The circle theorem and related theorems for Gauss-type quadrature rules

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The circle theorem for Jacobi weight functions

Jacobi weight function

$$w(t) = w^{(\alpha,\beta)}(t) := (1-t)^\alpha (1+t)^\beta,\ \alpha > -1,\ \beta > -1$$

Gauss-Jacobi quadrature formula

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \sum_{\nu=1}^{n} \lambda_\nu^G f(\tau_\nu^G) + R_n^G(f)$$

$$R_n^G(p) = 0 \quad \text{for all } \ p \in \mathbb{P}_{2n-1}$$

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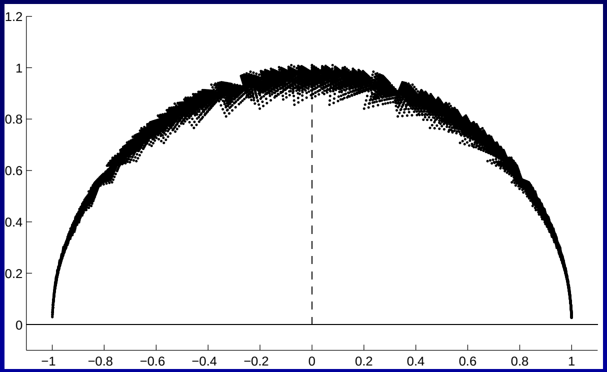
Davis and Rabinowitz (1961)

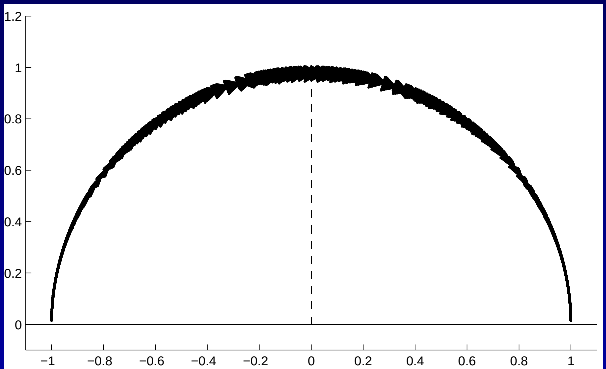
$$\frac{n\lambda_\nu^G}{\pi w^{(\alpha,\beta)}(\tau_\nu^G)} \sim \sqrt{1-(\tau_\nu^G)^2}, \quad n \to \infty$$

plots

$$\alpha, \beta = -.75(.25)1.0(.5)3.0, \quad \beta \geq \alpha$$

$n = 20 : 5 : 40$ $\qquad$ $n = 60 : 5 : 80$





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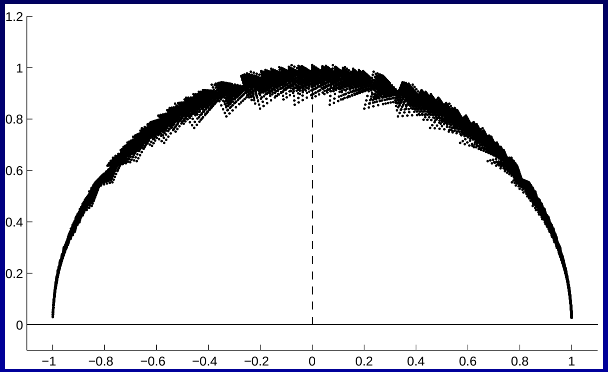

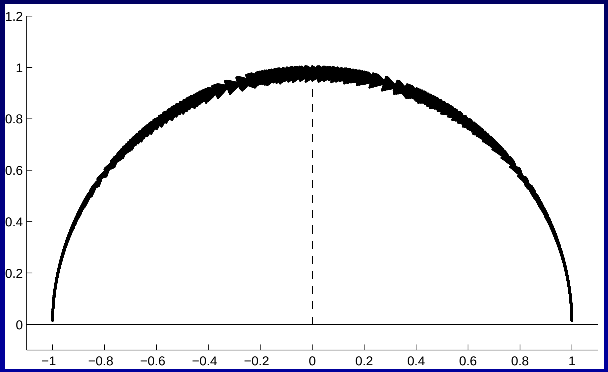

open questions

1. true for more general weight functions?
2. what about Gauss-Radau?
3. limiting curves other than the circle?

