The Hardy-Littlewood Function

An Exercise in Slowly Convergent Series

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OUTLINE

- The Hardy-Littlewood (HL) function

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- Summation by integration

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- A quadrature problem and its solution

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- Quadrature approximation of the HL-function
- Direct summation with acceleration

The function of Hardy and Littlewood

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k}, \quad x > 0$$

background

summation procedure of Lambert Hardy and Littlewood (1936): there exist infinitely many x with $x \to \infty$ such that

$$H(x) > C(\log \log x)^{1/2}$$

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complete monotonicity C. Berg and H. Alzer (work in progress): complete monotonicity for all m of $-[x^m \psi^{(m)}(x)]^{(m)}$ is equivalent to $H(x) \geq -\frac{\pi}{2}$

Ismail and Clark (2003): true if $m = 2, 3, \ldots, 16$

Summation by integration (G. and Milovanović, 1985)

$$S = \sum_{k=1}^{\infty} a_k, \quad a_k = (\mathcal{L}f)(k)$$

where

$$(\mathcal{L}f)(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

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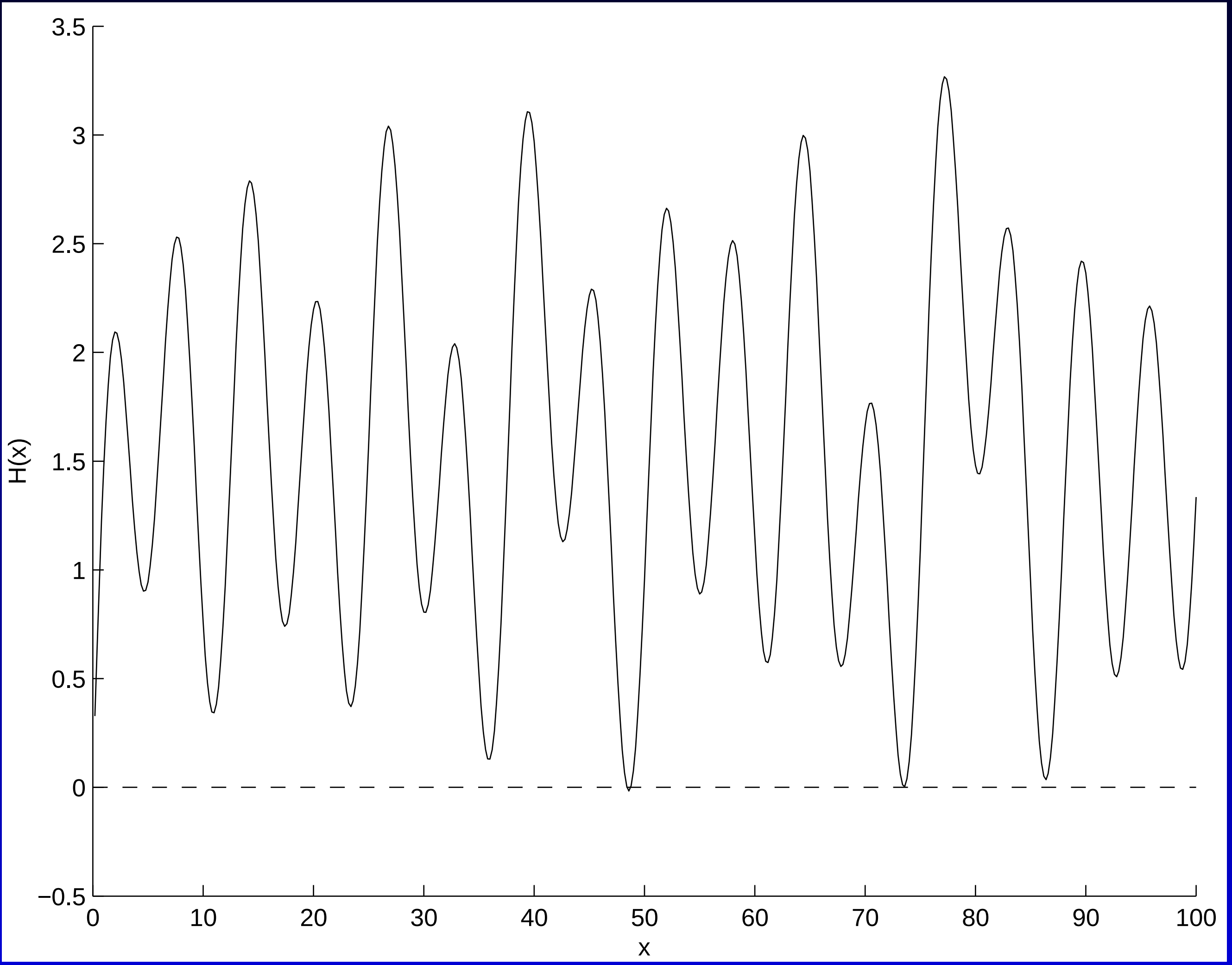
summation procedure

