Polynomials orthogonal with respect to densely oscillating and exponentially decaying weight functions

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Web Site

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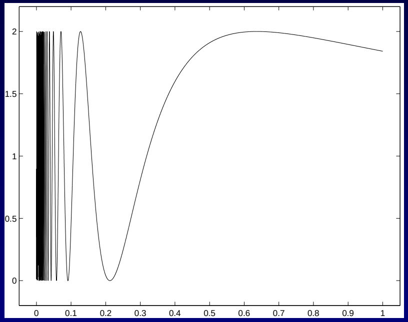
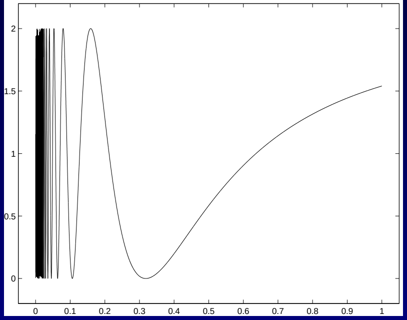
A gallery of weight functions

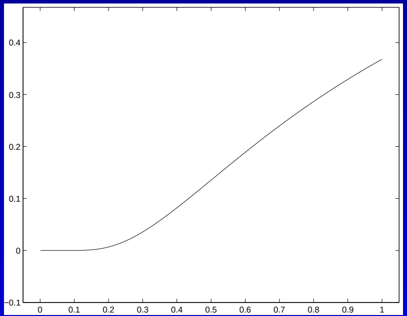
$1 + \sin(1/t)$



$1 + \cos(1/t)$

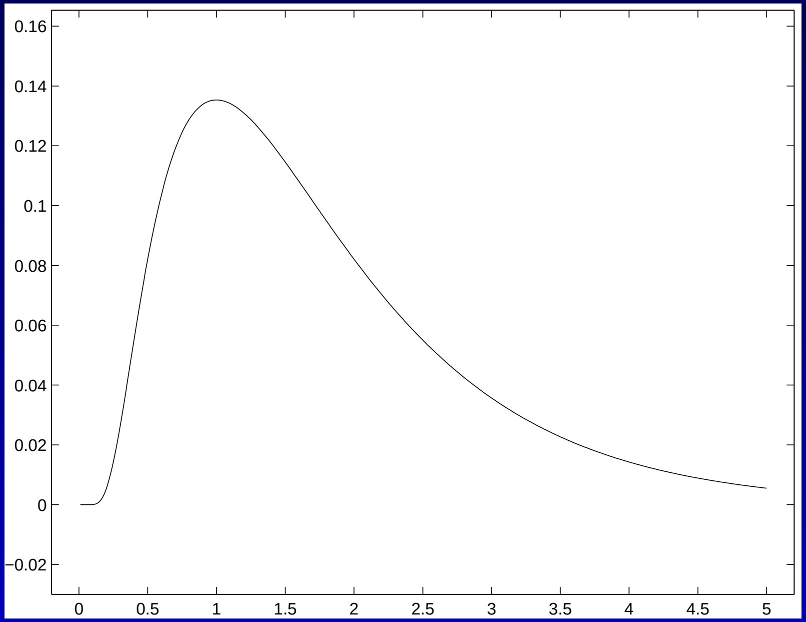


$\exp(-1/t)$

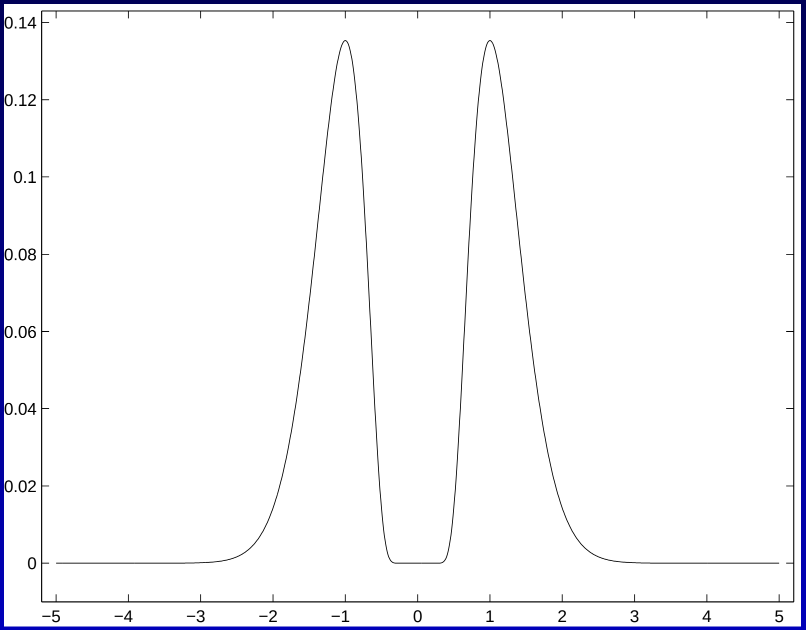


gallery (cont')

$$\exp\left(-1/t - t\right)$$



$$\exp\left(-1/t^2 - t^2\right)$$



Moment-based approach

The first $2n$ moments

$$\mu_k = \int_a^b t^k w(t)\mathrm{d}t, \quad k = 0, 1, \ldots, 2n-1,$$

of a weight function w on $[a, b]$ determine uniquely the first n recurrence coefficients α_k, β_k, $k = 0, 1, \ldots, n-1$ in the three-term recurrence relation for the orthogonal polynomials $\pi_k(\,\cdot\,; w)$,

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t),$$

$$k = 0, 1, \ldots, n-1,$$

$$\pi_{-1}(t) = 0, \quad \pi_0(t) = 1$$

Moment map

$$\texttt{mom} = [\mu_k]_{k=0}^{2n-1} \mapsto \texttt{ab} = [\alpha_k, \beta_k]_{k=0}^{n-1}$$

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Implementation

Chebyshev algorithm (1859)

Moment map

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Chebyshev algorithm (1859)

Matlab

```
ab=chebyshev(N,mom)
```

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Ill-conditioned!

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Implementation

Chebyshev algorithm (1859)

Matlab

