On Euler's attempt to compute logarithms by interpolation

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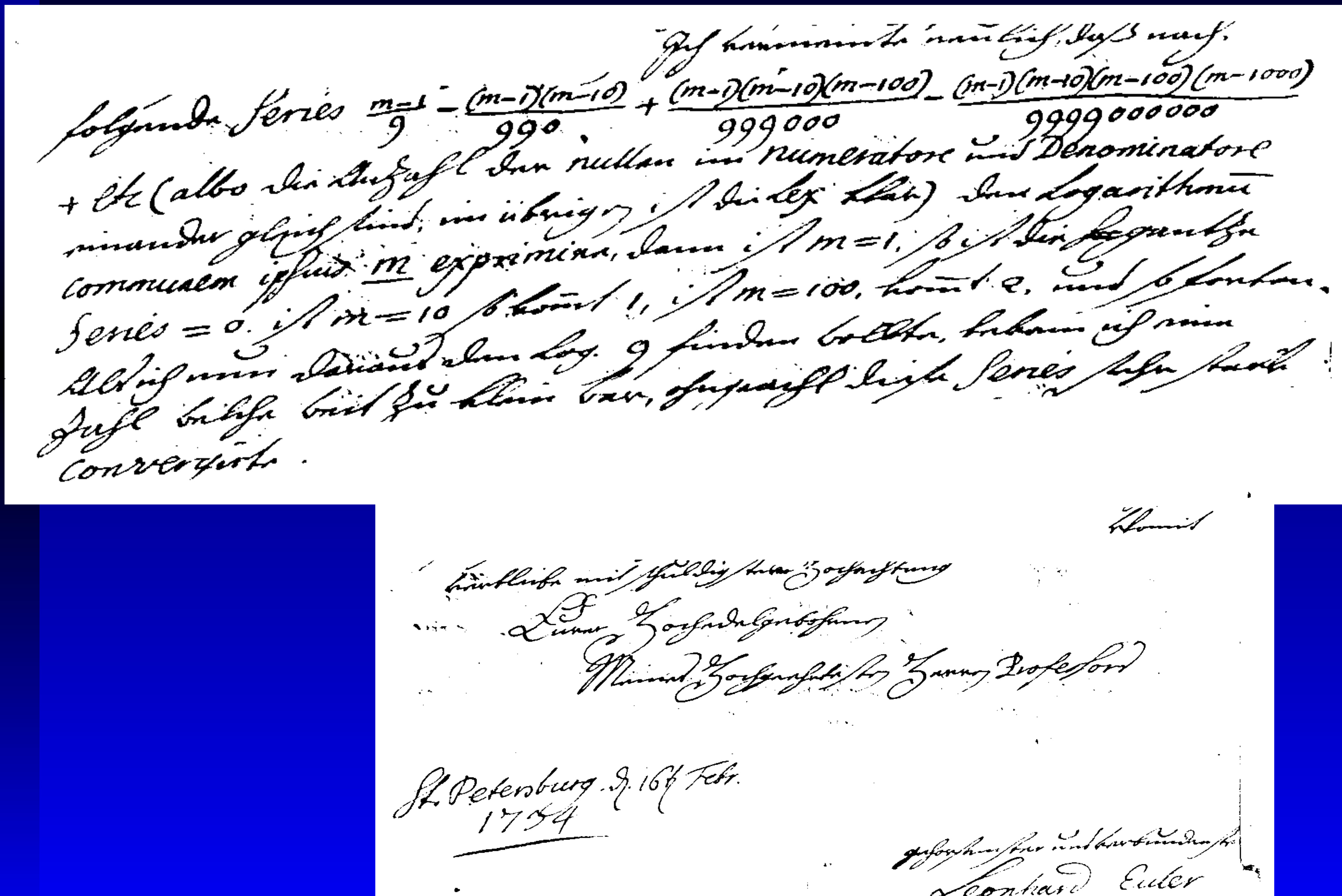
Leonhard Euler 1707–1783

