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The spiral of Theodorus, numerical analysis, and special functions☆

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ABSTRACT

Theodorus of Cyrene (ca. 460–399 B.C.), teacher of Plato und Theaetetus, is known for his proof of the irrationality of $\sqrt{n}$, $n = 2, 3, 5, \ldots, 17$. He may have known also of a discrete spiral, today named after him, whose construction is based on the square roots of the numbers $n = 1, 2, 3, \ldots$. The subject of this lecture is the problem of interpolating this discrete, angular spiral by a smooth, if possible analytic, spiral. An interesting solution was proposed in 1993 by P.J. Davis, which is based on an infinite product. The computation of this product gives rise to problems of numerical analysis, in particular the summation of slowly convergent series, and the identification of the product raises questions regarding special functions. The former are solved by a method of integration, in particular Gaussian integration, the latter by means of Dawson's integral und the Bose–Einstein distribution. Number-theoretic questions also loom behind this work.

1. Introduction

The topic of this lecture may be somewhat peripheral to the core of today's mathematical activities, yet it has a certain aesthetic appeal that may well compensate for its borderline status. The principal ideas go back to Greek antiquity, specifically to the 5th century B.C. mathematician and philosopher Theodorus of Cyrene (ca. 460–399 B.C.). He was born and grew up in Cyrene, then a sprawling Greek colony at the Northern coast of Africa (in what today is Libya), directly south of Greece. He also traveled to Athens, where he encountered Socrates. Not much, however, is known about his life and work. From the writings of Plato, who had been a student of Theodorus, in particular from his *Theaetetus*, we know about Theodorus's great fascination with questions of incommensurability. He was to have proved, for example, the irrationality of the square roots of the integers $n = 2, 3, 5, 6, 7, \ldots$, and, so Plato writes, for some reason he stopped at $n = 17$. This cryptic remark has given rise to all sorts of speculation as to what the reasons might have been. One of these, probably the least credible, will be mentioned later.

But let me first introduce the three topics mentioned in the title. First, the *spiral of Theodorus*, depicted in Fig. 1—a harmonious, very pleasing, and elegant spiral. The name "spiral of Theodorus", though, may be misleading, since Theodorus most certainly did not know of this spiral; it is a product of the late 20th century! Very likely, however, he was aware of, or even invented, a more primitive, angular precursor of this spiral, which we will call the "discrete spiral of Theodorus" (cf. Section 2) to distinguish it from the spiral in Fig. 1, which may be called the "analytic spiral of Theodorus".

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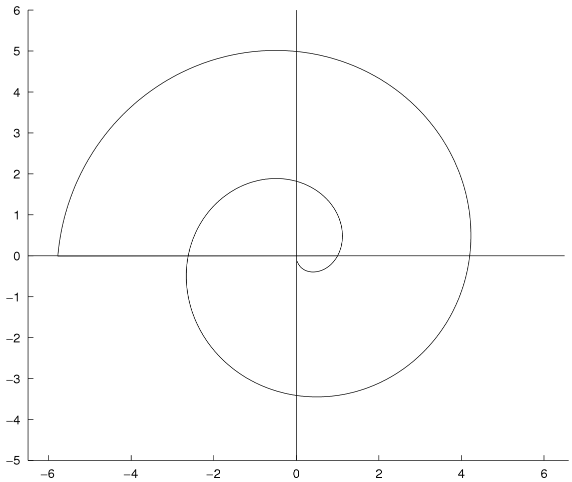

Fig. 1. Spiral of Theodorus.

The second topic – *numerical analysis* – has to do with the summation of slowly convergent series, in particular the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+k^{1/2}}. \tag{1}$$

It was in fact this series that gave the impetus to my interest in this area. I was visiting Brown University, early in the 1990s, where I was to give a colloquium lecture. Before the talk, I dropped by Prof. Philip Davis's office to chat a little about the newest mathematical gossip. I knew Prof. Davis well from our days at the (what was then called) National Bureau of Standards in Washington, DC. At one point during our conversation, he pulled out a crumpled envelope from his waste basket, scribbled the series (1) on the back of the envelope, and handed it to me with the words "compute it!". I responded that I couldn't do it on the spot, but promised to look at it once I was back home. (I already had an idea of how to go about it.) A few days later, I sent him back my answer,

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+k^{1/2}} = 1.860025079221190307180695915717143324666524121523451493049199503\ldots \tag{2}$$

if not to sixty-four digits, then at least to fifteen (or maybe twenty). This must have impressed Prof. Davis enough to let me in on what was behind this series, and what he was working on at the time: preparing for the Hedrick Lectures he was to give at the 75th anniversary meeting of the Mathematical Association of America. The theme of these lectures was spirals, not only those in mathematics, but also spirals as they occur in nature, in celestial mechanics, and elsewhere. An expanded version of these lectures later appeared in book form [1].