```
ab=chebyshev(N,mom)
```

Ill-conditioned!

Remedy: symbolic Matlab

```
ab=schebyshev(dig,N,mom)
```

The weight function $w(t) = 1 + \sin(1/t)$ **on** $[0, 1]$

moments

let

$$\mu_k^0 = \int_0^1 t^k \sin(1/t)\mathrm{d}t = \int_1^\infty t^{-(k+2)} \sin t \, \mathrm{d}t$$

then

$$\mu_{k+1}^0 = \frac{1}{k+2} \left[\frac{1}{k+1}(\cos 1 - \mu_{k-1}^0) + \sin 1 \right]$$

where

$$\mu_{-1}^0 = \frac{\pi}{2} - \mathrm{Si}(1), \quad \mu_0^0 = \sin 1 - \mathrm{Ci}(1)$$

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where

$$\mu_{-1}^0 = \frac{\pi}{2} - \mathrm{Si}(1), \quad \mu_0^0 = \sin 1 - \mathrm{Ci}(1)$$

thus

$$\mu_k = \int_0^1 t^k [1 + \sin(1/t)]\mathrm{d}t = \frac{1}{k+1} + \mu_k^0$$

Matlab routine

```
% SR_SINO Symbolic/variable-precision recurrence
%     coefficients for the weight function w(x)=
%     1+sin(1/x) on [0,1]
%
syms mom ab
digits(d); dig=d;
mom(1)='sin(1)-Ci(1)';
mom(2)='(Si(1)-pi/2+sin(1)+cos(1))/2';
for k=3:2*N
   mom(k)=(('cos(1)'-mom(k-2))/(k-1)+'sin(1)')/k;
end
for k=1:2*N
   mom(k)=mom(k)+1/k;
end
ab=schebyshev(dig,N,mom);
```

output (N=40, d=16)

```
ab =
[.5841029561609566, 1.504067061906928]
[.4634474607770499, .7094822535096882e-1]
[.4977629714178322, .7892077774694954e-1]
[.5356590088623750, .5547885019105795e-1]
[.4669144430825117, .6259489484316939e-1]
[.4951204560106238, .7262419731845188e-1]
        ...        ...        ...
[.5109646577717308, .5919672749794708e-1]
[.4919346767523436, .6218164541801303e-1]
[.4973070307106440, .6494413841533854e-1]
[.5100281141083978, .6093638294208964e-1]
```