From Euler's letter to Daniel Bernoulli



Ich bemerkte neulich, dass nach folgende Series $\frac{m-1}{9} - \frac{(m-1)(m-10)}{990} + \frac{(m-1)(m-10)(m-100)}{999000} - \frac{(m-1)(m-10)(m-100)(m-1000)}{9999000000}$ + etc (allwo die Anzahl der nullen im numeratore und Denominatore einander gleich sind, im übrigen ist die lex klar) den Logarithmum communem ipsius m exprimire, dann ist $m=1$, so ist die folgende Series $=0$. ist $m=10$ so kommt 1, ist $m=100$, kommt 2, und so fortan. Als ich nun daraus den log. 9 finden wollte, bekame ich eine Zahl welche viel zu klein war, ohngeachtet diese Series sehr stark convergirte.

kommt

verbleibe mit schuldigster Hochachtung

Euer Hochedelgebohren

Meines Hochgeehrtesten Herrn Professors

St. Petersburg d. 16 Febr. 1734

gehorsamster und verbundenster

Leonhard Euler

“Ich vermeinte neulich, daß nachfolgende *Series*

$$\frac{m-1}{9} - \frac{(m-1)(m-10)}{990} + \frac{(m-1)(m-10)(m-100)}{999\,000} - \frac{(m-1)(m-10)(m-100)(m-1000)}{9\,999\,000\,000} + \text{etc.}$$

(alwo die Anzahl der nullen im *Numeratore* und *Denominatore* einander gleich sind, im übrigen ist die *Lex* klar) den *Logarithmum communem ipsius* m exprimire, dann ist $m = 1$, so ist die gantze *Series* $= 0$, ist $m = 10$ so kommt 1, ist $m = 100$, kommt 2, und so fortan. Als ich nun daraus den *Log* [*arithmum*] 9 finden wollte, bekam ich eine Zahl welche weit zu klein war, ohngeacht diese *Series* sehr stark convergirte.”

Euler's idea

Compute $f(x) = \log x, \ \ 1 \leq x < 10,$ by interpolating f at $x_r = 10^r$, $r = 0, 1, 2, \ldots$

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Newton interpolation series

$$S(x) = \sum_{k=1}^{\infty} a_k (x-1)(x-10)\cdots(x-10^{k-1})$$

$$a_k = [x_0, x_1, \ldots, x_k]f, \qquad k = 1, 2, 3, \ldots$$

Euler:

$$a_1 = \tfrac{1}{9},\; a_2 = -\tfrac{1}{990},\; a_3 = \frac{1}{999\,000},\; a_4 = -\frac{1}{9\,999\,000\,000}$$

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in general (proof by induction)

$$a_k = \frac{(-1)^{k-1}}{10^{k(k-1)/2}(10^k - 1)}, \quad k = 1, 2, 3, \ldots$$

Convergence

$$S(x) = \sum_{k=1}^{\infty} t_k(x), \quad 1 \le x < 10$$

$$t_k(x) = \frac{x-1}{10^k - 1} \prod_{r=1}^{k-1} (1 - x/10^r)$$

$$|t_k(x)| < \frac{9}{10^k - 1} \left(1 - \frac{1}{10^{k-1}}\right) < \frac{9}{10^k}, \; k \ge 2$$

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remainder

$$|S(x) - \sum_{k=1}^{n} t_k(x)| \le |t_{n+1}(x)| + |t_{n+2}(x)| + \cdots$$

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fast convergence, but

$$S(9) = .8977\ldots, \quad \log 9 = .9542\ldots$$

?

Go complex!

$$S(z) = \sum_{k=1}^{\infty} t_k(z), \quad z \in \mathbb{C}$$

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Convergence is uniform on $|z| \leq R$

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Consequence: Let $\mathcal{D} = \{z \in \mathbb{C} : |\arg z| < \pi\}$ and $\log z$, $z \in \mathcal{D}$, denote the principal branch of the logarithm. Then $S(z)$ cannot be equal to $\log z$ on any set of z-values having an accumulation point in $\mathcal{D}\backslash\{\infty\}$.

The q-analogue of the logarithm

W. Van Assche and E. Koelink (2006)

$$S_q(x) = -\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} (x;q)_n \, , \quad 0 < q < 1$$

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Limit of Euler's interpolation series

$$S(x) = S_q(x), \quad q = \frac{1}{10}$$

Can Euler's idea be salvaged?

