The Spiral of Theodorus,

Numerical Analysis, and Special Functions

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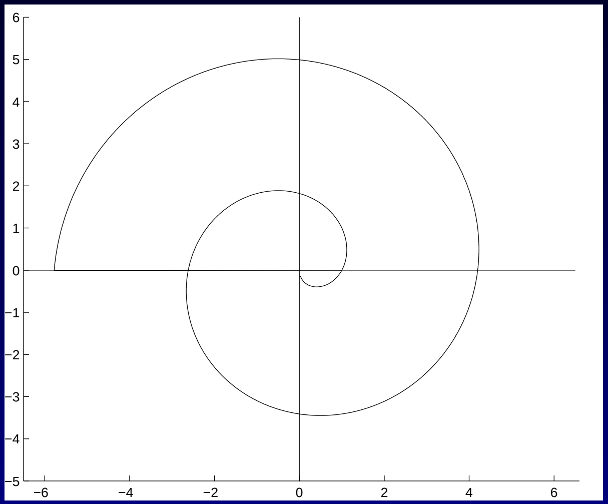
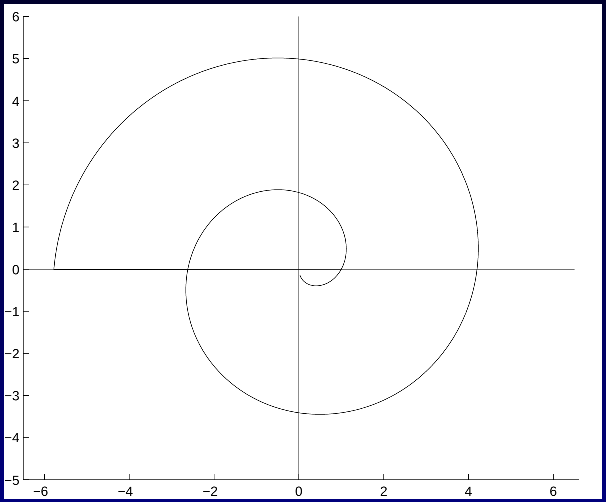
Theodorus of

ca. 460–399 B.C.

spiral of Theodorus



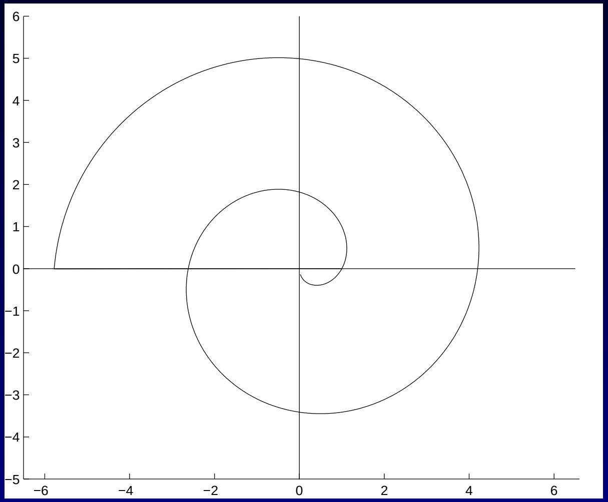
spiral of Theodorus



numerical analysis

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}}$$

spiral of Theodorus



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$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}}$$

$$= 1.86002507922119030718069591571714332466652412152345$$

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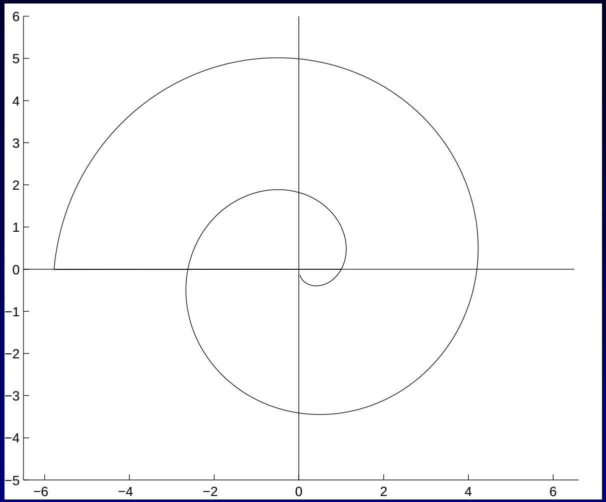