Example 1 $\int_0^1 \tan((\frac{1}{2}\pi - \delta)t) \sin(1/t) \mathrm{d}t$

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$$\int_0^1 f(t)\sin(1/t)\mathrm{d}t = \int_0^1 f(t)[1+\sin(1/t)]\mathrm{d}t - \int_0^1 f(t)\mathrm{d}t$$

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$$\int_0^1 f(t)\sin(1/t)\mathrm{d}t = \int_0^1 f(t)[1+\sin(1/t)]\mathrm{d}t - \int_0^1 f(t)\mathrm{d}t$$

```
% INTSINO Example 1
%
load -ascii absin0;
ab=absin0; ab1=r_jacobi01(40);
for n=4:4:36
   xw1=gauss(n,ab1); xw=gauss(n,ab);
   int1=sum(xw1(:,2).*tan((pi/2-delta)...
     .*xw1(:,1)));
   ints=sum(xw(:,2).*tan((pi/2-delta)...
     .*xw(:,1)));
   int=ints-int1
end
```

output $(\delta = .1)$

```
>> intsin0

   1.2716655036125e+00
   1.2957389942560e+00
   1.2961790099686e+00
   1.2961860624657e+00
   1.2961861691603e+00
   1.2961861708344e+00
   1.2961861708631e+00
   1.2961861708636e+00
   1.2961861708636e+00
```

output $(\delta = .1)$

```
>> intsin0

   1.2716655036125e+00
   1.2957389942560e+00
   1.2961790099686e+00
   1.2961860624657e+00
   1.2961861691603e+00
   1.2961861708344e+00
   1.2961861708631e+00
   1.2961861708636e+00
   1.2961861708636e+00
```

1000-point Gauss-Legendre
on [0,1] yields 6 correct digits

The weight function $w(t) = e^{-1/t}$ on $[0,1]$

moments

$$\mu_k = \int_0^1 t^k e^{-1/t} \mathrm{d}t = \int_1^\infty t^{-(k+2)} e^{-t} \mathrm{d}t = E_{k+2}(1)$$

recurrence

$$\mu_{k+1} = \frac{1}{k+2}\left(e^{-1} - \mu_k\right), \qquad \mu_0 = E_2(1)$$

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recurrence

$$\mu_{k+1} = \frac{1}{k+2}\,(e^{-1} - \mu_k), \qquad \mu_0 = E_2(1)$$

Matlab routine

```
% SR_EXPO Symbolic/variable-precision recurrence coefficients
%    for the weight function w(x)=exp(-1/x) on [0,1]
%
syms mom ab
digits(d); dig=d;
mom(1)=vpa('Ei(2,1)',d);
for k=2:2*N
   mom(k)=('exp(-1)'-mom(k-1))/k;
end
ab=schebyshev(dig,N,mom);
```

Example 2 $\int_0^1 \ln(1+t)e^{-1/t}\mathrm{d}t$

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Gauss quadrature with weight function $e^{-1/t}$

```
>> intexp0

   n        n-point Gauss
   2     8.1262554100479e-02
   4     8.1255735149253e-02
   6     8.1255733983155e-02
   8     8.1255733982820e-02
  10     8.1255733982819e-02
  12     8.1255733982819e-02
```

102-point Gauss–Laguerre quadrature of $e^{-1}(1+t)^{-2}\ln(1+(1+t)^{-1})$ yields the same limit value

The weight function $w(t) = e^{-1/t-t}$ on $[0, \infty]$

moments

$$\mu_k = \int_0^\infty t^k e^{-(1/t+t)} \mathrm{d}t = 2K_{k+1}(2)$$

recurrence

$$\mu_{k+1} = (k+1)\mu_k + \mu_{k-1}, \; \mu_{-1} = 2K_0(2), \; \mu_0 = 2K_1(2)$$