$$\begin{aligned} S &= \sum_{k=1}^{\infty} (\mathcal{L}f)(k) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kt} f(t)\mathrm{d}t \\ &= \int_0^{\infty} \sum_{k=1}^{\infty} e^{-(k-1)t} \cdot e^{-t} f(t)\mathrm{d}t \\ &= \int_0^{\infty} \frac{1}{1-e^{-t}} \cdot e^{-t} f(t)\mathrm{d}t = \int_0^{\infty} \frac{t}{1-e^{-t}} \frac{f(t)}{t} e^{-t}\mathrm{d}t \end{aligned}$$

poles at $\pm 2\mu i\pi$, $\mu = 1, 2, 3, \ldots$

Progress report #1



The HL-function for $0 \leq x \leq 100$

A quadrature problem

$$\int_{\mathbb{R}} g(t)\mathrm{d}\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu g(\tau_\nu) + R_n(g)$$

determine λ_ν, τ_ν such that $R_n(g) = 0$ if $g \in \mathbb{S}_{2n}$, where

$$\mathbb{S}_{2n} = \mathbb{Q}_m \oplus \mathbb{P}_{2n-m-1}, \quad 0 \le m \le 2n$$

$$\mathbb{P}_{2n-m-1} = \text{polynomials of degree } \le 2n - m - 1$$

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specifically:

$$\mathbb{Q}_m = \text{span}\left\{ r(t) = \tfrac{1}{1+\zeta_\mu t}, \ \ \mu = 1, 2, \ldots, m \right\}$$

$$\zeta_\mu \in \mathbb{C}, \quad \zeta_\mu \neq 0, \quad 1 + \zeta_\mu t \neq 0 \text{ on supp}(\mathrm{d}\lambda)$$

Theorem (G., 2000; Vanherwegen et al., 2000)

Let $\omega_m(t) = \prod_{\mu=1}^{m}(1+\zeta_\mu t)$. *Assume the existence of a* (polynomial) *Gauss formula*

$$\int_{\mathbb{R}} g(t)\frac{\mathrm{d}\lambda(t)}{\omega_m(t)} = \sum_{\nu=1}^{n} \lambda_\nu^G g(\tau_\nu^G), \quad g \in \mathbb{P}_{2n-1}.$$

Then

$$\tau_\nu = \tau_\nu^G, \quad \lambda_\nu = \omega_m(\tau_\nu^G)\lambda_\nu^G, \quad \nu = 1, 2, \ldots, n,$$

yields the desired formula.

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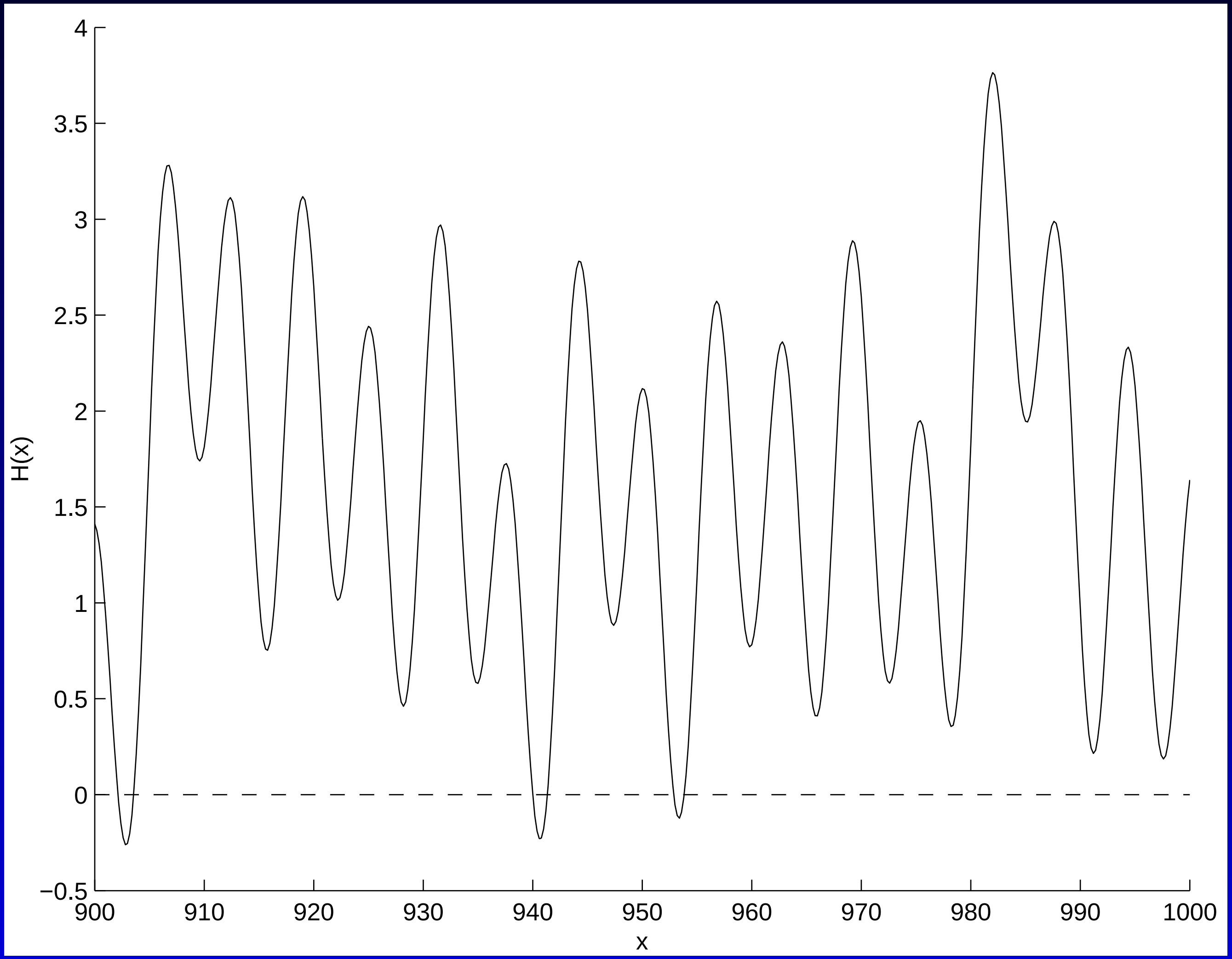
Example m even, $\zeta_\mu = \pm\frac{i}{2\mu\pi}$, $\mu = 1, 2, \ldots, m/2$