1. Take $x_r = \omega^r$, $r = 0, 1, 2, \ldots$, $\omega > 1$. Then

$$S(x;\omega) = \lim_{n\to\infty} S_n(x;\omega) = \log\omega \cdot S_q(x),\ q = \frac{1}{\omega}$$

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As a consequence

$$\lim_{\omega\downarrow 1, n\to\infty} S_n(x;\omega) = \log x$$

Euler (1750) studied $S_{1/\omega}(x)$, but missed the connection with the logarithm!

Convergence behavior

$$S(x;\omega) = \sum_{k=1}^{\infty} t_k(x;\omega), \qquad S_n(x;\omega) = \sum_{k=1}^{n} t_k(x;\omega)$$

If $\omega < 1$ then $\lim_{k\to\infty} |t_k| = \infty$: interpolation process diverges. Therefore, assume $\omega > 1$.

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Theorem The error $|S_n(x;\omega) - \log x|$, $1 \leq x < 10$, can be made arbitrarily small by taking $\omega > 1$ sufficiently close to 1 and n sufficiently large.

But ...

… $\max_{k\in\mathbb{N}} |t_k(x;\omega)|$ may become extremely large!

$x\backslash\omega$	1.1	1.05	1.025	1.0125	1.00625
2	.41	.42	.43	.43	.43
6	$.11\times10^{4}$	$.78\times10^{7}$	$.81\times10^{15}$	$.17\times10^{32}$	$.15\times10^{65}$
10	$.19\times10^{8}$	$.24\times10^{16}$	$.74\times10^{32}$	$.13\times10^{66}$	$.82\times10^{132}$

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Errors achievable in d-digit computation with n terms

$x\backslash\omega$	1.1	1.05	1.025	1.0125	1.00625
2	$.17\times10^{-12}$	$.14\times10^{-23}$	$.95\times10^{-46}$	$.18\times10^{-88}$	$.20\times10^{-174}$
6	$.24\times10^{-8}$	$.43\times10^{-15}$	$.22\times10^{-28}$	$.43\times10^{-55}$	$.11\times10^{-107}$
10	$.43\times10^{-4}$	$.12\times10^{-6}$	$.17\times10^{-11}$	$.54\times10^{-21}$	$.76\times10^{-40}$
d	40	50	60	100	200
n	100	200	400	800	1500

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If $1 \le x \le 5$ and $\omega = 1.1$, then $|t_k(x;\omega)| \le 72.2$ and $d = 14$, $n = 100$ yields 10-digit accuracy. For $5 < x < 10$, use $\log x = \log(x/2) + \log 2$.

If $1 \le x \le 2$, then $|t_k(x;\omega)| < 1$, and for $\omega = 1.1$ one obtains 10-digit accuracy with $n = 20$.

2. Take $x_r = 10^{\omega/(r+1)}$, $r = 0, 1, 2, \ldots, \omega > 0$, hence

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Theorem Let $f \in C^{\infty}[a, b]$ and $x_r \in [a, b]$, $r = 0, 1, 2, \ldots$. Then the interpolation series $S(x)$ converges to $f(x)$ for any $x \in [a, b]$, provided

$$\lim_{k \to \infty} \frac{(b-a)^k}{k!} M_k = 0,$$

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Proof Cauchy's formula for the remainder term of interpolation

$$S_n(x) - f(x) = \frac{f^{(n+1)}(\xi_n)}{(n+1)!} \prod_{r=0}^{n} (x - x_r) \quad \square$$

If $f(x) = \log x, \quad a \le x \le b$ then

$$\frac{(b-a)^k}{k!} M_k = \left(\frac{b}{a} - 1\right)^k / (k \ln 10) \to 0 \text{ iff } \left|\frac{b}{a} - 1\right| \le 1$$

geometric convergence with ratio $q < 1$ if $\frac{b}{a} \le 1 + q$

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Change of variable

Given $1 \le x < 10$, determine the integer $k_0 \ge 0$ such that $10^{k_0\omega} \le x < 10^{(k_0+1)\omega}$. Then $t := 10^{-k_0\omega} x$ satisfies $1 \le t < 10^\omega$, and $\log x = \log t + k_0\omega$.