The weight function $w(t) = e^{-1/t-t}$ on $[0, \infty]$

moments

$$\mu_k = \int_0^\infty t^k e^{-(1/t+t)} \mathrm{d}t = 2K_{k+1}(2)$$

recurrence

$$\mu_{k+1} = (k+1)\mu_k + \mu_{k-1}, \ \mu_{-1} = 2K_0(2), \ \mu_0 = 2K_1(2)$$

Matlab routine

```
% SR_EXPOINF Variable-precision recurrence coefficients
%     for the weight function w(x)=exp(-1/x-x) on [0,inf]
%
syms mom ab
digits(d); dig=d;
mom(1)=vpa('2*BesselK(1,2)',dig);
mom(2)=mom(1)+vpa('2*BesselK(0,2)',dig);
for k=3:2*N
   mom(k)=(k-1)*mom(k-1)+mom(k-2);
end
ab=schebyshev(dig,N,mom);
```

output (N=40, d=16)

```
ab =
[ 1.814307758763789, .279731763633044 9]
[ 3.647885050815283, 1.336902874017094]
[ 5.563608408242503, 4.576187502809998]
[ 7.510248881089434, 9.776110045536486]
[ 9.472385776425876, 16.953645182917 04]
[ 11.44360258233455, 26.11622048172850]
           ...                 ...
[ 73.23900952424970, 1300.294223771922]
[ 75.23687505008481, 1373.374066736622]
[ 77.23481475899122, 1448.453190623454]
[ 79.23282425676095, 1525.531620678047]
```

Example 3 $\int_0^\infty J_0(t)e^{-1/t-t}\mathrm{d}t$

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Gauss quadrature with weight function $e^{-1/t-t}$

```
>> intexp0inf
```

n	n-point Gauss
4	1.1162402700893e-01
8	1.1153340191221e-01
12	1.1153288987809e-01
16	1.1153289176609e-01
20	1.1153289176207e-01
24	1.1153289176207e-01

1000-point Gauss–Laguerre quadrature of $J_0(t)e^{-1/t}$ yields eleven correct digits

The weight function $w(t) = e^{-1/t^2 - t^2}$ on $\mathbb{R}$

moments

$$\mu_{2k+1} = 0, \quad \mu_{2k} = 2K_{k+1/2}(2)$$

recurrence

$$\mu_{2k+2} = (k + \tfrac{1}{2})\mu_{2k} + \mu_{2k-2}, \quad \mu_{-2} = \mu_0 = 2K_{1/2}(2)$$

The weight function $w(t) = e^{-1/t^2 - t^2}$ on $\mathbb{R}$

moments

$$\mu_{2k+1} = 0, \quad \mu_{2k} = 2K_{k+1/2}(2)$$

recurrence

$$\mu_{2k+2} = (k + \tfrac{1}{2})\mu_{2k} + \mu_{2k-2}, \quad \mu_{-2} = \mu_0 = 2K_{1/2}(2)$$

Matlab routine

```
% SR_EXPMINFPINF Symbolic/variable-precision recurrence
%    coefficients for the weight function
%    w(x)=exp(-1/x^2-x^2) on [-inf,inf]
%
syms mom ab
digits(d); dig=d;
mom(2:2:2*N)=vpa(0,dig);
mom(1)=vpa('2*BesselK(1/2,2)',dig);
mom(3)=3*mom(1)/2;
for k=3:2*N
   mom(2*k-1)=(k-3/2)*mom(2*k-3)+mom(2*k-5);
end
ab=schebyshev(dig,N,mom);
```

output (N=40, d=16)

```
ab =
[0, .239875543936 1229]
[0, 1.5000000000000000]
[0, .6666666666666667]
[0, 2.5833333333333333]
[0, 1.4758064516129003]
[0, 3.6594394500026441]
...        ...
[0, 16.499636356 31090]
[0, 20.316049338 08658]
[0, 17.467111691 29393]
[0, 21.342812825 04185]
```

Example 4 $\int_{-\infty}^{\infty} e^{-t} \cos t \, e^{-1/t^2 - t^2} \mathrm{d}t$

Example 4 $\int_{-\infty}^{\infty} e^{-t} \cos t \, e^{-1/t^2-t^2} \mathrm{d}t$

Gauss quadrature with weight function e^{-1/t^2-t^2}

```
>> intexpminfpinf

  n        n-point Gauss
  2     1.504037427987 6e-01
  4     1.130819395794 3e-01
  6     1.134279622780 3e-01
  8     1.134269584074 5e-01
 10     1.134269598159 2e-01
 12     1.134269598147 5e-01
 14     1.134269598147 5e-01
```