$$\omega_m(t) = \prod_{\mu=1}^{m/2}\left(1+\frac{t^2}{4\mu^2\pi^2}\right) > 0 \text{ on } \mathbb{R}$$

Progress report #2



The HL-function for $900 \leq x \leq 1000$

Construction of Gauss quadrature rules

$$\int_{\mathbb{R}} g(t)\mathrm{d}\sigma(t) = \sum_{\nu=1}^{n} \sigma_\nu g(\tau_\nu), \qquad g \in \mathbb{P}_{2n-1}$$

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orthogonal polynomials

$$\pi_k(t) = \pi_k(t; \mathrm{d}\sigma) : \int_{\mathbb{R}} \pi_k(t)\pi_\ell(t)\mathrm{d}\sigma(t) = 0 \text{ if } k \neq \ell$$

$$\tau_\nu = \text{zeros of } \pi_n(\,\cdot\,; \mathrm{d}\sigma)$$

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three-term recurrence relation

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t),$$

$$\pi_{-1}(t) = 0, \quad \pi_0(t) = 1$$

where $\alpha_k = \alpha_k(\mathrm{d}\sigma)$, $\beta_k = \beta_k(\mathrm{d}\sigma)$, and by convention $\beta_0 = \int_{\mathbb{R}} \mathrm{d}\sigma(t)$

Jacobi matrix

$$\boldsymbol{J}_n(\mathrm{d}\sigma) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{0} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{0} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