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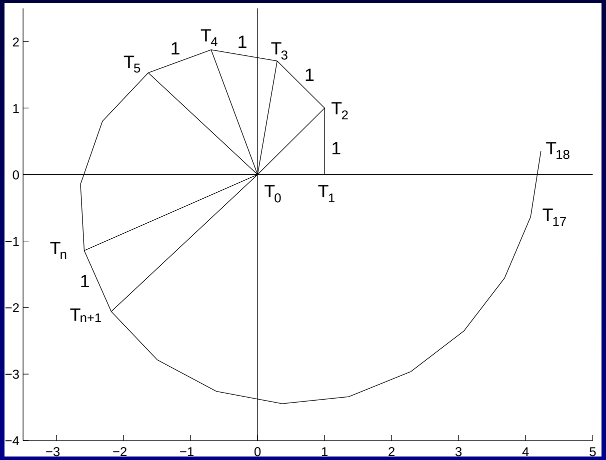
special functions

$$F(x) = e^{-x^2} \int_0^x e^{t^2} \mathrm{d}t \quad \text{Dawson's integral}$$

discrete “spiral of Theodorus” (also known as “Quadratwurzelschnecke”; Hlawka, 1980)



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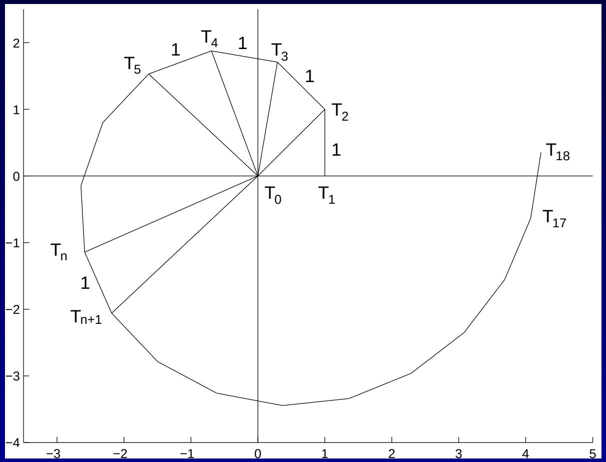

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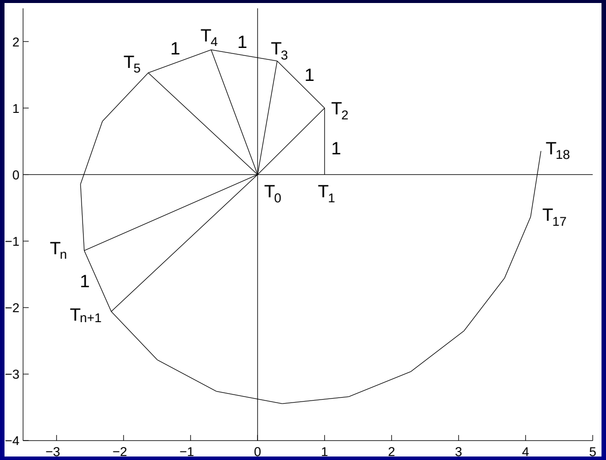

parametric representation $T(\alpha) \in \mathbb{C}, \quad \alpha \geq 0$

defining properties

$$\left.\begin{array}{l} T(n) = T_n \\ |T_n| = \sqrt{n} \\ |T_{n+1} - T_n| = 1 \end{array}\right\} \quad n = 0, 1, 2, \ldots$$

problem: Interpolate the discrete Theodorus spiral by a smooth (or even analytic) spiral

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$$T(\alpha) = \prod_{k=1}^{\infty} \frac{1 + \mathrm{i}/\sqrt{k}}{1 + \mathrm{i}/\sqrt{k + \alpha - 1}}, \quad \alpha \geq 0 \ \ (\mathrm{i} = \sqrt{-1})$$

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$$T(\alpha+1) = \left(1 + \frac{\mathrm{i}}{\sqrt{\alpha}}\right) T(\alpha)$$

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(Gronau, 2004) Davis's function is the unique solution of the above difference equation with $|T(\alpha)|$ and $\arg T(\alpha)$ monotonically increasing, and $T(1) = 1$.

a little bit of number theory

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distribution of the angles

$$\varphi_n = \angle T_1 T_0 T_{n+1} = \sum_{k=1}^{n} \sin^{-1} \frac{1}{\sqrt{k+1}}, \quad n = 1, 2, 3, \ldots$$

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more generally

$$\varphi_n(\alpha) = \angle T(\alpha) T_0 T(\alpha + n) = \sum_{k=1}^{n} \sin^{-1} \frac{1}{\sqrt{k+\alpha}}$$

$$1 < \alpha < 2, \; n = 1, 2, 3, \ldots$$

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The sequence $\{\varphi_n(\alpha)\}_{n=1}^{\infty}$ is equidistributed mod 2π for any α with $1 < \alpha < 2$ (Niederreiter, email Feb. 3, 2009)

logarithmic derivative of $T(\alpha)$

$$\frac{T'(\alpha)}{T(\alpha)} = \sum_{k=1}^{\infty} \frac{1+\mathrm{i}/\sqrt{k+\alpha-1}}{1+\mathrm{i}/\sqrt{k}} \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1+\mathrm{i}/\sqrt{k}}{1+\mathrm{i}/\sqrt{k+\alpha-1}}\right)$$