1000-point Gauss–Hermite quadrature of $e^{-t-1/t^2} \cos t$ yields eight correct digits

Circle theorem for Gaussian quadrature

(Davis and Rabinowitz, 1961)

Theorem The nodes τ_ν and weights λ_ν of the n-point Gaussian quadrature formula on $[-1,1]$ satisfy

$$n\lambda_\nu/(\pi w(\tau_\nu)) \sim \sqrt{1-\tau_\nu^2}, \quad n \to \infty.$$

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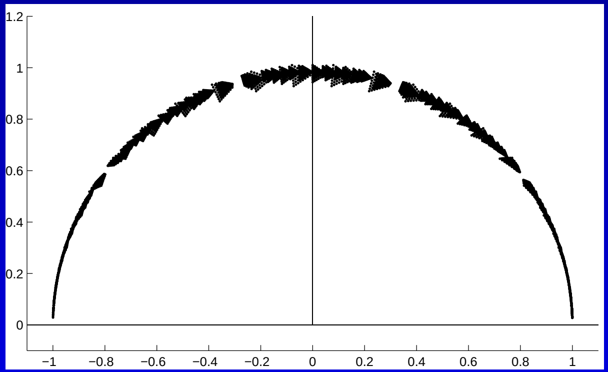
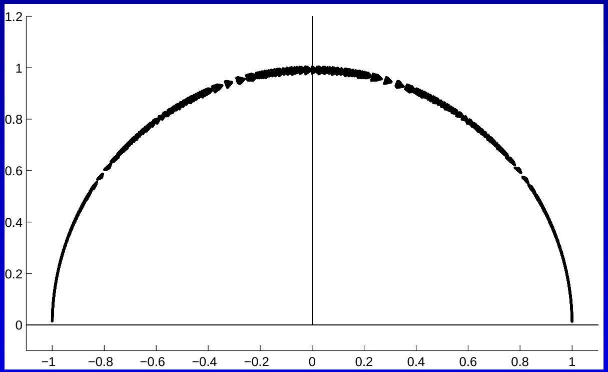
Example Gauss-Jacobi formula, with Jacobi parameters

α, β=–.75:.25:1.0, $\beta \geq \alpha$



$n = 20 : 5 : 40$



$n = 60 : 5 : 80$

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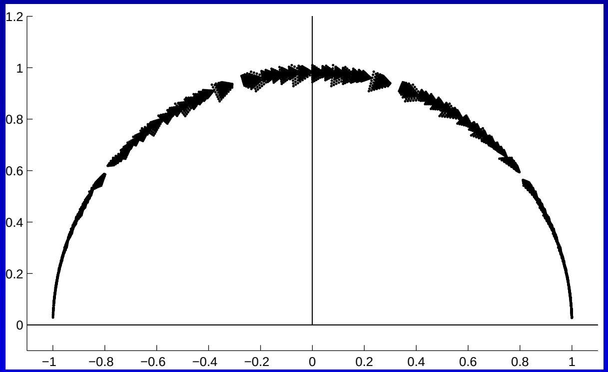