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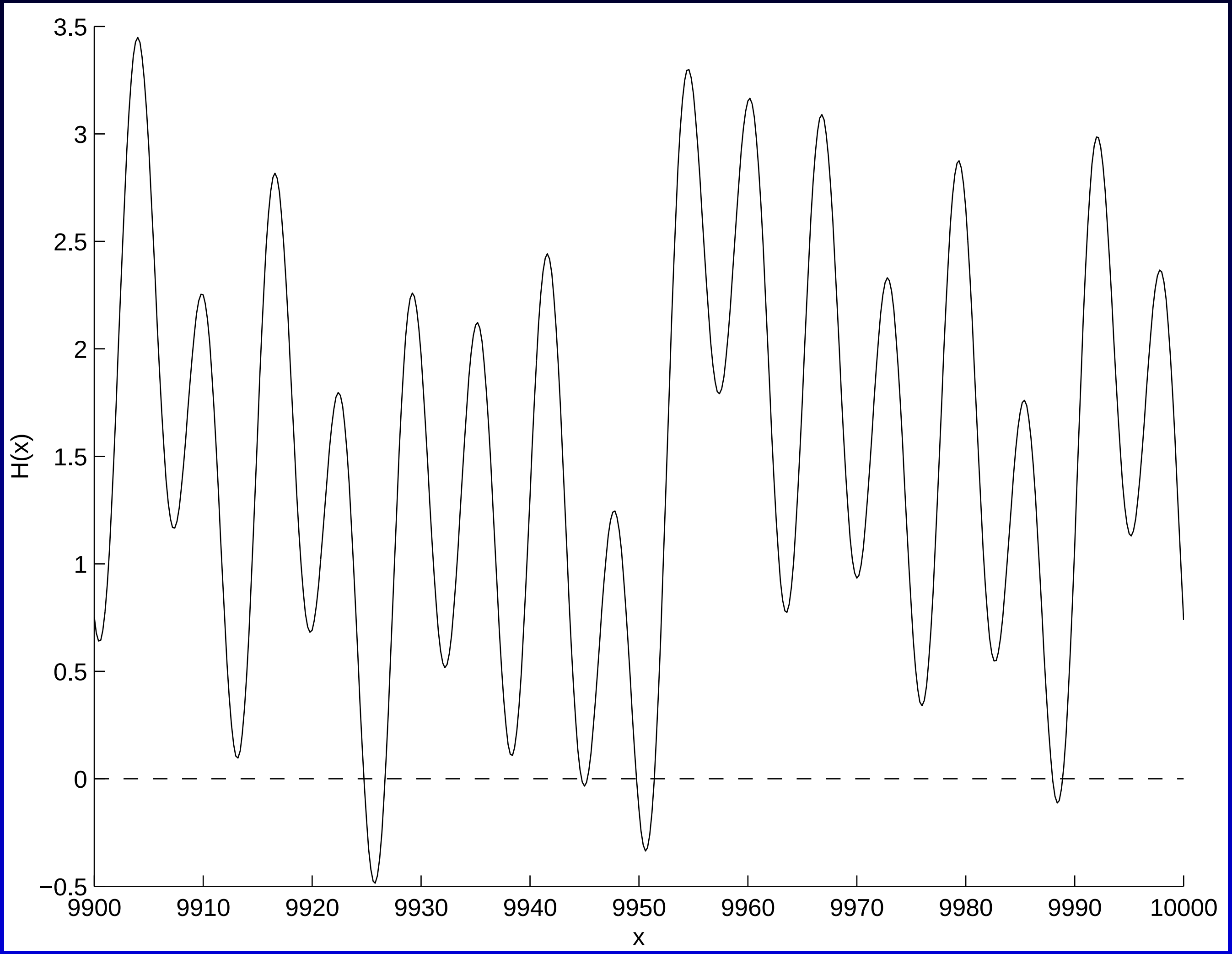
Theorem (Golub & Welsch, 1969) *The Gauss nodes τ_ν are the eigenvalues of $\boldsymbol{J}_n$,*

$$\boldsymbol{J}_n(\mathrm{d}\sigma)\boldsymbol{v}_\nu = \tau_\nu \boldsymbol{v}_\nu, \quad \boldsymbol{v}_\nu^T \boldsymbol{v}_\nu = 1,$$

and the Gauss weights σ_ν given by

$$\sigma_\nu = \beta_0 \boldsymbol{v}_{\nu,1}^2, \quad \boldsymbol{v}_\nu = [\boldsymbol{v}_{\nu,1}, \ldots]^T.$$

Progress report #3



The HL-function for $9,900 \leq x \leq 10,000$

Computation of recurrence coefficients

discretization method

approximation of $\mathrm{d}\sigma$ by a discrete N-point measure $\mathrm{d}\sigma_N$

$$\int_{\mathbb{R}} p(t)\mathrm{d}\sigma(t) \approx \sum_{k=1}^{N} w_k p(t_k) =: \int_{\mathbb{R}} p(t)\mathrm{d}\sigma_N(t)$$

then

$$\alpha_k(\mathrm{d}\sigma) \approx \alpha_k(\mathrm{d}\sigma_N), \quad \beta_k(\mathrm{d}\sigma) \approx \beta_k(\mathrm{d}\sigma_N)$$

(discrete) inner product

$$(u,v)_N = \int_{\mathbb{R}} u(t)v(t)\mathrm{d}\sigma_N(t) = \sum_{k=1}^{N} w_k u(t_k)v(t_k)$$

Darboux's formulae

$$(I) \quad \begin{cases} \alpha_k = \frac{(t\pi_k,\pi_k)_N}{(\pi_k,\pi_k)_N}, & k = 0, 1, \ldots, n-1, \\ \beta_0 = (\pi_0, \pi_0)_N, & \\ \beta_k = \frac{(\pi_k,\pi_k)_N}{(\pi_{k-1},\pi_{k-1})_N}, & k = 1, 2, \ldots, n-1 \end{cases}$$

