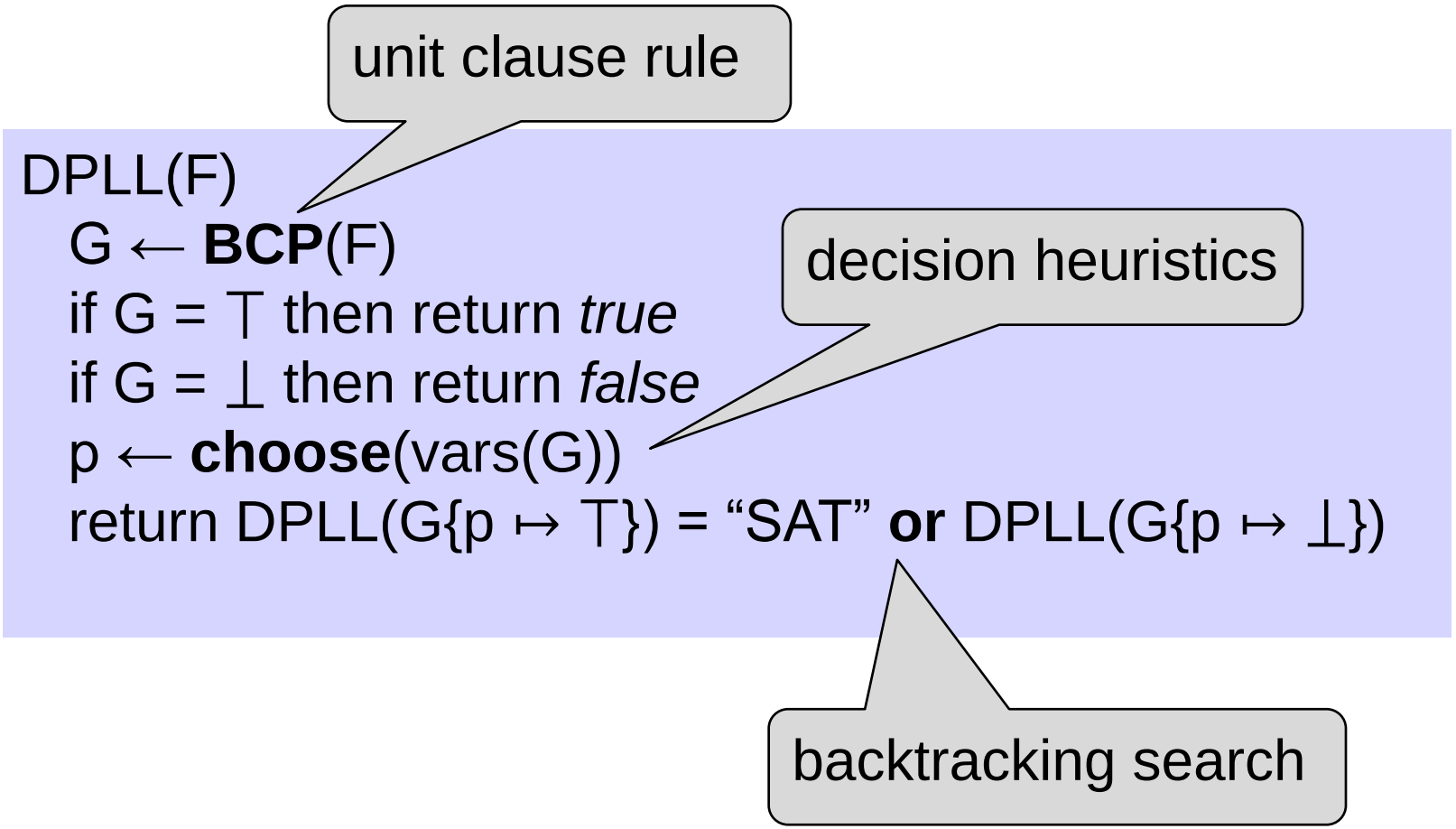


Implication Graph



Unit-clause rule with backtrack search

DPLL SAT Solver

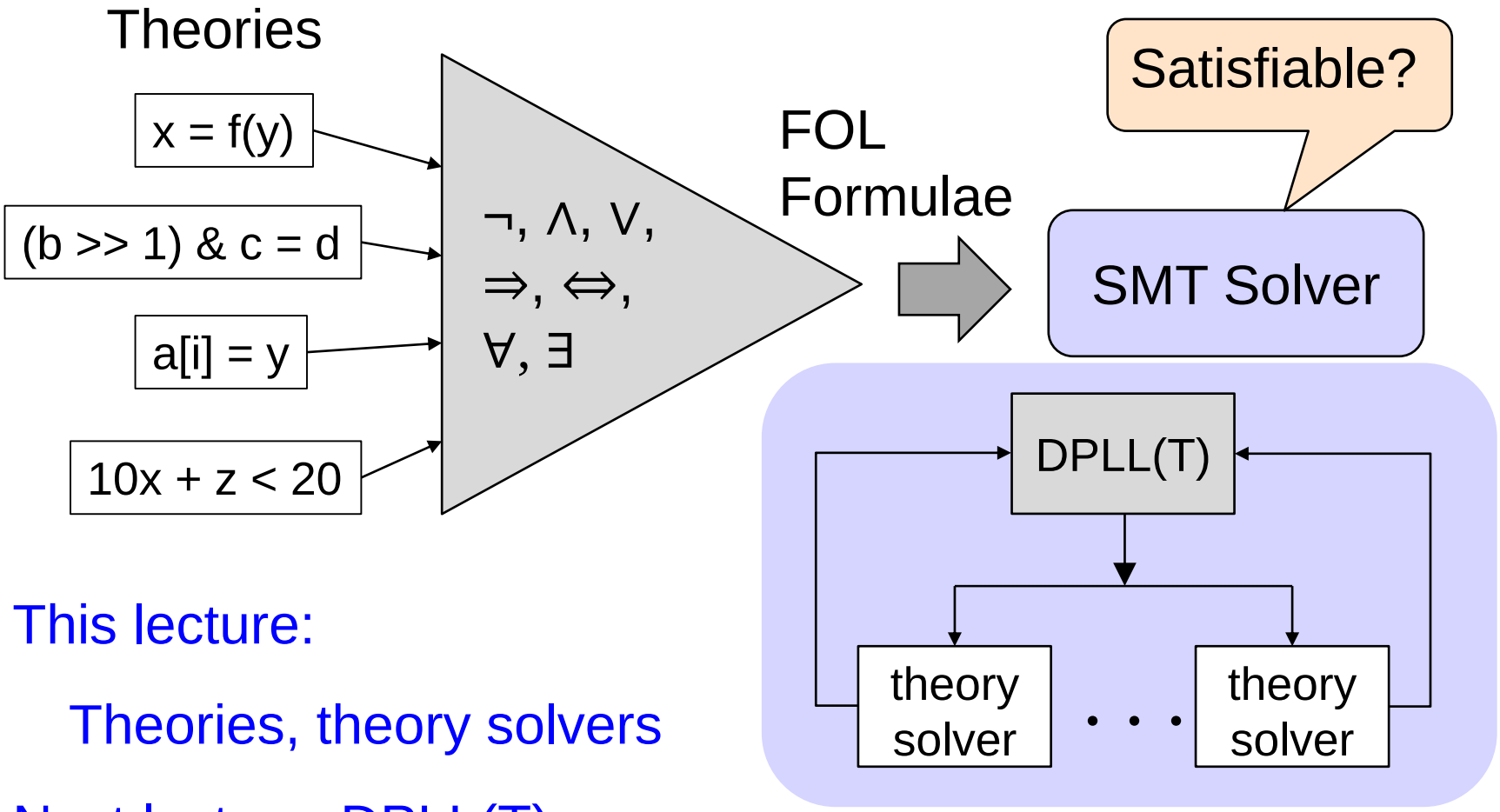


Satisfiability Modulo Theories (SMT)

Slides modified from those by Aarti Gupta

Textbook: The Calculus of Computation by A. Bradley and Z. Manna

Satisfiability Modulo Theory (SMT)



This lecture:

Theories, theory solvers

Next lecture: DPLL(T)

First-Order Theories

Software manipulates structures

- Numbers, arrays, lists, bitvectors,…

Software (and hardware) verification

- Involve reasoning about such structures

First-order theories

- Formalize structures to enable reasoning about them
- Validity is *sometimes* decidable

- Note: Validity of FOL is undecidable

First-order theories

Recall: FOL

- **Logical symbols**
  - Connectives: ¬, ∧, ∨, ⇒, ⇔
  - Quantifiers: ∀, ∃
- **Non-logical symbols**
  - Variables: x, y, z
  - N-ary functions: f, g
  - N-ary predicates: p, q
  - Constants: a, b, c

-Every dog has its day
-Some dogs have more days than others
-All cats have more days than dogs
-Triangle length theory

Interpretation of a FOL formula:
$\forall x \forall y\ x>0 \wedge y>0 \Rightarrow add(x,y)>0$

First-order theory *T* is defined by:

- **Signature Σ_T**
  - set of constant, function, and predicate symbols
- **Set of *T-Models***
  - models that fix the interpretation of symbols of Σ_T
  - alternately, can use Axioms A_T *(closed Σ_T formulae)* to provide meaning to symbols of Σ_T

Examples of FO theories

`x = f(y)`

Equality (and uninterpreted functions)

- = stands for the usual equality
- f is not interpreted in T-model

`(b >> 1) & c = d`

Fixed-width bitvectors

- >> is shift operator (function)
- & is bit-wise-and operator (function)
- 1 is a constant

`10x + z < 20`

Linear arithmetic (over R and Z)

- + is arithmetic plus (function)
- < is less-than (predicate)
- 10 and 20 are constants

`a[i] = y`

Arrays

- a[i] can be viewed as *select(a, i)* that selects the i-th element in array a

Satisfiability Modulo Theory

First-order theory *T* is defined by:

- **Signature Σ_T**
  - set of constant, function, and predicate symbols
- **Set of *T-Models***
  - models that fix the interpretation of symbols of Σ_T
  - alternately, can use Axioms A_T *(closed* Σ_T *formulae)* to provide meaning to symbols of Σ_T

A formula F is **T-satisfiable (satisfiable modulo T)** iff
M ⊨ F for some T-model M.