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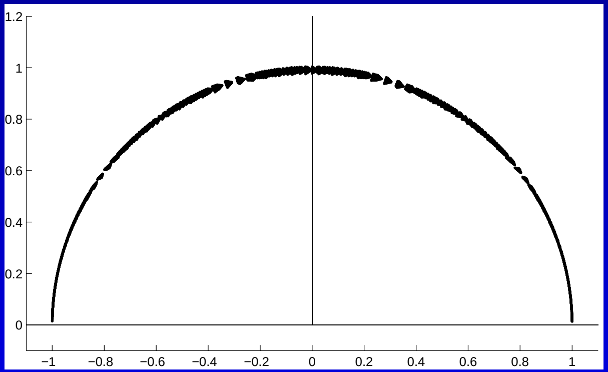
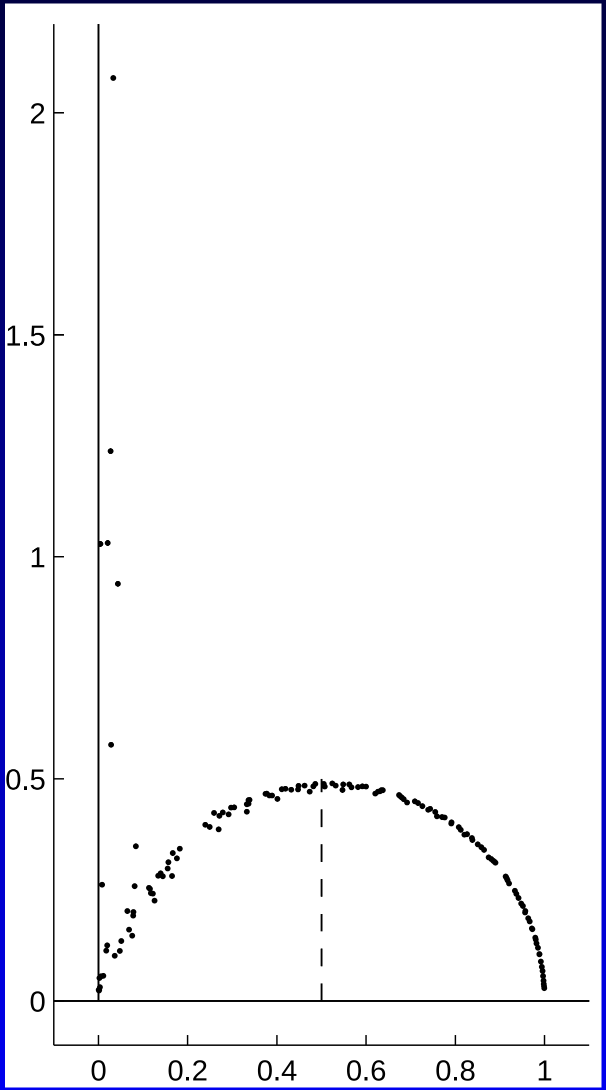


$n = 60 : 5 : 80$

Query True for $1 + \sin(1/t)$ or $\exp(-1/t)$?