The circle theorem for weight functions in the Szegö class

$$w \in \mathcal{S} \quad \text{iff} \quad \int_{-1}^{1} \frac{\ln w(t)}{\sqrt{1-t^2}}\, \mathrm{d}t > -\infty$$

Theorem If $w \in \mathcal{S}$ and $1/(\sqrt{1-t^2}\, w(t)) \in L_1[\Delta]$, where Δ is any compact subinterval of $(-1, 1)$, then

$$n\lambda_\nu/(\pi w(\tau_\nu)) \sim \sqrt{1-\tau_\nu^2} \text{ on } \Delta, \quad n \to \infty.$$

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Remarks

1. equality holds if $w(t) = (1-t^2)^{-1/2}$
2. pointwise convergence almost everywhere holds if $w \in \mathcal{S}$ locally,

$$\operatorname{supp} w = [-1,1], \quad \int_\Delta \ln w(t)\mathrm{d}t > -\infty$$

Proof

Christoffel function

$$\lambda_n(x;w) := \min_{\substack{p\in\mathbb{P}_{n-1}\\ p(x)=1}} \int_{\mathbb{R}} p^2(t)w(t)\mathrm{d}t$$

Nevai (1979)

$$\frac{n\lambda_n(x;w)}{\pi w(x)} \sim \sqrt{1-x^2}, \quad n\to\infty$$

uniformly for $x \in \Delta$

$$\lambda_n(\tau_\nu^G;w) = \lambda_\nu^G, \quad \nu = 1,2,\ldots,n$$

$\square$

Example Pollaczek weight function (not in $\mathcal{S}$)

$$w(t; a, b) = \frac{2\exp(\omega \arccos(t))}{1 + \exp(\omega\pi)}\,, \ |t| \le 1,\ a > |b|$$

where

$$\omega = \omega(t) = (at + b)(1 - t^2)^{-1/2}$$

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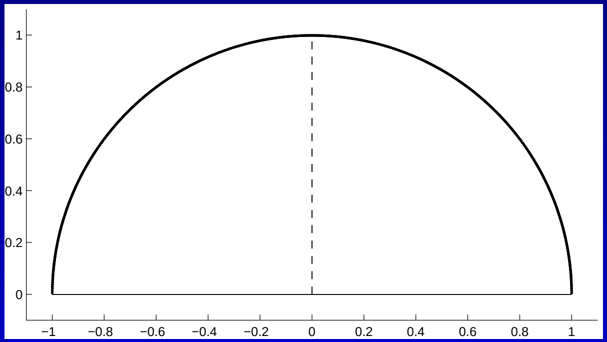

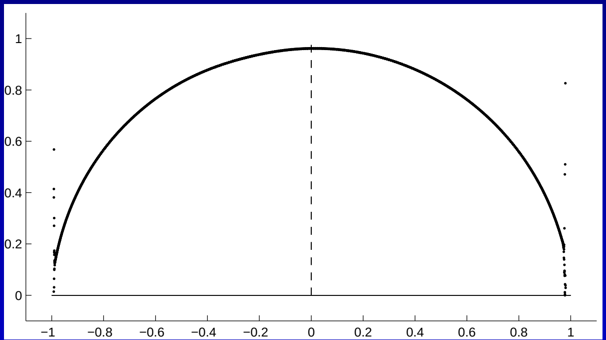

$a = b = 0$ $\qquad\qquad$ $a = 4, b = 1$

$n = 380(5)400$

Gauss-Radau formula

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \lambda_0^R f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^R f(\tau_\nu^R) + R_n^R(f)$$

Theorem Under the conditions of the previous theorem, the Gauss-Radau formula admits a circle theorem.

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Proof Let $w^R(t) = (t+1)w(t)$. Then

$$\tau_\nu^R \text{ are the zeros of } \; \pi_n(\,\cdot\,; w^R)$$

Define

$$\ell_\nu^*(t) = \prod_{\mu\neq\nu} \frac{t-\tau_\mu^R}{\tau_\nu^R - \tau_\mu^R}, \qquad \nu = 1, 2, \ldots, n$$

proof (cont') Gauss-Radau is interpolatory,

$$\lambda_\nu^R = \int_{-1}^1 \frac{(t+1)\pi_n(t;w^R)}{(\tau_\nu^R+1)(t-\tau_\nu^R)\pi_n'(\tau_\nu^R;w^R)}\, w(t)\mathrm{d}t$$
$$= \int_{-1}^1 \frac{(t+1)\ell_\nu^*(t)}{\tau_\nu^R+1}\, w(t)\mathrm{d}t.$$

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$$= \int_{-1}^{1} \frac{(t+1)\ell_\nu^*(t)}{\tau_\nu^R+1}\, w(t)\mathrm{d}t.$$

Let λ_ν^*, $\nu = 1, 2, \ldots, n$, be the Gauss weights for w^R,

$$\lambda_\nu^* = \int_{-1}^{1} \ell_\nu^*(t) w^R(t)\mathrm{d}t = \int_{-1}^{1} \ell_\nu^*(t)(t+1)w(t)\mathrm{d}t$$
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Then

$$\frac{n\lambda_\nu^R}{\pi w(\tau_\nu^R)} = \frac{n\lambda_\nu^*}{\pi(\tau_\nu^R+1)w(\tau_\nu^R)} = \frac{n\lambda_\nu^*}{\pi w^R(\tau_\nu^R)} \sim \sqrt{1-(\tau_\nu^R)^2} \quad \square$$