A formula F is **T-valid (valid modulo T)** iff
M ⊨ F for all T-models M.

Theory *T* is decidable
if *validity modulo T* is decidable
for every Σ_T-formula *F.*

There is an algorithm
that always terminates
with “yes” if *F* is *T*-valid,
and “no” if *F* is *T*-invalid.

Fragment of a Theory

Fragment of a theory *T*

is a syntactically restricted subset of formulae of the theory

Example

- *Quantifier-free fragment* (QFF) of theory *T* is the set of formulae without quantifiers
- Quantifier-free *conjunctive* fragment of theory T is the set of formulae without quantifiers and *disjunction*

Fragments

- can be decidable, even if the full theory isn't
- can have a decision procedure of lower complexity than for full theory

Theory of Equality T_E

Signature

$$\Sigma_E : \{=, a, b, c, \ldots, f, g, h, \ldots, p, q, r, \ldots\}$$

consists of

- a binary predicate "=" that is interpreted using axioms
- constant, function, and predicate symbols

Axioms of T_E

1. $\forall x.\ x=x$ (reflexivity)
2. $\forall x,y.\ x=y \rightarrow y=x$ (symmetry)
3. $\forall x,y,z.\ x=y \wedge y=z \rightarrow x=z$ (transitivity)
4. for each n-ary function symbol f,

   $\forall x_1,\ldots,x_n,y_1,\ldots,y_n.\ \bigwedge_i (x_i=y_i) \rightarrow$

   $f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$ (function congruence)
5. for each n-ary predicate symbol p,

   $\forall x_1,\ldots,x_n,y_1,\ldots,y_n.\ \bigwedge_i (x_i=y_i) \rightarrow$

   $(p(x_1,\ldots,x_n) \leftrightarrow p(y_1,\ldots,y_n))$ (predicate congruence)

Decidability of T_E

Bad news

- T_E is undecidable

Good news

- Quantifier-free fragment of T_E is decidable
- Very efficient algorithms for QFF conjunctive fragment
  - Based on congruence closure

Theory solver for T_E

- In 1954 Ackermann showed that the theory of equality and uninterpreted functions is decidable.
- In 1976 Nelson and Oppen implemented an $O(m^2)$ algorithm based on congruence closure computation.

- Modern implementations are based on the union-find data structure (*data structures again!)*
- Efficient: $O(n \log n)$

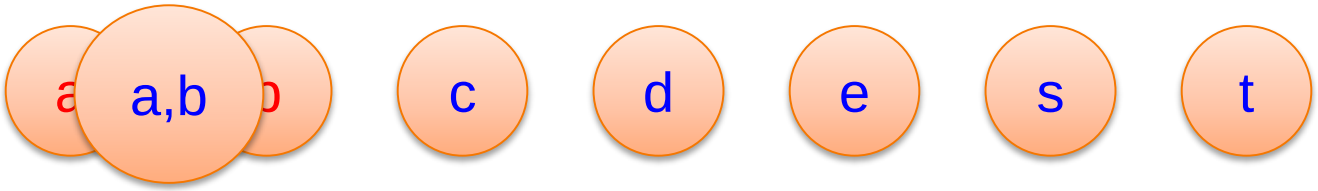
Deciding Equality

$a = b,\ b = c,\ d = e,\ b = s,\ d = t,\ a \neq e,\ a \neq s$

Note: Quantifier-free, Conjunctive

Deciding Equality

$a = b,\ b = c,\ d = e,\ b = s,\ d = t,\ a \neq e,\ a \neq s$



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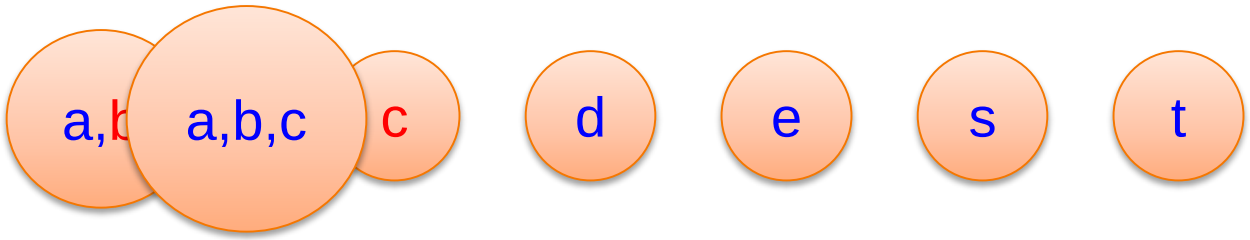

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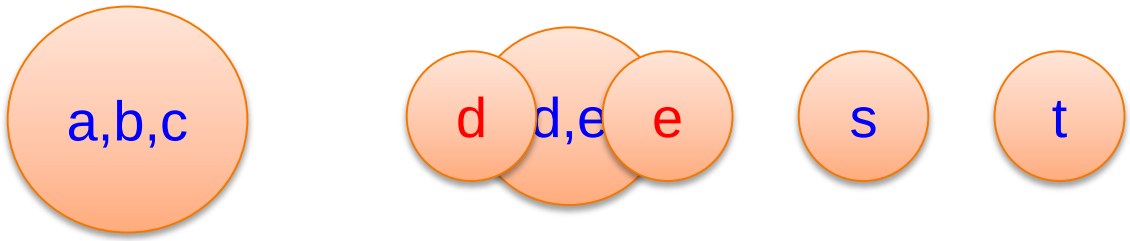

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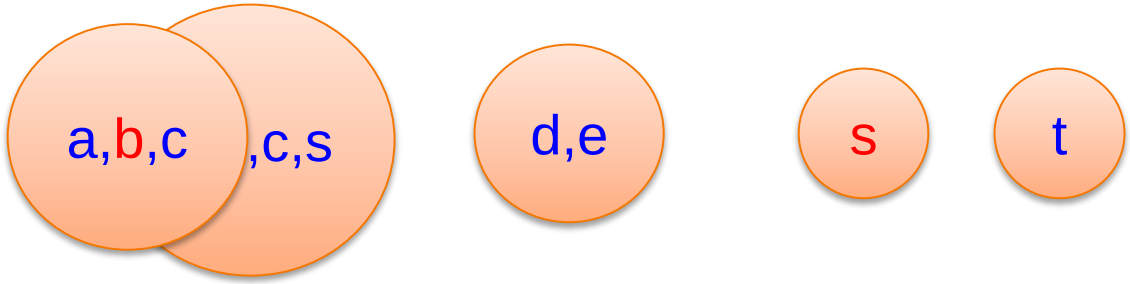

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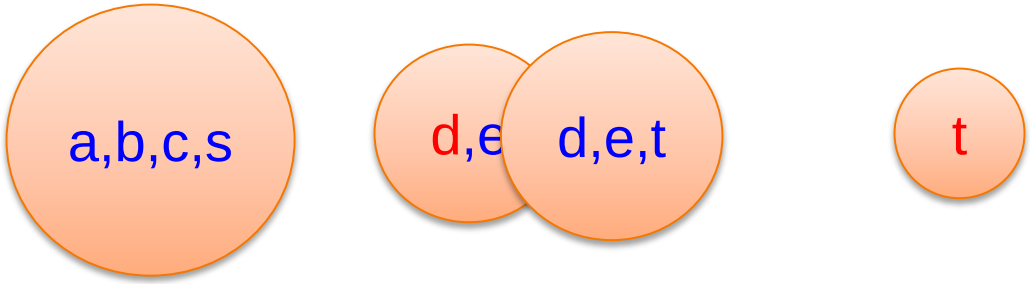

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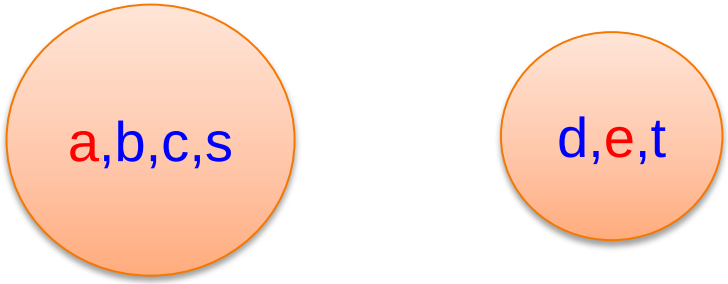