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$$= \sum_{k=1}^{\infty} (1+\mathrm{i}/\sqrt{k+\alpha-1}) \frac{\mathrm{i}}{2} \frac{(k+\alpha-1)^{-3/2}}{(1+\mathrm{i}/\sqrt{k+\alpha-1})^2}$$

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$$= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\frac{\sqrt{k+\alpha-1}-\mathrm{i}}{(k+\alpha-1)(k+\alpha)}$$

logarithmic derivative of $T(\alpha)$ (cont')

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)(k+\alpha)} + \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)^{3/2}+(k+\alpha-1)^{1/2}}$$

logarithmic derivative of $T(\alpha)$ (cont')

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$$= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k+\alpha-1}-\frac{1}{k+\alpha}\right) + \frac{\mathrm{i}}{2}U(\alpha)$$

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$$= \frac{1}{2\alpha} + \frac{\mathrm{i}}{2}U(\alpha)$$

where

$$U(\alpha) = \sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)^{3/2}+(k+\alpha-1)^{1/2}}$$

$$\frac{T'(\alpha)}{T(\alpha)} = \frac{1}{2\alpha} + \frac{\mathrm{i}}{2}\,U(\alpha)$$

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integrate from 1 to $\alpha > 1$

$$\ln T(\alpha) = \ln(\alpha^{1/2}) + \frac{\mathrm{i}}{2} \int_1^{\alpha} U(\alpha) \mathrm{d}\alpha$$

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polar representation of $T(\alpha)$

$$T(\alpha) = \sqrt{\alpha}\,\exp\left(\frac{\mathrm{i}}{2}\int_1^{\alpha} U(\alpha)\mathrm{d}\alpha\right), \quad \alpha > 1$$

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since $T'(1) = \frac{1}{2} + \frac{\mathrm{i}}{2}\,U(1)$, the slope of the tangent vector to the spiral at $\alpha = 1$ is

$$U(1) = \sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}} \quad (\text{Theodorus constant})$$

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numerical analysis and special functions: compute and identify

$$U(\alpha) = \sum_{k=1}^{\infty} \frac{1}{(k+\alpha-1)^{3/2} + (k+\alpha-1)^{1/2}}, \quad \int_1^\alpha U(\alpha) \mathrm{d}\alpha \ \text{ for } 1 < \alpha < 2$$

first digression

summation by integration (G. & Milovanović, 1985)

$$s = \sum_{k=1}^{\infty} a_k, \quad a_k = (\mathcal{L}f)(k)$$

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$$= \int_0^{\infty} \left(\sum_{k=1}^{\infty} e^{-kt} \right) f(t) \mathrm{d}t = \int_0^{\infty} \frac{t}{e^t - 1} \frac{f(t)}{t} \mathrm{d}t$$

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Thus

$$\sum_{k=1}^{\infty} a_k = \int_0^{\infty} \frac{f(t)}{t} \varepsilon(t) \mathrm{d}t, \quad f = \mathcal{L}^{-1} a$$

$$\varepsilon(t) = \frac{t}{e^t - 1} \quad \text{Bose} - \text{Einstein distribution}$$

Theodorus: $a_k = \frac{1}{k^{3/2} + k^{1/2}} = \frac{k^{-1/2}}{k+1}$

convolution theorem for Laplace transform

$$\mathcal{L}g \cdot \mathcal{L}h = \mathcal{L}g * h, \quad (g * h)(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau$$

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application to a_k

$$\begin{aligned} k^{-1/2} &= \left(\mathcal{L}\,\frac{t^{-1/2}}{\sqrt{\pi}}\right)(k) \\ \frac{1}{k+1} &= \left(\mathcal{L}\,e^{-t}\right)(k) \end{aligned}$$

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$$\begin{aligned} a_k &= \left(\mathcal{L}\,\frac{t^{-1/2}}{\sqrt{\pi}}\right)(k) \cdot \left(\mathcal{L}\,e^{-t}\right)(k) \\ &= \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\int_0^t \tau^{-1/2} e^{-(t-\tau)}\mathrm{d}\tau\right)(k) \end{aligned}$$

Theodorus (cont')

$$f(t) = \frac{1}{\sqrt{\pi}} e^{-t} \int_0^t \tau^{-1/2} e^{\tau} \mathrm{d}\tau$$

$$= \frac{2}{\sqrt{\pi}} e^{-t} \int_0^{\sqrt{t}} e^{x^2} \mathrm{d}x = \frac{2}{\sqrt{\pi}} F(\sqrt{t})$$