Example $w(t) = t^\alpha \ln(1/t)$ on $[0, 1]$, $\alpha > -1$
Gauss-Radau

$$\int_0^1 f(t)t^\alpha \ln(1/t)\mathrm{d}t = \lambda_0 f(0) + \sum_{\nu=1}^n \lambda_\nu f(\tau_\nu) + R_n($$

circle theorem (transformed to $[0, 1]$)

$$\frac{n\lambda_\nu}{\pi\tau_\nu^\alpha \ln(1/\tau_\nu)} \sim \sqrt{(\tfrac{1}{2})^2 - (\tau_\nu - \tfrac{1}{2})^2}, \quad n \to \infty.$$

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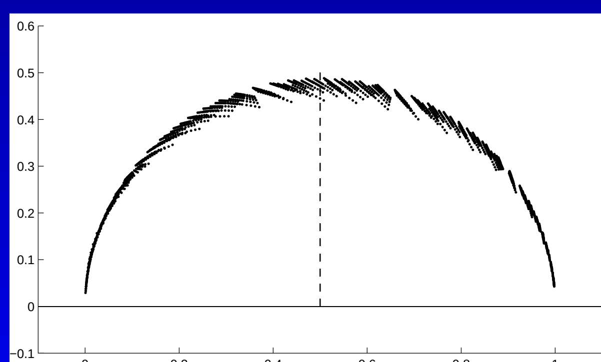
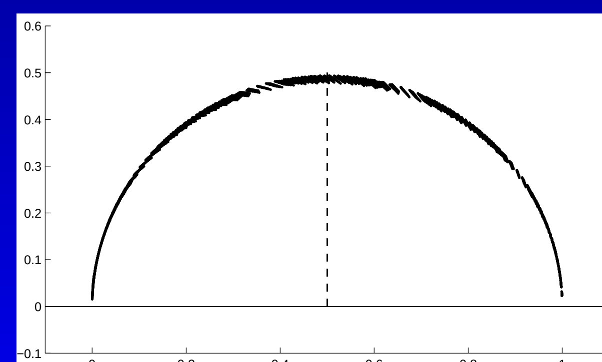
plot ($\alpha = -.75(.25)1.0(.5)3$)



$n = 20(5)40$



$n = 60(5)80$

Gauss-Lobatto formula

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \lambda_0^L f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^L f(\tau_\nu^L)$$
$$+\lambda_{n+1}^L f(1) + R_n^L(f)$$

Theorem Under the conditions of the previous theorem, the Gauss-Lobatto formula admits a circle theorem.

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Theorem Under the conditions of the previous theorem, the Gauss-Lobatto formula admits a circle theorem.

Proof Similar to Gauss-Radau:

$$\frac{n\lambda_\nu^L}{\pi w(\tau_\nu^L)} = \frac{n\lambda_\nu^*}{\pi(1-(\tau_\nu^L)^2)w(\tau_\nu^L)} = \frac{n\lambda_\nu^*}{\pi w^L(\tau_\nu^L)} \sim \sqrt{1-(\tau_\nu^L)^2} \quad \square$$

Gauss-Kronrod formula

$$(\mathrm{GK}) \qquad \int_{-1}^{1} f(t)w(t)\mathrm{d}t = \sum_{\nu=1}^{n} \lambda_{\nu}^{K} f(\tau_{\nu}^{G}) + \sum_{\mu=1}^{n+1} \lambda_{\mu}^{*K} f(\tau_{\mu}^{K}) + R_n^K(f)$$

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Theorem Assume (GK) exists with τ_μ^K distinct nodes in $(-1,1)$ and $\tau_\mu^K \neq \tau_\nu^G$, all μ,ν, and the Gauss formula for w admits a circle theorem. Then, under some additional (technical) conditions,

$$\frac{2n\lambda_\nu^K}{\pi w(\tau_\nu^G)} \sim \sqrt{1-(\tau_\nu^G)^2}, \quad \frac{2n\lambda_\mu^{*K}}{\pi w(\tau_\mu^K)} \sim \sqrt{1-(\tau_\mu^K)^2}$$