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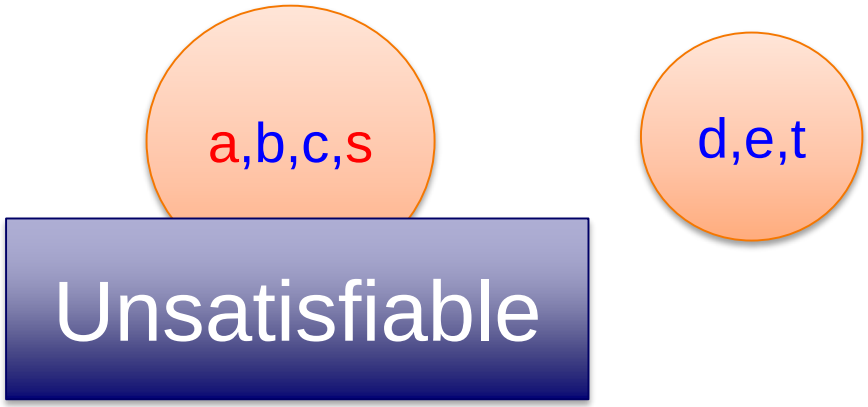
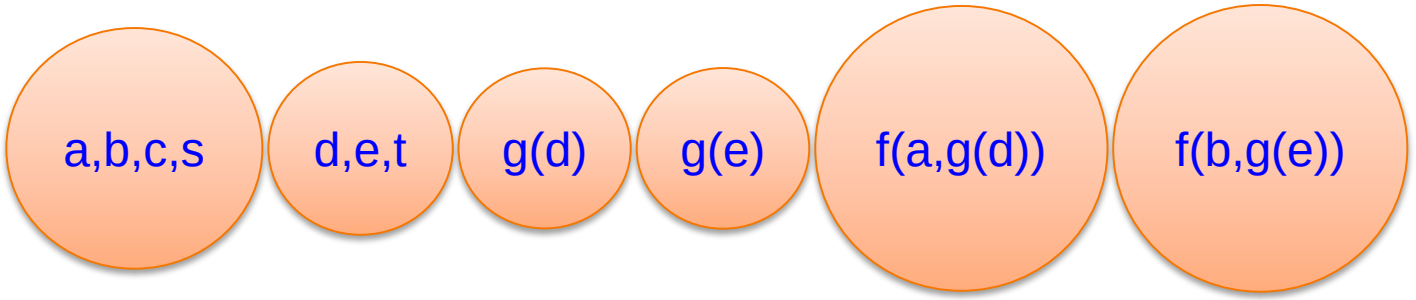


Deciding Equality + (uninterpreted) Functions

$a = b,\ b = c,\ d = e,\ b = s,\ d = t,\ f(a, g(d)) \neq f(b, g(e))$



Congruence Rule:

$x_1 = y_1, \ldots, x_n = y_n$ implies $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$

Deciding Equality + (uninterpreted) Functions

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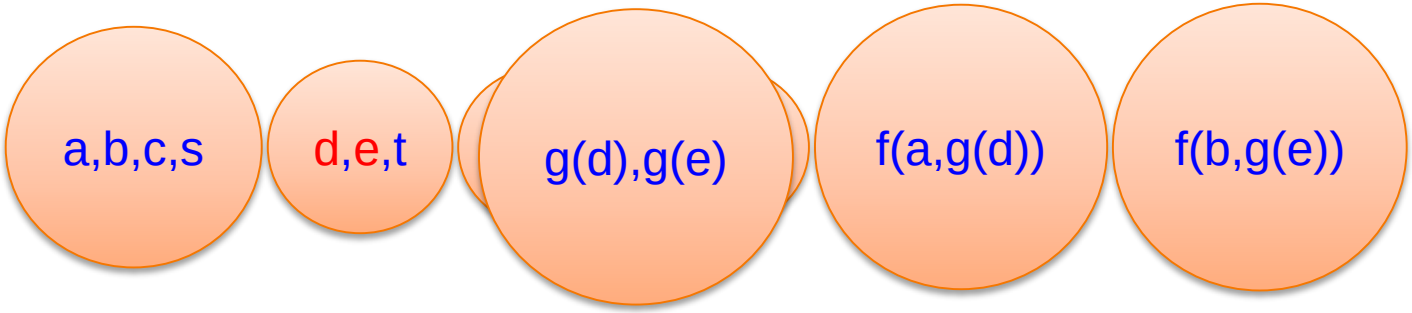

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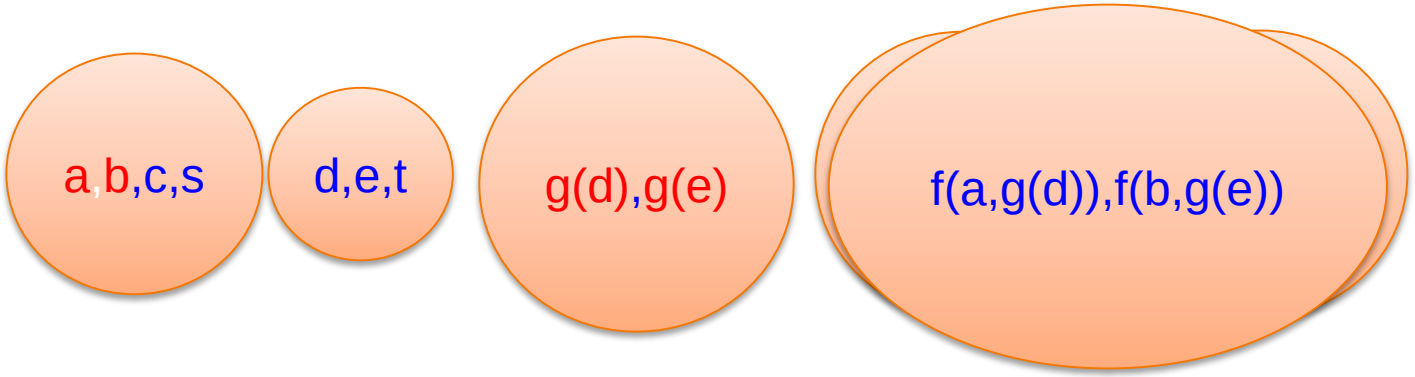

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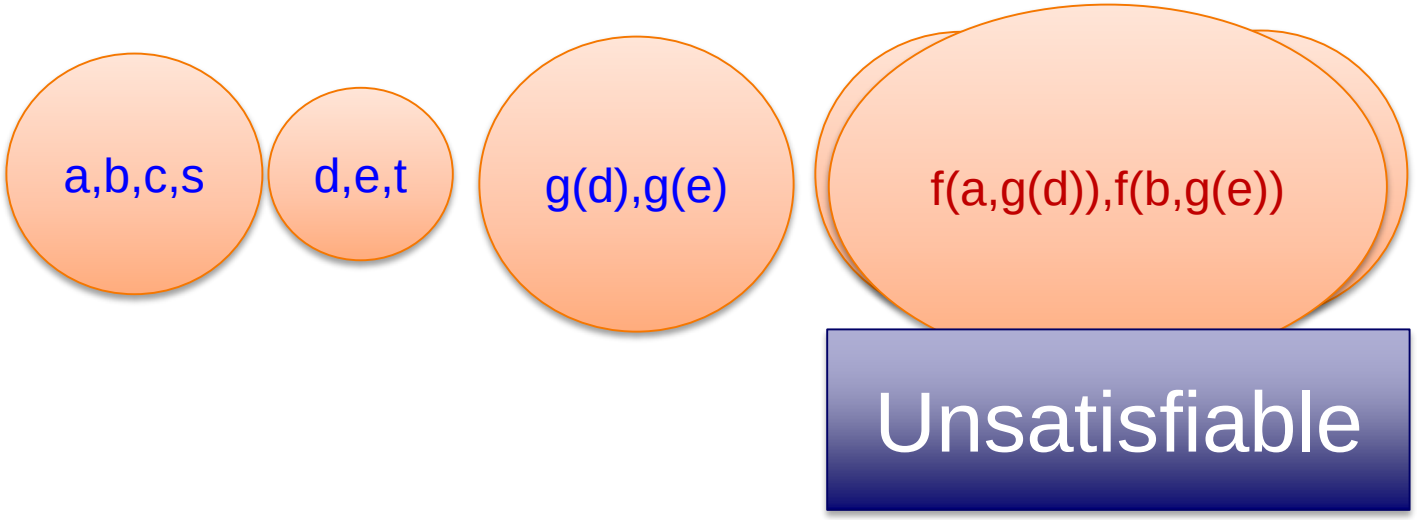


Efficient implementation using Union-Find data structure

Example: program equivalence

```
int fun1(int y) {
  int x, z;
  z = y;
  y = x;
  x = z;
  return x*x;
}

int fun2(int y) {
  return y*y;
}
```

T_E formula that is satisfiable iff programs are not equivalent:

$(z1 = y0 \wedge y1 = x0 \wedge x1 = z1 \wedge r1 = x1*x1) \wedge$
$(r2 = y0* y0) \wedge$
$\neg(r2 = r1)$

quantifier-free conjunctive fragment

Using 32-bit integers,
and interpreting * as multiplication,
a SAT solver fails to return an answer in 1 minute.