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recurrence relation

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recurrence relation

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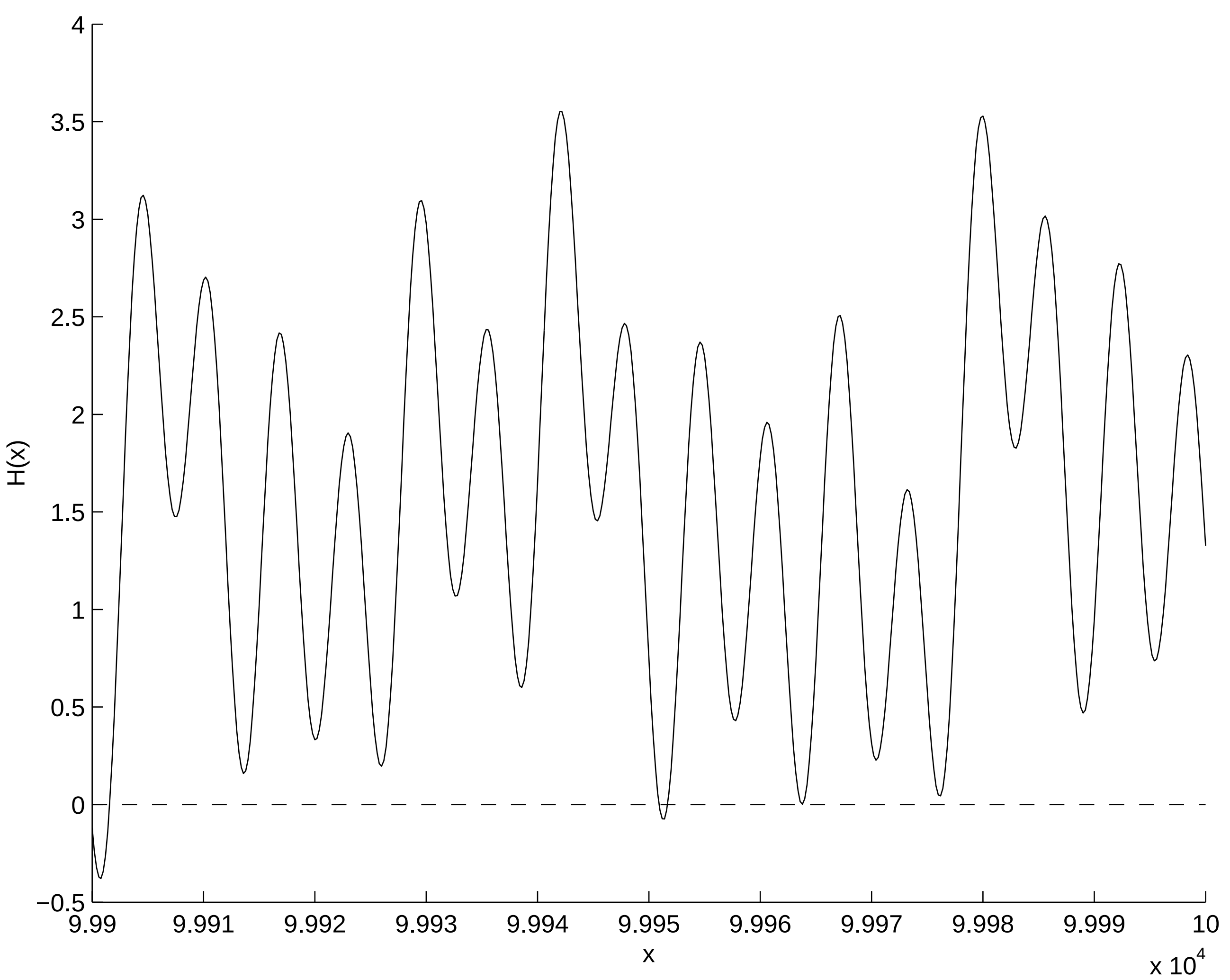
Stieltjes's procedure

$$\pi_0 = 1 \stackrel{(I)}{\Longrightarrow} \alpha_0, \beta_0 \stackrel{(II)}{\Longrightarrow} \pi_1 \stackrel{(I)}{\Longrightarrow} \alpha_1, \beta_1 \stackrel{(II)}{\Longrightarrow} \cdots$$
$$\stackrel{(I)}{\Longrightarrow} \alpha_{n-1}, \beta_{n-1}$$

Progress report #4



The HL-function for $99,900 \leq x \leq 100,000$

Back to Hardy-Littlewood

$$H(x) = \sum_{k=1}^{\infty} a_k(x), \quad a_k(x) = \frac{1}{k} \sin \frac{x}{k}$$

general term as a Laplace transform

$$\frac{1}{s} e^{x/s} = \left(\mathcal{L}_{(t)} I_0(2\sqrt{xt})\right)(s), \; I_0 = \text{modified Bessel}$$

$$\frac{1}{s} \sin \frac{x}{s} = \frac{1}{s} \frac{1}{2i} \left(e^{ix/s} - e^{-ix/s}\right)$$

$$\Longrightarrow \quad a_k(x) = (\mathcal{L} f)(k),$$

$$f(t) = f(t; x)$$

$$= \frac{1}{2i} \left[I_0(2\sqrt{ixt}) - I_0(2\sqrt{-ixt}) \right]$$

The function f

power series

$$f(t;x) = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{(2k+1)!^2}, \qquad u = xt$$

$$\lim_{t \to 0} \frac{f(t;x)}{t} = x$$

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integral representation

$$f(t;x) = \frac{1}{\pi} \int_0^{\pi} e^{\sqrt{2u}\cos\theta} \sin(\sqrt{2u}\cos\theta)\mathrm{d}\theta$$

composite trapezoidal rule

HL-function (cont')

$$H(x) = \int_0^\infty \frac{t}{1-e^{-t}} \frac{f(t;x)}{t} e^{-t}\mathrm{d}t$$

quadrature approximation

$$H(x) \approx \sum_{\nu=1}^n \lambda_\nu \frac{\tau_\nu}{1-e^{-\tau_\nu}} \frac{f(\tau_\nu;x)}{\tau_\nu}$$