The third topic – *special functions* – finally involves Dawson's integral

$$F(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t, \tag{3}$$

probably better known with the opposite signs in the exponents of the exponentials, which then becomes the familiar Gaussian error function.

The theme of this lecture is to show how these three seemingly disparate topics hang together.

2. The discrete spiral of Theodorus

As is well known, in the mathematics of Greek antiquity, numbers and algebraic expressions were thought of differently than they are today. A number like 3 was viewed not so much as a numerical value but as a geometric object: a straight line that has three units in length. Likewise, $\sqrt{2}$ was viewed as the length of the diagonal of a unit square. Since Theodorus was concerned with square roots of successive numbers, he must have viewed them also in geometric terms. Almost inevitably, then, he must have arrived at the construction indicated in Fig. 2. Here, the points $T_0, T_1, T_2, \ldots$ ("T" for "Theodorus") are constructed as follows: T_0 is the origin, and T_1 on the real axis a distance of 1 away from T_0. Thus, the distance $|T_1T_0|$ is $1 = \sqrt{1}$. From T_1 one proceeds in a perpendicular upward direction a distance of 1 to T_2, so $|T_2T_0| = \sqrt{2}$. Then again, perpendicularly, one proceeds a distance of 1 to T_3 and has $|T_3T_0| = \sqrt{2+1} = \sqrt{3}$. Continuing in this manner, the points $T_4, T_5, T_6, \ldots$ so obtained have distances from the origin that are $|T_nT_0| = \sqrt{n}$, $n = 4, 5, 6, \ldots$. One can therefore interpret the successive square roots $\sqrt{n}$ geometrically as being the radial distances of the vertices T_n of the spiral-like construct of

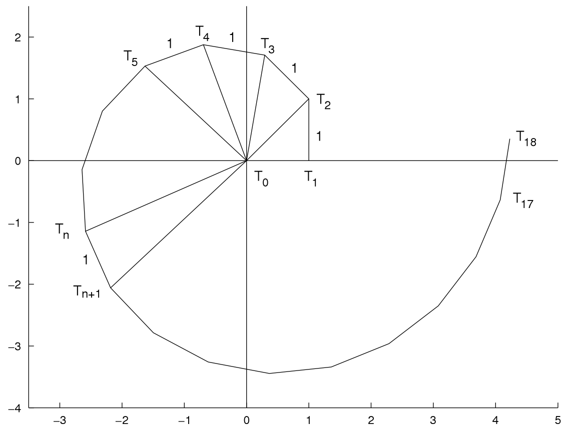

Fig. 2. Discrete spiral of Theodorus.

Fig. 2. It is natural to call it the *discrete spiral of Theodorus*. (It is also known as the "Quadratwurzelschnecke", a term given it by Hlawka in [2].)

It is convenient to view this spiral as a curve in the complex plane, represented parametrically by a complex-valued function

$$T(\alpha) \in \mathbb{C}, \quad \alpha \geq 0.$$

We want this function for integer values of the parameter to produce the vertices of the spiral, $T(n) = T_n$, $n = 0, 1, 2, \ldots$. These are uniquely defined by the relations

$$\left.\begin{array}{l} |T_n| = \sqrt{n} \\ |T_{n+1} - T_n| = 1 \end{array}\right\} \quad n = 0, 1, 2, \ldots \tag{4}$$

with $T_1 = 1$. Linear interpolation between integer-valued parameters then defines $T(\alpha)$ for all $\alpha \geq 0$.

Why did Theodorus stop at $n = 17$? The graph in Fig. 2 gives a clue: The line from T_{17} to T_0, whose length is $\sqrt{17}$, can be drawn without any obstruction. Not so for the line from T_{18} to T_0, and all subsequent lines, which intersect part of the figure already drawn. Since legend has it that geometers in antiquity drew their lines in sand, such intersections become messy, and that's why Theodorus stopped at 17. As I indicated before, *se non é vero, é ben trovato*! ["If it's not true, it's a good story!"]

3. The analytic spiral of Theodorus

3.1. Definition and properties

Davis in [1] posed the problem of interpolating the discrete Theodorus spiral by a smooth, if possible analytic, curve. This is an interpolation problem involving an infinite number of data points, a problem of the type Euler already faced in 1729 when he tried to interpolate the successive factorials on the real line. Ingeniously, Euler discovered the gamma function (now also called the second Eulerian integral) as a valid analytic interpolant. In addition, he derived a number of properties of the gamma function involving product representations, including an infinite product formula for the reciprocal of the gamma function. Davis, who knew Euler's work very well (cf. [3]), used it as a source of inspiration and came up with an interpolant, also expressed as an infinite product,

$$T(\alpha) = \prod_{k=1}^{\infty} \frac{1 + \mathrm{i}/\sqrt{k}}{1 + \mathrm{i}/\sqrt{k + \alpha - 1}}, \quad \alpha \geq 0. \tag{5}$$

Since the general term of the product is $\sim 1 + k^{-3/2}$ as $k \to \infty$, and the series $\sum_{k=1}^{\infty} k^{-3/2}$ converges (absolutely, though slowly), the same is true for the infinite product, as follows from well-known theorems.