Example: program equivalence

```
int fun1(int y) {
  int x, z;
  z = y;
  y = x;
  x = z;
  return x*x;
}

int fun2(int y) {
  return y*y;
}
```

T_E formula that is satisfiable iff programs are not equivalent:

$(z1 = y0 \wedge y1 = x0 \wedge x1 = z1 \wedge r1 = \mathbf{sq}(x1) \wedge (r2 = \mathbf{sq}(y0)) \wedge \neg(r2 = r1)$

uninterpreted function symbol **sq** (abstraction of *)

Using T_E (with uninterpreted functions), SMT solver proves unsat in a fraction of a second.

Example: program equivalence

```
int fun1(int y) {
  int x, z;
  x = x ^ y;
  y = x ^ y;
  x = x ^ y;
  return x*x;
}

int fun2(int y) {
  return y*y;
}
```

Is the uninterpreted function abstraction going to work in this case?

No, we need the theory of fixed-width bitvectors to reason about ^ (xor).

Theory of fixed-width bitvectors T_{BV}

Signature

- constants
- fixed-width words (bitvectors) for modeling machine ints, longs, etc.
- arithmetic operations (+, -, *, /, etc.) (functions)
- bitwise operations (&, |, ^, etc.) (functions)
- comparison operators (<, >, etc.) (predicates)
- equality (=)

Theory of fixed-width bitvectors is decidable

- Bit-flattening to SAT: NP-complete complexity

Bit-Vector Logic: Syntax

$$
\begin{array}{rcl}
\mathit{formula} & : & \mathit{formula} \vee \mathit{formula} \mid \neg \mathit{formula} \mid \mathit{atom} \\
\mathit{atom} & : & \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{Boolean\text{-}Identifier} \mid \mathit{term}[\,\mathit{constant}\,] \\
\mathit{rel} & : & = \mid < \\
\mathit{term} & : & \mathit{term}\ \mathit{op}\ \mathit{term} \mid \mathit{identifier} \mid \sim \mathit{term} \mid \mathit{constant} \mid \\
 & & \mathit{atom}?\mathit{term} : \mathit{term} \mid \\
 & & \mathit{term}[\,\mathit{constant} : \mathit{constant}\,] \mid \mathit{ext}(\,\mathit{term}\,) \\
\mathit{op} & : & + \mid - \mid \cdot \mid / \mid << \mid >> \mid \& \mid \mid \mid \oplus \mid \circ
\end{array}
$$

Bit-Vector Logic: Syntax

$$
\begin{array}{rcl}
formula & : & formula \vee formula \mid \neg formula \mid atom \\
atom & : & term\ rel\ term \mid Boolean\text{-}Identifier \mid term[\, constant \,] \\
rel & : & = \mid < \\
term & : & term\ op\ term \mid identifier \mid \sim term \mid constant \mid \\
 & & atom?term:term \mid \\
 & & term[\, constant : constant \,] \mid ext(\, term \,) \\
op & : & + \mid - \mid \cdot \mid / \mid << \mid >> \mid \& \mid \mid \mid \oplus \mid \circ
\end{array}
$$

$\sim x$: bit-wise negation of x

$ext(x)$: sign- or zero-extension of x

$x << d$: left shift with distance d

$x \circ y$: concatenation of x and y

A simple decision procedure

Transform Bit-Vector Logic to Propositional Logic

Most commonly used decision procedure

Also called '*bit-blasting*'

A simple decision procedure

Transform Bit-Vector Logic to Propositional Logic

Most commonly used decision procedure

Also called '*bit-blasting*'

Bit-Vector Flattening

1. Convert propositional part as before
2. Add a *Boolean variable for each bit* of each sub-expression (term)
3. Add *constraint* for each sub-expression

We denote the new Boolean variable for i of term t by t_i

Bit-vector flattening

What constraints do we generate for a given term?

What constraints do we generate for a given atom

This is easy for the bit-wise operators.

Example for $t=a \mid b$

$$\bigwedge_{i=0}^{l-1} t_i \quad = (a_i \vee b_i))$$

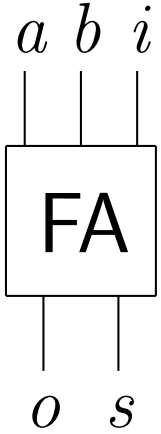
What about x=y

Flattening bit-vector arithmetic

How to flatten $s=a+b$

How to flatten $s=a+b$

$\longrightarrow$ we can build a *circuit* that adds them!



Full Adder

$$s \equiv (a+b+i) \bmod 2 \equiv a \oplus b \oplus i$$

$$o \equiv (a+b+i) \operatorname{div} 2 \equiv a \cdot b + a \cdot i + b \cdot i$$

The full adder in CNF:

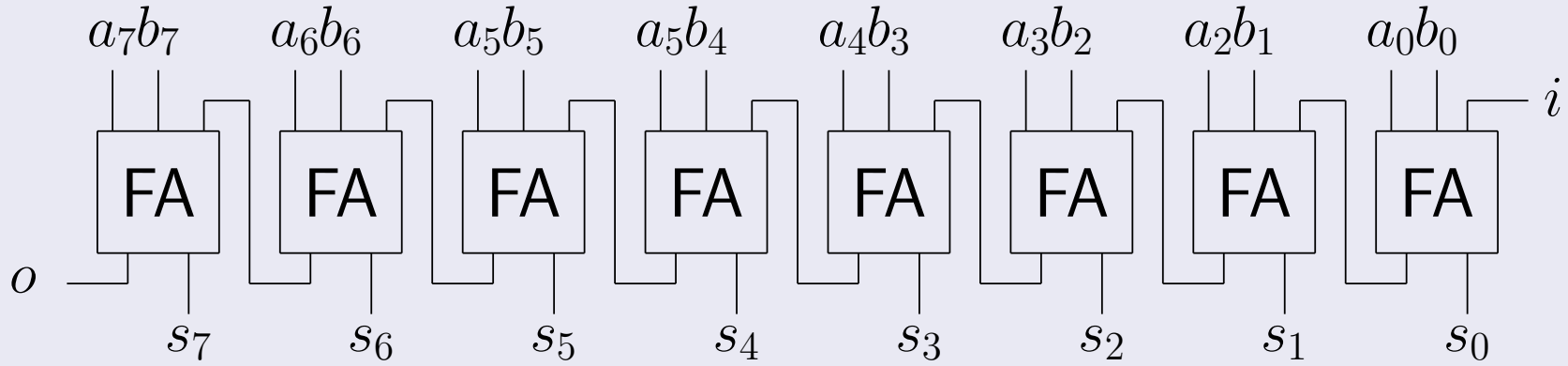
$$(a \vee b \vee \neg o) \wedge (a \vee \neg b \vee i \vee \neg o) \wedge (a \vee \neg b \vee \neg i \vee o) \wedge$$
$$(\neg a \vee b \vee i \vee \neg o) \wedge (\neg a \vee b \vee \neg i \vee o) \wedge (\neg a \vee \neg b \vee o)$$

Flattening bit-vector arithmetic

Ok, this is good for one bit! How about more?

Ok, this is good for one bit! How about more?

8-Bit ripple carry adder (RCA)



Also called *carry chain adder*

Adds l variables

Adds $10 * l$ clauses 6 for o , 4 for s

Multipliers

Multipliers result in very hard formulas

Example:

$$a \cdot b = c \wedge b \cdot a \neq c \wedge x < y \wedge x > y$$

CNF: About 11000 variables, unsolvable for current SAT solvers

Similar problems with division, modulo

Multipliers

Multipliers result in very hard formulas

Example:

$$a \cdot b = c \wedge b \cdot a \neq c \wedge x < y \wedge x > y$$

CNF: About 11000 variables, unsolvable for current SAT solvers

Similar problems with division, modulo

Theories for Arithmetic

Natural numbers $\mathbb{N}$ = {0,1,2,…}

Integers $\mathbb{Z}$ = {…,-2,-1,0,1,2,…}

Three theories: (Axioms in [BM Ch. 3])

Peano arithmetic T_{PA}

- Natural numbers with addition (+), multiplication (*), equality (=)
- T_{PA}-satisfiability and T_{PA}-validity are undecidable

Presburger arithmetic $T_{\mathbb{N}}$

- Natural numbers with addition (+), equality (=)
- $T_{\mathbb{N}}$-satisfiability and $T_{\mathbb{N}}$-validity are decidable

Theory of integers $T_{\mathbb{Z}}$

- Integers with addition (+), subtraction (-), comparison (>), equality (=), *multiplication by constants*
- $T_{\mathbb{Z}}$-satisfiability and $T_{\mathbb{Z}}$-validity are decidable

Theory of Integers $T_{\mathbb{Z}}$

$\Sigma_{\mathbb{Z}}$: {…,-2,-1,0,1,2,…,-3*,-2*,2*,3*,…,+,-,=,>}

where

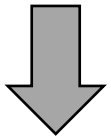
- …,-2,-1,0,1,2,… are constants
- …,-3*,-2*,2*,3*,… are unary functions
  (intended meaning: 2*x is x+x, -3*x is -x-x-x)
- +,-,>,= have the usual meaning