HL-function (cont')

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polynomial/rational Gauss with

$$\mathbb{Q}_m : \quad m = 2\lfloor (n+1)/2 \rfloor$$

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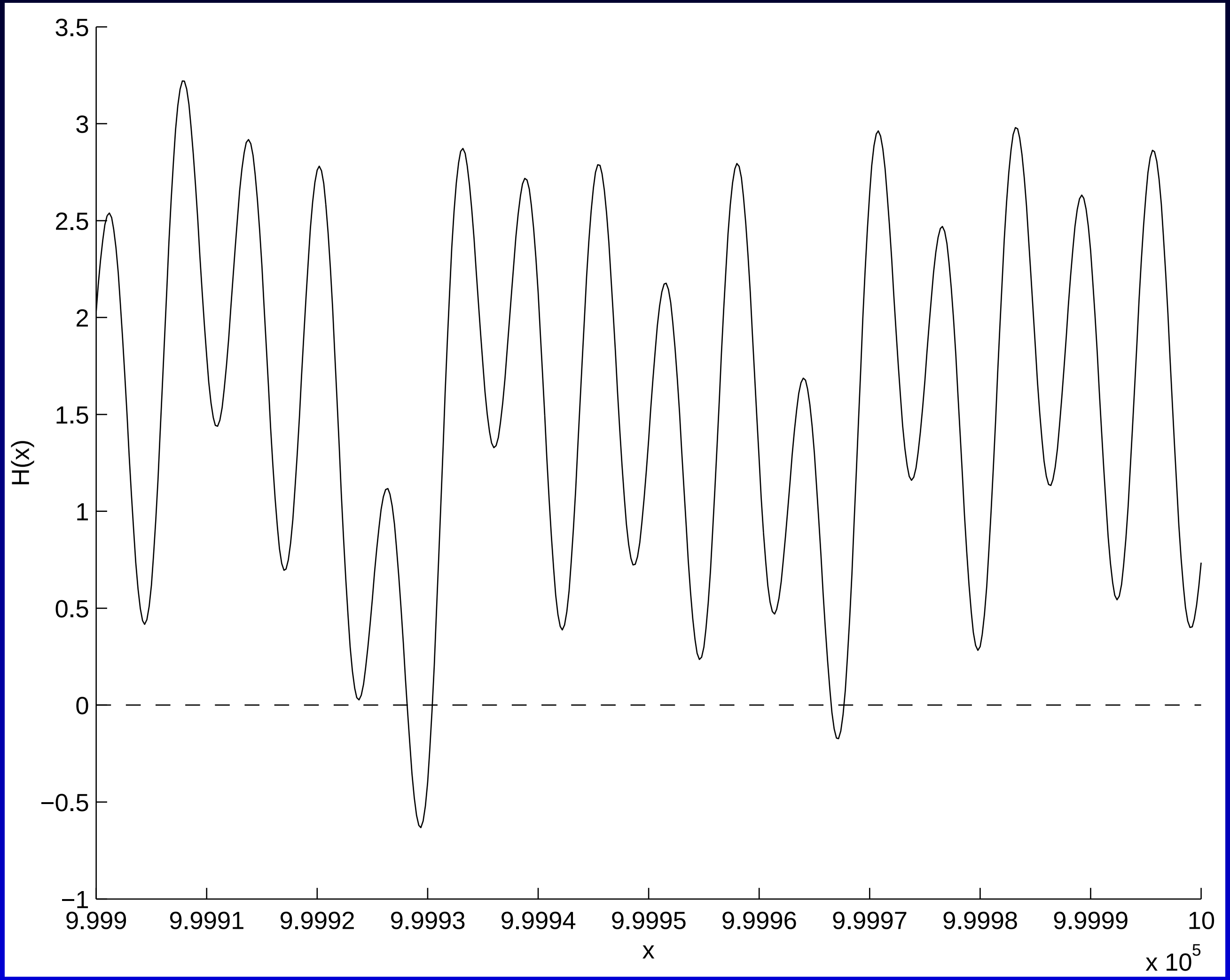
performance with error tolerance $\frac{1}{2}10^{-6}$

x	10	25	50	75	100
n	20	35	55	75	95
# of digits lost	1	4	10	15	20

Progress report #5 (smoking gun?)