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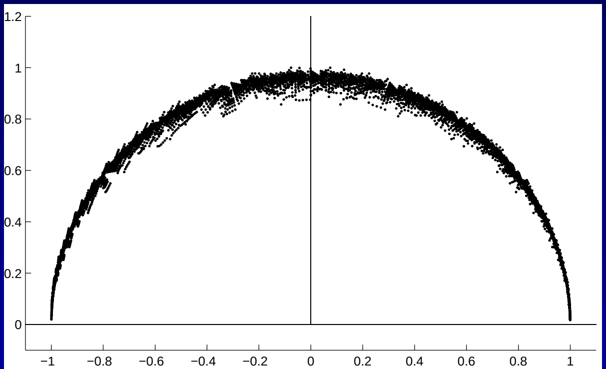
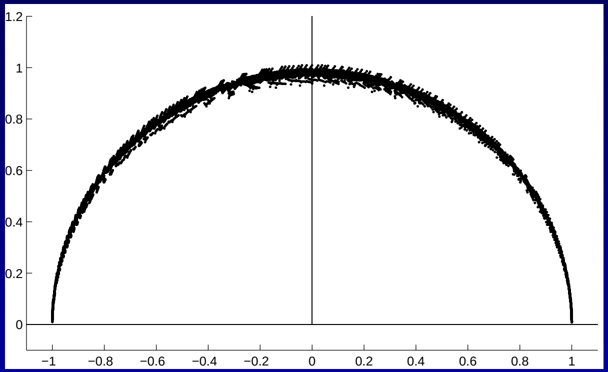
Proof   Similar to previous proofs. $\square$

Example Jacobi weight $w^{(\alpha,\beta)}$, $\alpha, \beta \in [0, \frac{5}{2})$
Peherstorfer and Petras (2003)





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$$\alpha, \beta = 0(.4)2,\ \beta \geq \alpha$$

Connection with potential theory

density of the equilibrium measure $\omega_{[-1,1]}$ of the interval $[-1, 1]$

$$\omega'_{[-1,1]}(t) = \frac{1}{\pi} \frac{1}{\sqrt{1-t^2}}\,, \quad 0 < t < 1$$

Christoffel function and equilibrium measure

$$\frac{n\lambda_n(t;w)}{w(t)} \sim \frac{1}{\omega'_{[-1,1]}(t)}, \quad n \to \infty$$

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In general, for $E \subset \mathbb{R}$ a regular set and $\Delta \subset E$ a compact set on which w satisfies the Szegö condition (Totik, 2000),

$$\frac{n\lambda_\nu^G}{w(\tau_\nu^G)} \sim \frac{1}{\omega'_E(\tau_\nu^G)} \quad \text{on } \Delta$$

Example weight function supported on two intervals

$$w(t) = \begin{cases} |t|^\gamma (t^2 - \xi^2)^p (1-t^2)^q, & t \in [-1,-\xi] \cup [\xi, 1], \\ 0 \quad \text{elsewhere}, & \end{cases}$$

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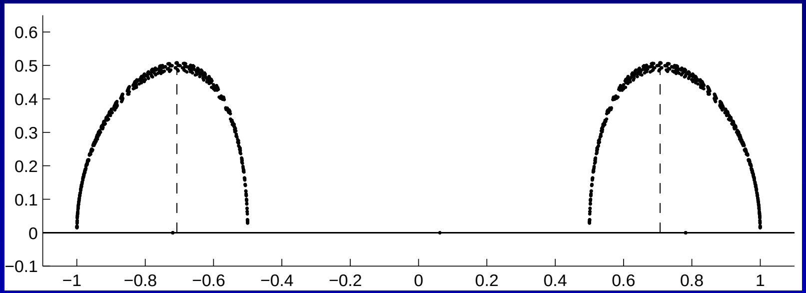
plot (for $\xi = \frac{1}{2}$ and $p = q = \pm\frac{1}{2}$, $\gamma = \pm 1$)



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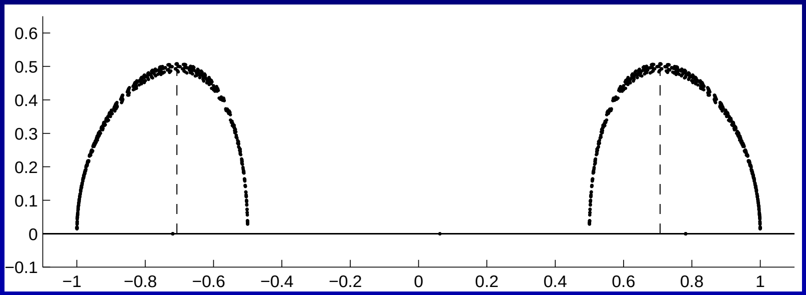

$$n = 60(5)80$$

density of equilibrium measure

$$\omega'_{[-1,-\xi]\cup[\xi,1]}(t) = \pi^{-1}|t|(t^2 - \xi^2)^{-1/2}(1 - t^2)^{-1/2}$$