$T_{\mathbb{N}}$ and $T_{\mathbb{Z}}$ have the same expressiveness

- Every $\Sigma_{\mathbb{Z}}$-formula can be reduced to $\Sigma_{\mathbb{N}}$-formula
- Every $\Sigma_{\mathbb{N}}$-formula can be reduced to $\Sigma_{\mathbb{Z}}$-formula

Example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```



```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

A $T_{\mathbb{Z}}$ formula that is satisfiable iff this transformation is invalid:

$(i \geq 1) \wedge (i \leq 10) \wedge$
$(j + i = j)$

quantifier-free conjunctive fragment

Theory of Reals $T_{\mathbb{R}}$ and Theory of Rationals $T_{\mathbb{Q}}$

$\Sigma_{\mathbb{R}}$: {0, 1, +, –, *, =, ≥ } with multiplication

$\Sigma_{\mathbb{Q}}$: {0, 1, +, –, =, ≥ } without multiplication

Both are decidable

- High time complexity

Quantifier-free fragment of $T_{\mathbb{Q}}$ is efficiently decidable

Theory of Arrays T_A

Σ_A : {*select*, *store*, =}

where

- *select*(*a*,*i*) is a binary function:
  - read array *a* at index *i*
- *store*(*a*,*i*,*v*) is a ternary function:
  - write value *v* to index *i* of array *a*

Axioms of T_A

1. $\forall a,i,j.\ i = j \rightarrow select(a,i) = select(a,j)$ (array congruence)

2. $\forall a,v,i,j.\ i = j \rightarrow select(store(a,i,v),j) = v$ (select-store 1)

3. $\forall a,v,i,j.\ i \neq j \rightarrow select(store(a,i,v),j) = select(a,j)$ (select-store 2)

T_A is undecidable

Quantifier-free fragment of T_A is decidable

Note about T_A

Equality (=) is only defined for array elements…

- Example:
  $select(a,i) = e \rightarrow \forall j.\ select(\ store(a,i,e),\ j) = select(a,j)$
  is T_A-valid

…and not for whole arrays

- Example:
  $select(a,i)=e \rightarrow store(a,i,e)=a$
  is not T_A-valid

A program:

```
A[1]=-1;
A[2]=1;
k=unknown( );
if (A[k]==1) ...
```

Summary of Decidability Results

[BM Ch. 3, Page 90]

	Theory	Quantifiers Decidable	QFF Decidable
T_E	Equality	NO ✗	YES
T_{PA}	Peano Arithmetic	NO ✗	NO ✗
$T_{\mathbb{N}}$	Presburger Arithmetic	YES	YES
$T_{\mathbb{Z}}$	Linear Integer Arithmetic	YES	YES
$T_{\mathbb{R}}$	Real Arithmetic	YES	YES
$T_{\mathbb{Q}}$	Linear Rationals	YES	YES
T_A	Arrays	NO ✗	YES

QFF: Quantifier Free Fragment

Summary of Complexity Results

[BM Ch. 3, Pages 90, 91]

Theory		Quantifiers	QF Conjunctive
T_E	Equality	–	O(*n* log *n*)
$T_{\mathbb{N}}$	Presburger Arithmetic	O(*2^2^2^(kn)*)	NP-complete
$T_{\mathbb{Z}}$	Linear Integer Arithmetic	O(*2^2^2^(kn)*)	NP-complete
$T_{\mathbb{R}}$	Real Arithmetic	O(*2^2^(kn)*)	O(*2^2^(kn)*)
$T_{\mathbb{Q}}$	Linear Rationals	O(*2^2^(kn)*)	PTIME
T_A	Arrays	–	NP-complete

n – input formula size; k – some positive integer

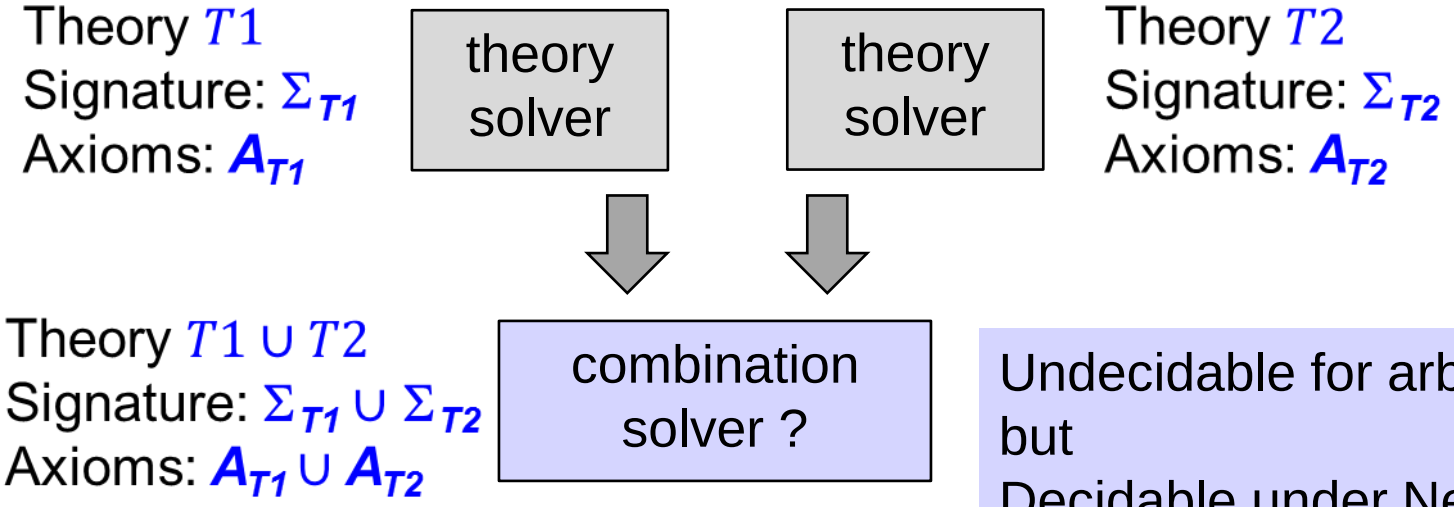
Note: Quantifier-free Conjunctive fragments look good!

Combination of Theories

Many applications require reasoning over a combination of theories

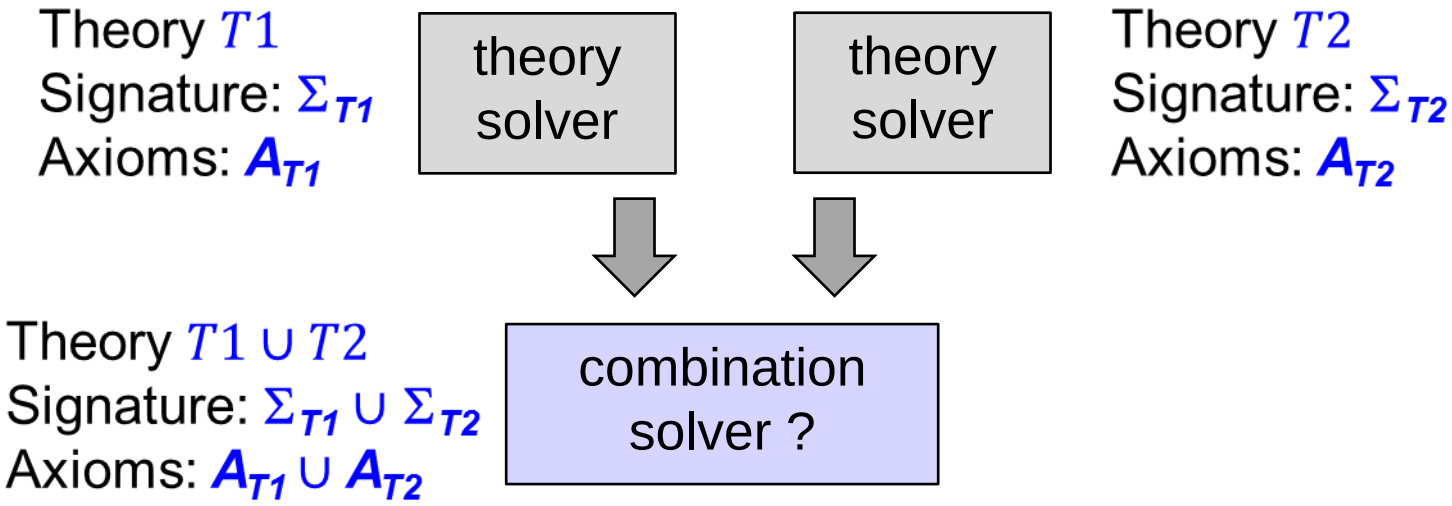
- Example: f(x) + 10 = g(y) belongs to $T_E \cup T_{\mathbb{Z}}$

Given decision procedures for theories T1 and T2, can we decide satisfiability of formulae in $T1 \cup T2$?



Undecidable for arbitrary T1, T2 but
Decidable under Nelson-Oppen restrictions

Decision Procedure for Combination of Theories



Nelson-Oppen Procedure for deciding satisfiability

- If both T1 and T2 are *quantifier-free (conjunctive) fragments*
- If "=" is the only symbol common to their signatures
- If T1 and T2 meet certain other technical restrictions

Note: Quantifier-free Conjunctive fragments look good!

Theories in SMT Solvers

Modern SMT solvers support many useful theories

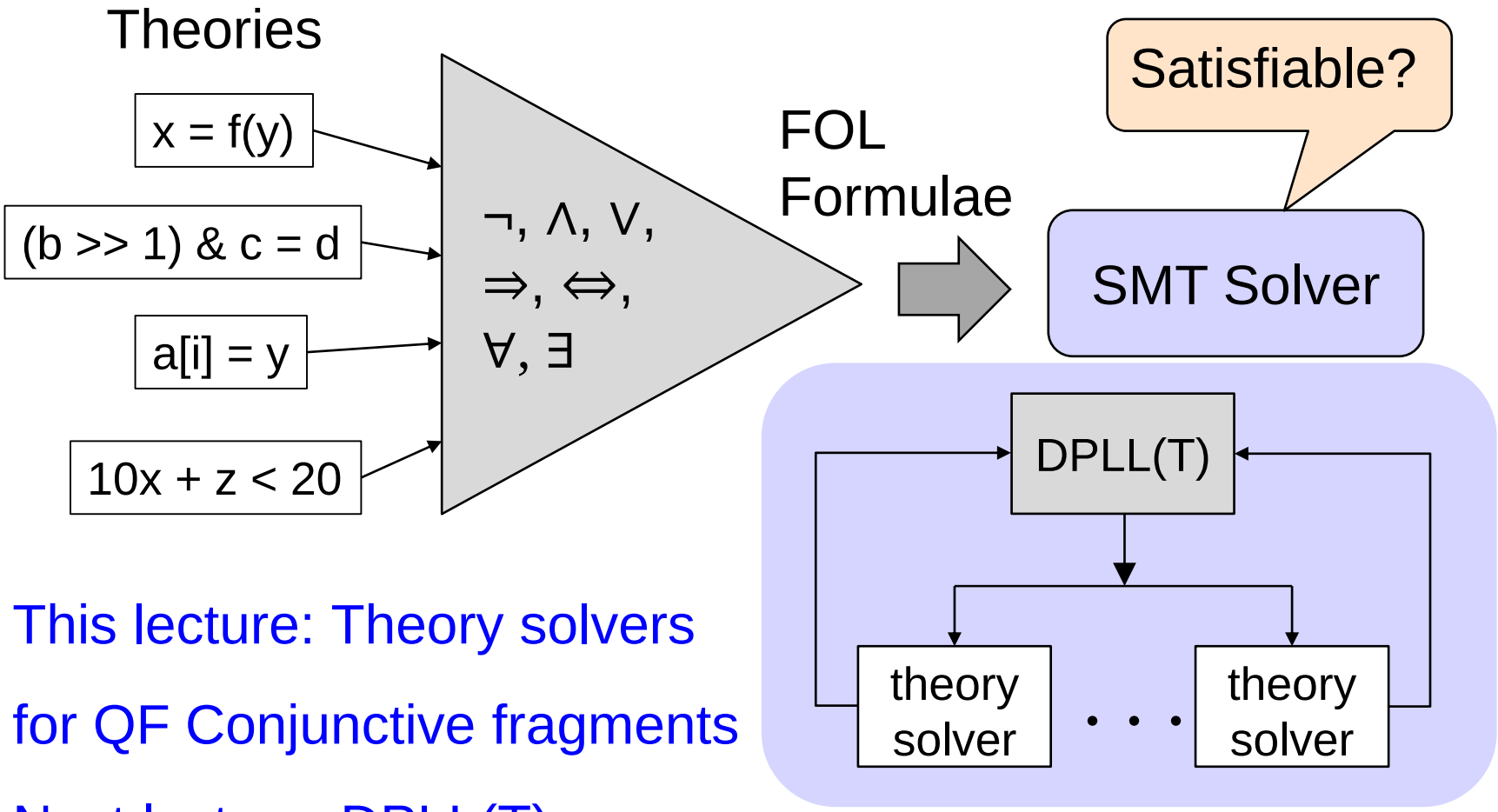
- QF_UF: Quantifier-free Equality with Uninterpreted Functions
- QF_LIA: Quantifier-free Linear Integer Arithmetic
- QF_LRA: Quantifier-free Linear Real Arithmetic
- QF_BV: Quantifier-free Bit Vectors (fixed-width)
- QF_A: Quantifier-free Arrays

… and many combinations

Check out more info at

http://smtlib.cs.uiowa.edu/index.shtml
http://smtlib.cs.uiowa.edu/logics.shtml

summary



This lecture: Theory solvers

for QF Conjunctive fragments

Next lecture: DPLL(T)

Z3 SMT Solver

http://rise4fun.com/z3/

Input format is an extension of SMT-LIB standard

Commands

- `declare-const` – declare a constant of a given type
- `declare-fun` – declare a function of a given type
- `assert` – add a formula to Z3's internal stack
- `check-sat` – determine if formulas currently on stack are satisfiable
- `get-model` – retrieve an interpretation
- `exit`

SMT solvers: DPLL(T)

Main idea: combine DPLL SAT solving with theory solvers