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$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}} = \int_0^{\infty} \frac{f(t)}{t} \varepsilon(t) \mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{F(\sqrt{t})}{\sqrt{t}} w(t) \mathrm{d}t$$

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$$w(t) = t^{-1/2} \varepsilon(t) = \frac{t^{1/2}}{e^t - 1}$$

second digression

Gaussian quadrature n-point quadrature formula

$$\int_0^\infty g(t)w(t)\mathrm{d}t = \sum_{k=1}^{n} \lambda_k^{(n)} g(\tau_k^{(n)}), \qquad g \in \mathbb{P}_{2n-1}$$

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orthogonal polynomials

$$(\pi_k, \pi_\ell) = 0,\ k \neq \ell,\ \text{ where }\ (u,v) = \int_0^\infty u(t)v(t)w(t)\mathrm{d}t$$

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three-term recurrence relation

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t),\ k = 0,1,2,\ldots$$
$$\pi_{-1}(t) = 0,\ \ \pi_0(t) = 1$$

where $\alpha_k = \alpha_k(w) \in \mathbb{R},\ \beta_k = \beta_k(w) > 0,\ \beta_0 = \int_0^\infty w(t)\mathrm{d}t$

Jacobi matrix

$$\boldsymbol{J}_n(w) = \begin{bmatrix} \alpha_0 & \beta_1 & & & \mathbf{0} \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \beta_2 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ \mathbf{0} & & & \beta_{n-1} & \alpha_{n-1} \end{bmatrix}$$

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Gaussian nodes and weights (Golub & Welsch, 1969)

$$\tau_k^{(n)} = \text{eigenvalues of } \boldsymbol{J}_n, \quad \lambda_k^{(n)} = \beta_0 \boldsymbol{v}_{k,1}^2$$

$$\boldsymbol{v}_{k,1} = \text{first component of (normalized) eigenvector } \boldsymbol{v}_k$$

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moments of w

$$\mu_k = \int_0^\infty t^k w(t) \mathrm{d}t, \quad k = 0, 1, \ldots, 2n-1$$

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Chebyshev algorithm

$$\{\mu_k\}_{k=0}^{2n-1} \mapsto \{\alpha_k, \beta_k\}_{k=0}^{n-1}$$

numerical results for Gaussian quadrature (in 15D-arithmetic)

$$\mu_k = \int_0^\infty t^k w(t) \mathrm{d}t = \int_0^\infty \frac{t^{k+1/2}}{e^t - 1} \mathrm{d}t = \Gamma(k+3/2)\zeta(k+3/2)$$

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$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty [F(\sqrt{t})/\sqrt{t}\,]\, w(t)$$

$$s_n = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{n} \lambda_k^{(n)} F\left(\sqrt{\tau_k^{(n)}}\right) / \sqrt{\tau_k^{(n)}}$$

numerical results for Gaussian quadrature (in 15D-arithmetic)

$$\mu_k = \int_0^\infty t^k w(t) \mathrm{d}t = \int_0^\infty \frac{t^{k+1/2}}{e^t - 1} \mathrm{d}t = \Gamma(k+3/2)\zeta(k+3/2)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty [F(\sqrt{t})/\sqrt{t}\,]\, w(t)$$

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n	s_n
5	1.85997...
15	1.86002507922117...
25	1.86002507922119030718068 9...
35	1.860025079221190307180695915717141...
45	1.8600250792211903071806959157171433246665235...
55	1.86002507922119030718069591571714332466652412155
65	1.86002507922119030718069591571714332466652412155
75	1.86002507922119030718069591571714332466652412155

computation and identification of $U(\alpha)$ and $\int_1^\alpha U(\alpha)\mathrm{d}\alpha$

$$\frac{(k+\alpha-1)^{-1/2}}{(k+\alpha-1)+1} = \frac{1}{\sqrt{\pi}} \left(\mathcal{L} \int_0^t \tau^{-1/2} e^{-(t-\tau)} \mathrm{d}\tau \right) (k+\alpha-1)$$

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applying the shift property for Laplace transform

$$(\mathcal{L}g)(s+b) = \left(\mathcal{L}\, e^{-bt} g(t)\right)(s)$$

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hence

$$f(t) = \frac{1}{\sqrt{\pi}}e^{-\alpha t}\int_0^t \tau^{-1/2}e^{\tau}\mathrm{d}\tau = \frac{2}{\sqrt{\pi}}e^{-(\alpha-1)t}F(\sqrt{t})$$

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- summation theory can be generalized to series

$$s_+ = \sum_{k=1}^{\infty} k^{\nu-1} r(k), \; s_- = \sum_{k=1}^{\infty} (-1)^k k^{\nu-1} r(k), \quad 0 < \nu < 1$$

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$r > 0$: spiral of Theodorus
$r < 0$: analytic continuation of the spiral

twin-spiral of Theodorus