The HL-function for $999,900 \leq x \leq 1,000,000$

Direct summation with acceleration

Assume $x \gg 1$ and let $n = \lfloor x \rfloor$

$$H(x) = \sum_{k=1}^{n} \frac{1}{k} \sin\frac{x}{k} + \sum_{k=n+1}^{\infty} \frac{1}{k} \sin\frac{x}{k}$$

$$= \sum_{k=1}^{n} \frac{1}{k} \sin\frac{x}{k} + \sum_{k=n+1}^{\infty} \frac{1}{k} \left(\sin\frac{x}{k} - \frac{x}{k} + \frac{1}{6} \left(\frac{x}{k}\right)^3 \right)$$

$$+x \left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right) - \frac{x^3}{6} \left(\frac{\pi^4}{90} - \sum_{k=1}^{n} \frac{1}{k^4} \right)$$

Euler-Maclaurin summation

$$\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \simeq \frac{B_0}{n+1} - \frac{B_1}{(n+1)^2} + \frac{B_2}{(n+1)^3} + \frac{B_4}{(n+1)^5} + \cdots + \frac{B_{10}}{(n+1)^{11}}$$

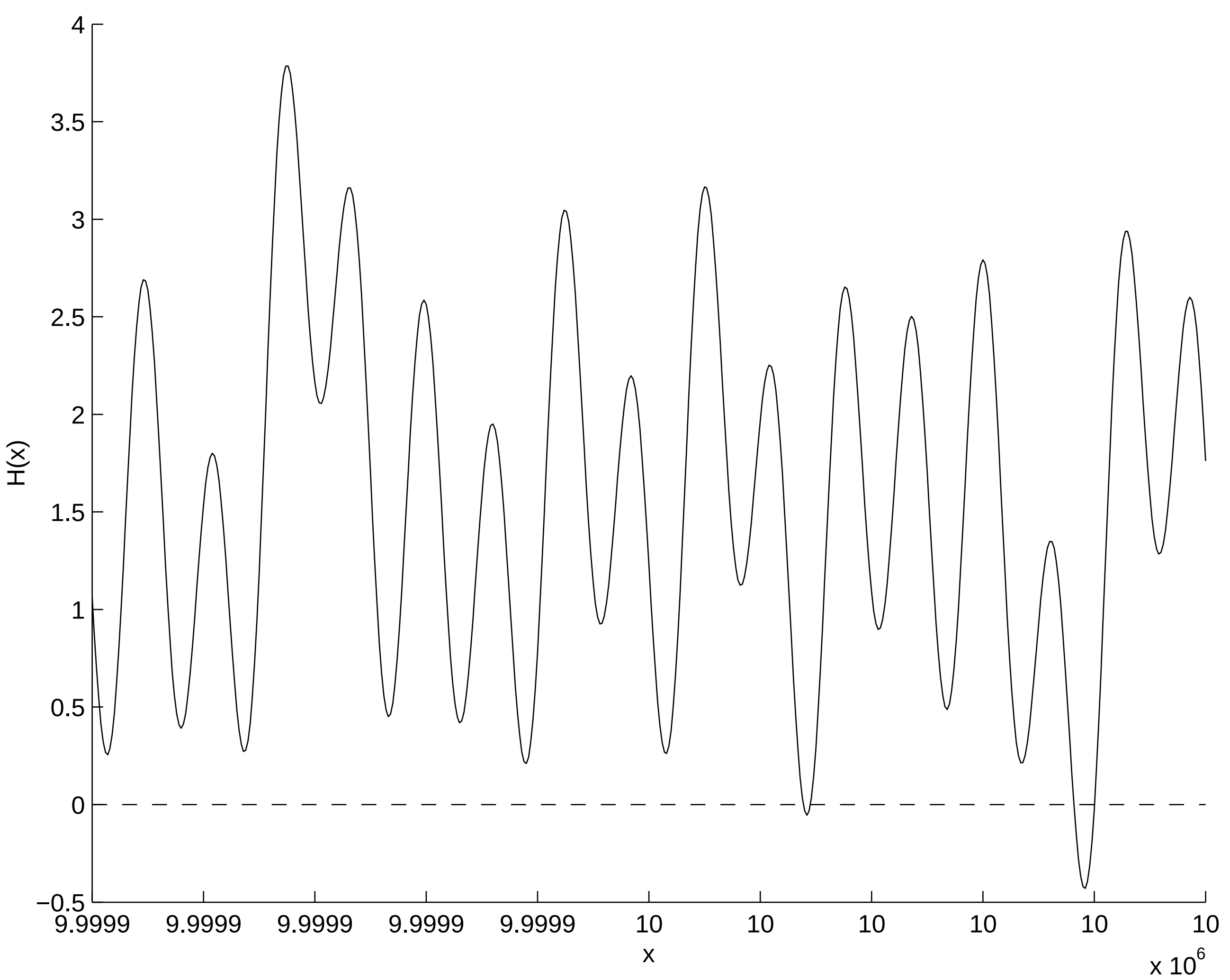
$$\frac{\pi^4}{90} - \sum_{k=1}^{n} \frac{1}{k^4} \simeq \frac{B_0}{3(n+1)^3} - \frac{B_1}{(n+1)^4} + \frac{2B_2}{(n+1)^5} + \frac{5B_4}{(n+1)^7} + \frac{28B_6}{3(n+1)^9} + \frac{3B_8}{(n+1)^{11}} + \frac{22B_{10}}{(n+1)^{13}}$$