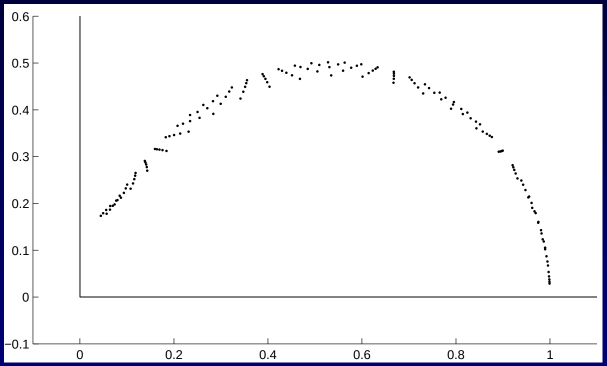
The circle theorem gone berserk

$$w(t) = 1 + \sin(1/t) \text{ on } [0, 1]$$



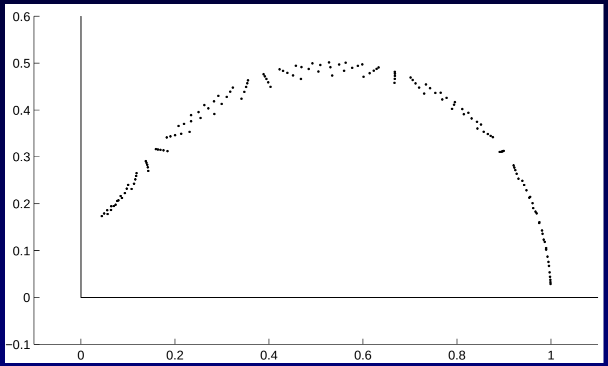
The circle theorem impaired

$$w(t) = \exp(-1/t) \text{ on } [0,1]$$



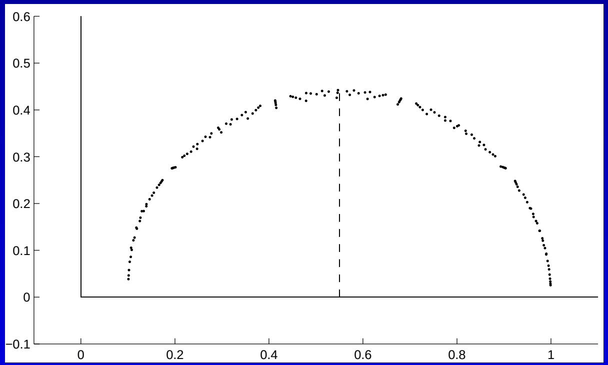
The circle theorem impaired

$$w(t) = \exp(-1/t) \text{ on } [0, 1]$$



. . . and repaired

$$w(t) = \exp(-1/t) \text{ on } [c, 1], c = .1$$



The circle theorem for weight functions in the Szegö class

Theorem (Nevai, 1979). If $\log w(t)/\sqrt{1-t^2} \in L_1[-1,1]$ and $1/(\sqrt{1-t^2}\, w(t)) \in L_1[\Delta]$, where Δ is any compact subinterval of $(-1,1)$, then

$$n\lambda_\nu/(\pi w(\tau_\nu)) \sim \sqrt{1-\tau_\nu^2} \text{ on } \Delta, \quad n \to \infty.$$

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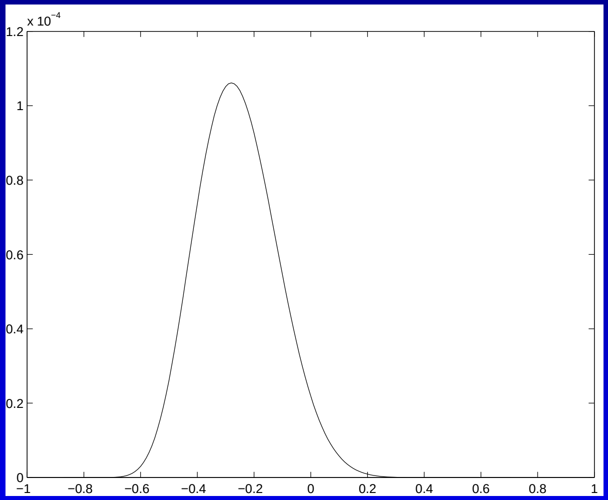
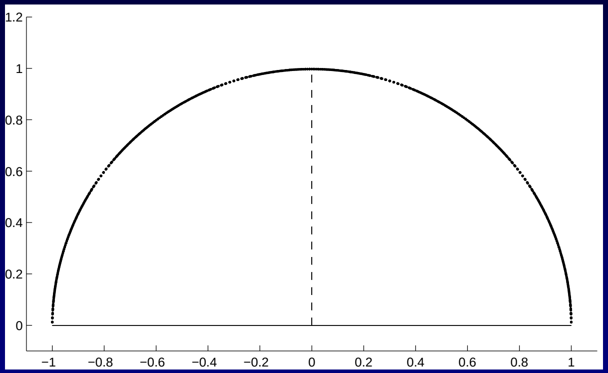
$$n\lambda_\nu/(\pi w(\tau_\nu)) \sim \sqrt{1-\tau_\nu^2} \text{ on } \Delta, \quad n \to \infty.$$

Example Pollaczek weight function $w(\,\cdot\,; a, b)$, $a \geq |b|$

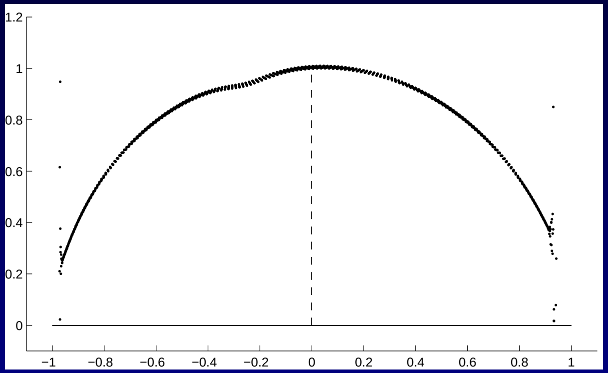


$$a = 4, b = 1$$

“Circle theorem” for Pollaczek weight function



$a = b = 0$



$a = 4, b = 1$

$$n = 180 : 5 : 200$$