- DPLL-based SAT over the *Boolean structure* of the formula
- theory solver handles the *conjunctive fragment*

- Recall: SAT solvers use many techniques to prune the exponential search space

This is called DPLL(T)

- T could also be a combination of theories (using Nelson-Oppen)

DPLL(T): main idea

SAT solver handles Boolean structure of the formula

- Treats each atomic formula as a propositional variable
- Resulting formula is called a *Boolean abstraction (B)*

Example

F: $(x=z) \wedge ((y=z \wedge x = z+1) \vee \neg (x=z))$

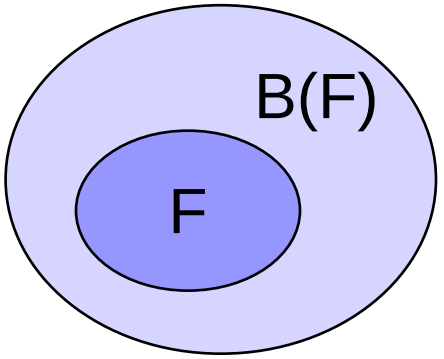
- B(F): $b1 \wedge ((b2 \wedge b3) \vee \neg b1)$
- Boolean abstraction (B) defined inductively over formulas
- B is a bijective function, B^{-1} also exists
  - B^{-1} $(b1 \wedge b2 \wedge b3)$: $(x=z) \wedge (y=z) \wedge (x=z+1)$
  - B^{-1} $(b1 \vee b2')$: $(x=z) \vee \neg(y=z)$

DPLL(T): main idea

Example

F: (x=z) ∧ ((y=z ∧ x = z+1) ∨ ¬ (x=z))

b1 b2 b3 b1



- B(F): b1 ∧ ((b2 ∧ b3) ∨ ¬b1)

B(F) is an *over-approximation* of F

- Use DPLL SAT solver to decide satisfiability of B(F)
  - If B(F) is Unsat, then F is Unsat
  - If B(F) has a satisfying assignment A, F may still be Unsat

Example

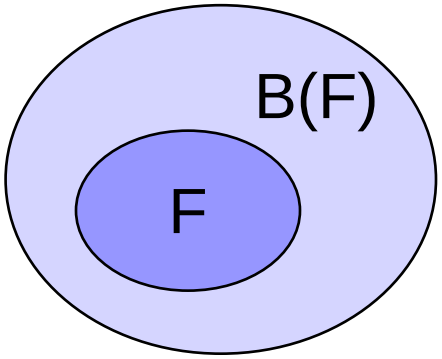
- SAT solver finds a satisfying assignment A: b1 ∧ b2 ∧ b3
- But, $B^{-1}(A)$ is unsatisfiable modulo theory
  - (x=z) ∧ (y=z) ∧ (x=z+1) is not satisfiable

DPLL(T): main idea

Example

F: (x=z) ∧ ((y=z ∧ x = z+1) ∨ ¬ (x=z))

b1 b2 b3 b1

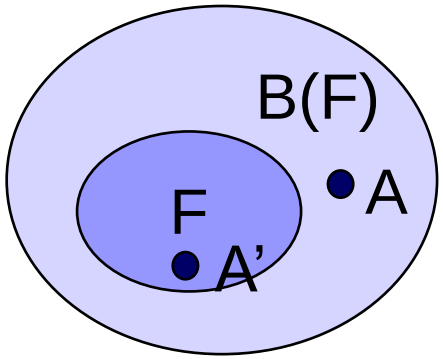


- B(F): b1 ∧ ((b2 ∧ b3) ∨ ¬b1)

B(F) is an *over-approximation* of F

- Use DPLL SAT solver to decide satisfiability of B(F)
  - If B(F) is Unsat, then F is Unsat
  - If B(F) has a satisfying assignment A
- Use theory solver to check if $B^{-1}(A)$ is satisfiable modulo T
  - Note $B^{-1}(A)$ is in conjunctive fragment
  - If $B^{-1}(A)$ is satisfiable modulo theory T, then F is satisfiable

DPLL(T): simple version



- Generate a Boolean abstraction B(F)
- Use DPLL SAT solver to decide satisfiability of B(F)
  - If B(F) is Unsat, then F is Unsat
  - If B(F) has a satisfying assignment A
- Use theory solver to check $B^{-1}(A)$ is satisfiable modulo T
  - If $B^{-1}(A)$ is satisfiable modulo theory T, then F is satisfiable
  - Because A satisfies the Boolean structure, and is *consistent with T*
- What if $B^{-1}(A)$ is unsatisfiable modulo T? Is F Unsat?
- No!
  - There may be other assignments *A'* that satisfy the Boolean structure and are consistent with T
- *Add* ¬*A* to B(F), and *backtrack* in DPLL SAT to find other assignments
  - Until there are no more satisfying assignments of B(F)

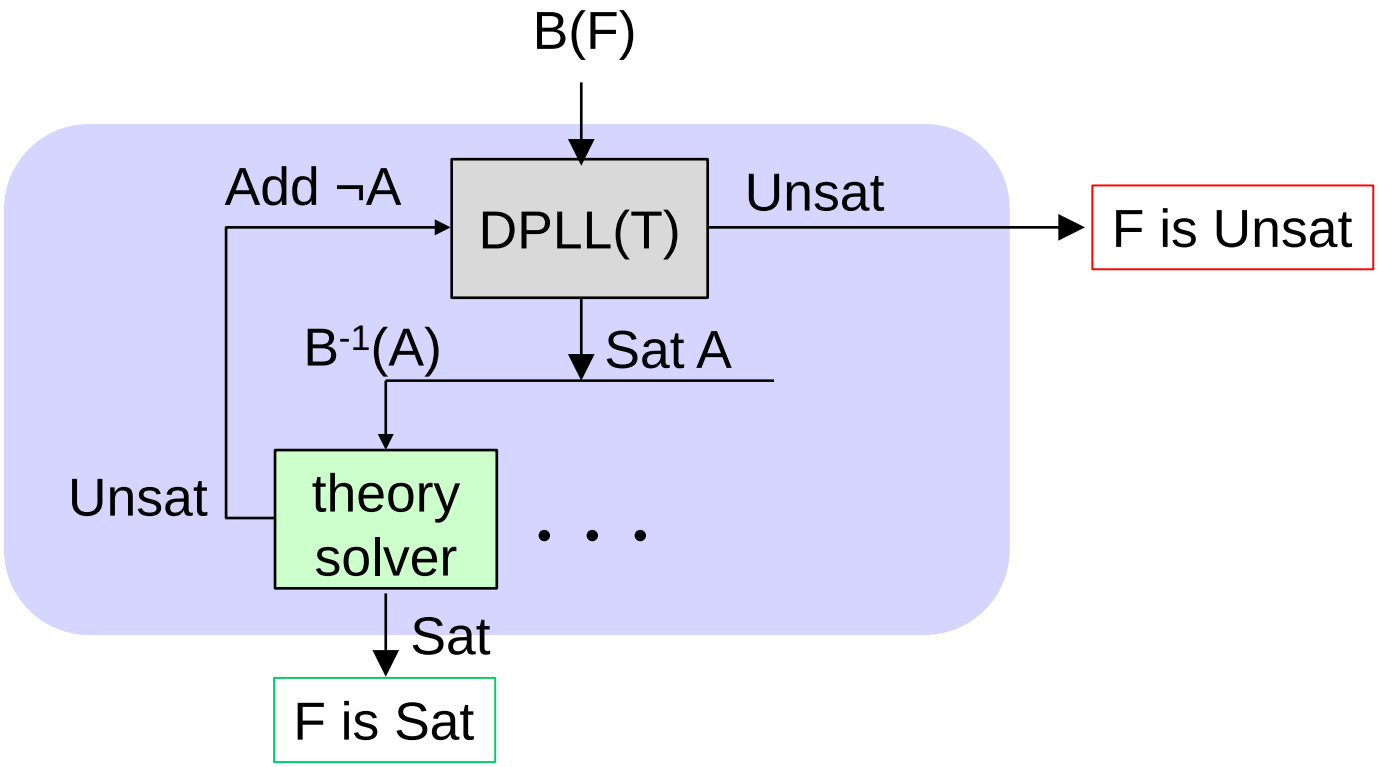
Like a conflict clause (due to a theory conflict)

DPLL(T): simple version recap

1. Generate a Boolean abstraction B(F)
2. Use DPLL SAT solver to decide satisfiability of B(F)