where

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6}, \ldots, B_{10} = \frac{5}{66}$$

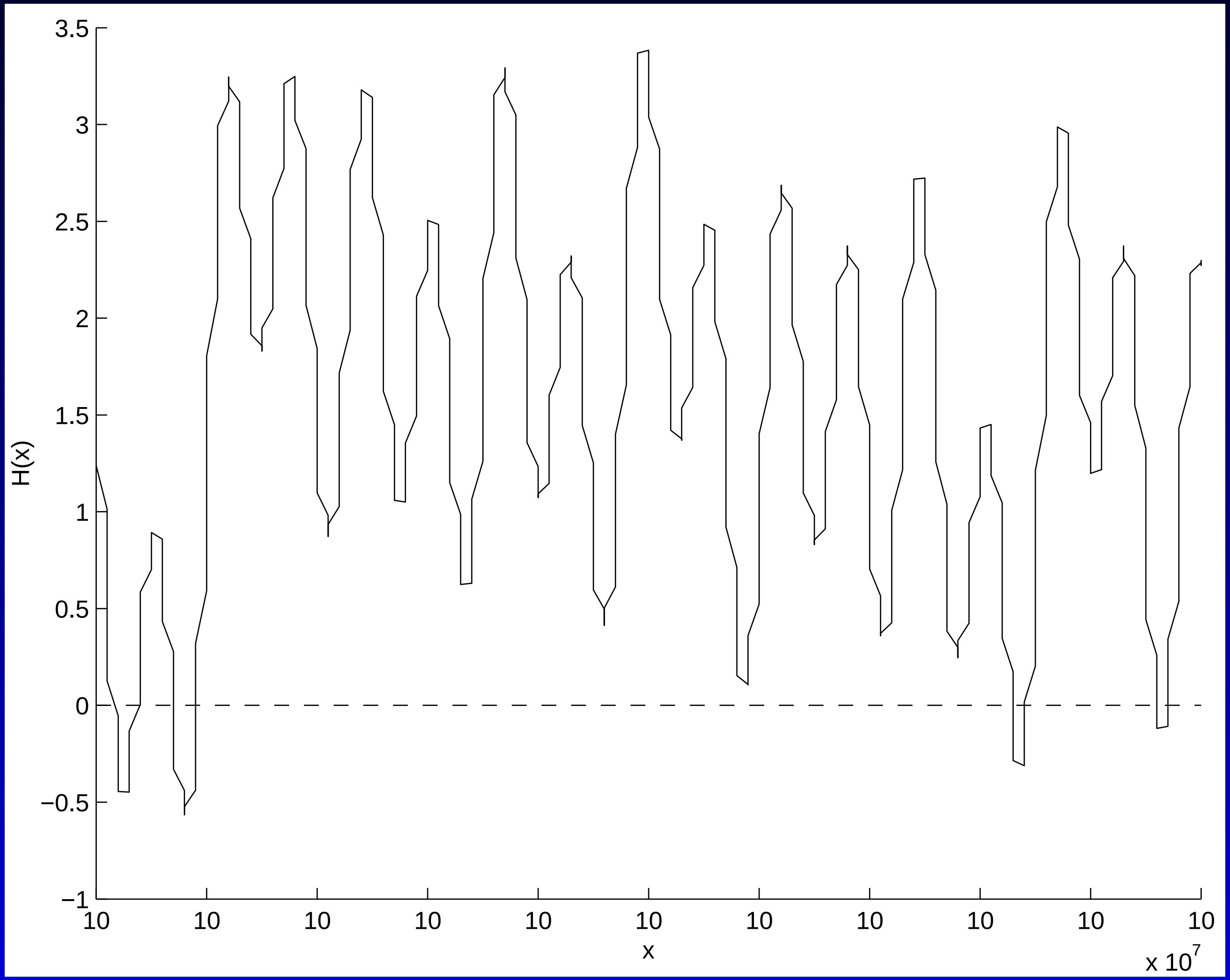
are the Bernoulli numbers

Progress report #6



The HL-function for $9,999,900 \leq x \leq 10,000,000$

Progress report #7



The HL-function for $99,999,900 \leq x \leq 100,000,000$

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