   - If B(F) is Unsat, then F is Unsat
   - Otherwise, find a satisfying assignment A
3. Use theory solver to check if $B^{-1}(A)$ is satisfiable modulo T
   - If $B^{-1}(A)$ is satisfiable modulo theory T, then F is satisfiable
   - Otherwise, $B^{-1}(A)$ is unsatisfiable modulo T
     Add ¬*A* to B(F), and backtrack in DPLL SAT

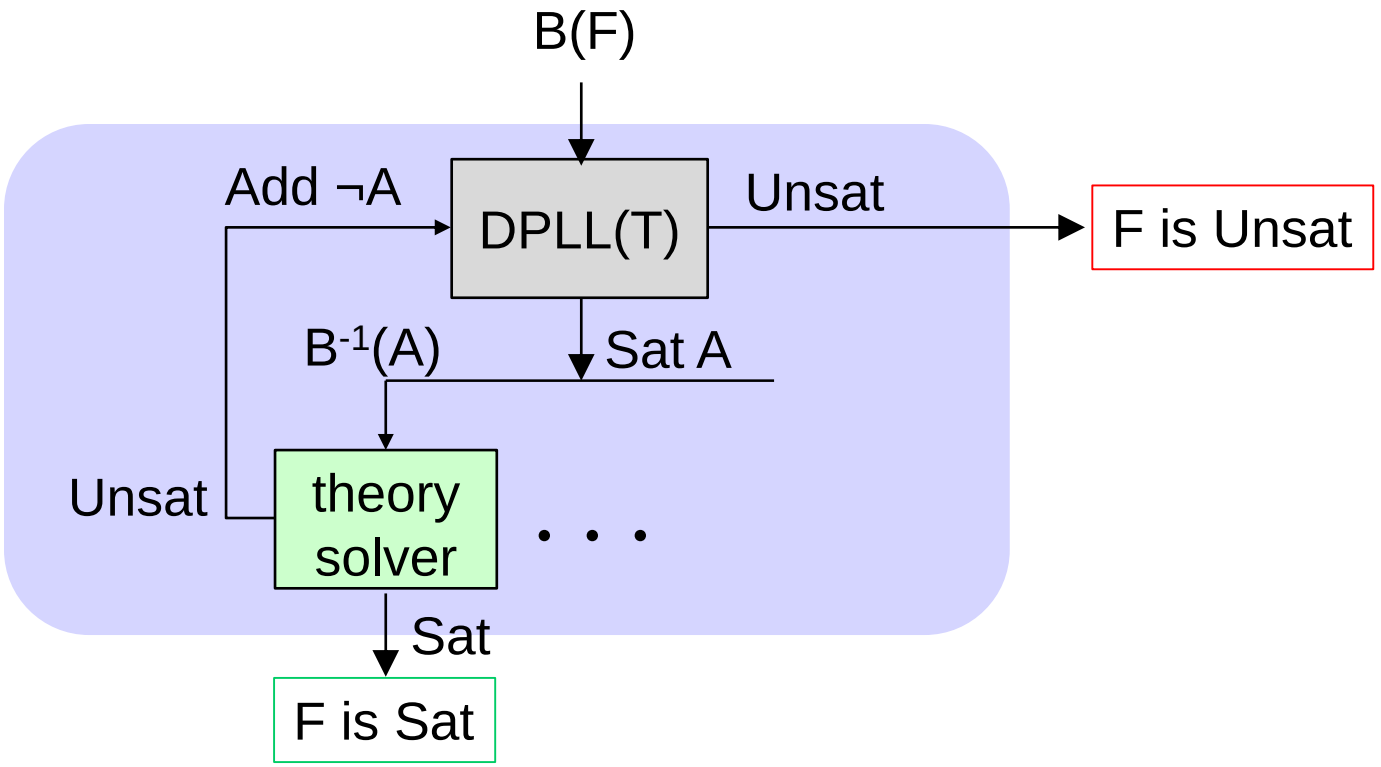
Repeat (2, 3) until there are no more satisfying assignments

DPLL(T): simple version example



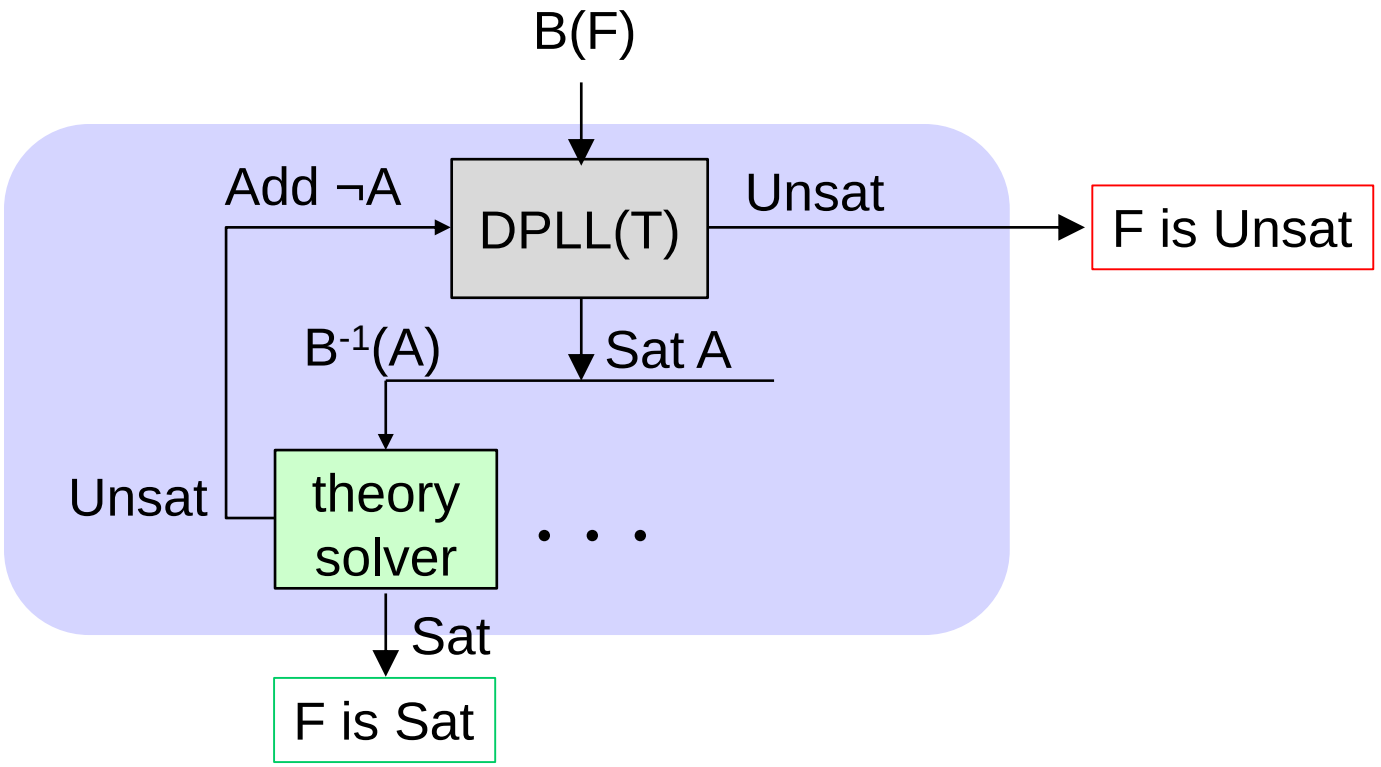
- Example F: (x=z) ∧ ((y=z ∧ x = z+1) ∨ ¬ (x=z))
  - B(F): b1 ∧ ((b2 ∧ b3) ∨ ¬b1)
  - DPLL finds A: b1 ∧ b2 ∧ b3, $B^{-1}(A)$: (x=z) ∧ (y=z) ∧ (x=z+1)
  - Theory solver checks $B^{-1}(A)$, this is unsat modulo T, therefore add ¬*A*
  - DPLL finds B(F) ∧ ¬*A:* b1 ∧ ((b2 ∧ b3) ∨ ¬b1) ∧ (b1' + b2' + b3') is Unsat
  - Therefore, F is Unsat

DPLL(T): simple version



- Correctness
  - When it says “F is Sat”, there is an assignment that satisfies the Boolean structure *and* is consistent with theory
  - When it says “F is Unsat”, the formula is unsatisfiable because B(F) ∧ ¬*A* is also an over-approximation of F
    - $B^{-1}(A)$ is not consistent with T, i.e., $B^{-1}(\neg A)$ is T-valid

DPLL(T): simple version



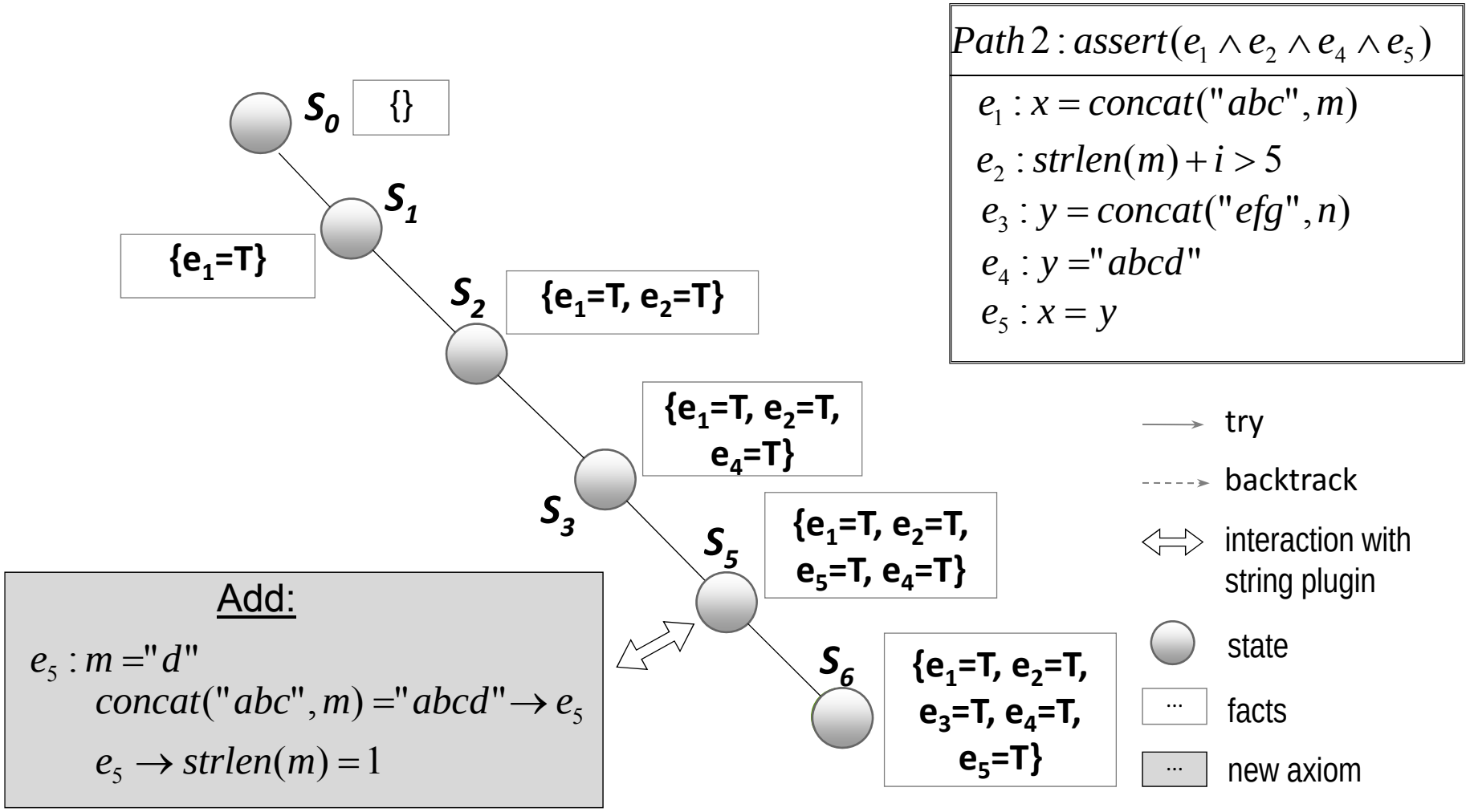
- Termination
  - B(F) has only a finite number of satisfying assignments
  - When ¬*A* is added to B(F), the assignment A will never be generated again
  - Either some satisfying assignment of B(F) is also T-satisfiable (F is SAT), or all satisfying assignments of B(F) are not T-satisfiable (F is Unsat)

An Example of Symbolic Analysis and DPLL(T)

1. m=getstr();
2. n=getstr();
3. i=getint();
4. x=strcat(“abc”,m)
5. if (strlen(m)+i>5)
6. y=“abcd”
7. else
8. y=strcat(“efg”,n);
9. if (x==y) …

$Path1 : assert(e_1 \wedge !e_2)$
$e_1 : x = concat("abc", m)$
$e_2 : strlen(m) + i > 5$
$e_3 : y = concat("efg", n)$
$e_4 : y = "abcd"$
$e_5 : x = y$

An Example of Symbolic Analysis and DPLL(T)



Recent Trends

- Hybrid analysis
- Model counting
  - Quantifying information flow
  - Side channel analysis
- Develop specialized theories
  - E.g., theories for regular expressions

In-class Exercise 1

- Model the following statement to a formula. Decide its validity by formulating it as a satisfiability problem and solving it. Please show the parse tree and the value assignment process.
  - If I study, then I will not fail basket weaving 101. If I do not play cards to often, then I will study. I failed basket weaving 101. Therefore, I played cards too often.

In-class Exercise 2

- Describe the execution of DPLL on the following formulae

(a) $(P \vee \neg Q \vee \neg R) \wedge (Q \vee \neg P \vee R) \wedge (R \vee \neg Q)$

(b) $(P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (Q \vee \neg R)$