Simple calculations will show that the function in (5) satisfies (cf. also (12))

$$|T(\alpha)| = \sqrt{\alpha} \tag{6}$$

and the first-order difference equation

$$T(\alpha + 1) = \left(1 + \frac{\mathrm{i}}{\sqrt{\alpha}}\right) T(\alpha). \tag{7}$$

As a consequence of (6) and (7) one also has

$$|T(\alpha+1)-T(\alpha)| = \left|\frac{\mathrm{i}}{\sqrt{\alpha}}T(\alpha)\right| = |\mathrm{i}| = 1. \tag{8}$$

The relations (6) and (8), for integer values $\alpha = n$, coincide exactly with the analogous relations (4) for the discrete spiral of Theodorus, and since $T(1) = 1$, the function $T(\alpha)$ does indeed interpolate the discrete spiral of Theodorus.

The arc $T(\alpha)$, $1 \le \alpha < 2$, may be considered the "heart" of the spiral; it completely determines the entire spiral, the infinite outer part corresponding to $2 \le \alpha < \infty$ by repeated forward application of (7), and the inner part corresponding to $0 < \alpha < 1$ by a backward application of (7). In the limit as $\alpha \downarrow 0$, one gets $T(0) = 0$.

Recall that Euler's gamma function also satisfies a first-order difference equation, the much simpler $y(\alpha+1) = \alpha y(\alpha)$. Harold Bohr and Johannes Mollerup in 1921 proved the beautiful result that this difference equation has no other solution, with $y(1) = 1$, than the gamma function, if one requires it to be logarithmically convex; cf. [4]. Davis posed the question of whether his own function $T(\alpha)$ in (5), as a solution of the difference equation (7), has a similar uniqueness property. This was answered in 2004 by Gronau [5], who proved, among other things, that $T(\alpha)$ is the only solution of the difference equation (7) with $T(1) = 1$, if one requires $|T(\alpha)|$ to be monotonic and $\arg T(\alpha)$ monotonic and continuous. In the same way as the Bohr–Mollerup result reinforces the legitimacy and importance of the gamma function, the Gronau result does the same for Davis's function.

3.2. Some number theory

An interesting number-theoretic question regards the distribution of the angles $\varphi_n = \angle T_1T_0T_{n+1}$ in the discrete spiral of Theodorus. From the geometry of Fig. 2, it is easily seen that

$$\varphi_n = \sum_{k=1}^{n} \sin^{-1}\frac{1}{\sqrt{k+1}}, \quad n = 1, 2, 3, \ldots. \tag{9}$$

Considering $\varphi_n \bmod 2\pi$, Hlawka in [2] proved that the sequence $\{\varphi_n\}_{n=1}^{\infty}$ is equidistributed mod 2π. In his book [6], Hlawka gives a very elegant proof based on an analytic equidistribution criterion of Fejér (cf. [7, Part II, Probl. 174, p. 281] and [8, pp. 843–844]).

The author, when preparing this lecture, wondered whether a similar equidistribution result holds for the angles $\varphi_n(\alpha) = \angle T(\alpha)T_0T(\alpha+n)$, $1 < \alpha < 2$, in Davis's analytic spiral of Theodorus. These are, from (13), $\varphi_n(\alpha) = \varphi(\alpha+n)-\varphi(\alpha)$, and by analogy with the discrete spiral one suspects that

$$\varphi_n(\alpha) = \sum_{k=1}^{n} \sin^{-1}\frac{1}{\sqrt{k+\alpha}}, \quad n = 1, 2, 3, \ldots, \tag{10}$$

which for $\alpha = 1$ in fact reduces to (9). We shall prove (10) in Section 3.3. The answer to the question of equidistribution was provided by Harald Niederreiter, a former Ph.D. student of Hlawka, and communicated to the author by email on February 3, 2009: The sequence $\{\varphi_n(\alpha)\}_{n=1}^{\infty}$ is indeed also equidistributed mod 2π for any fixed α with $1 < \alpha < 2$ (in fact, for any $\alpha > 0$), and the proof is a simple extension of the proof given by Hlawka in [6].

3.3. Polar representation

When dealing with spirals, it is useful to have a polar representation thereof. For the spiral in Fig. 1, this can be nicely obtained by logarithmic differentiation of $T(\alpha)$. Since $T(\alpha)$ is a product, its logarithmic derivative is the sum of the logarithmic derivatives of the factors,

$$\begin{aligned}
\frac{T'(\alpha)}{T(\alpha)} &= \sum_{k=1}^{\infty} \frac{1+\mathrm{i}/\sqrt{k+\alpha-1}}{1+\mathrm{i}/\sqrt{k}}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1+\mathrm{i}/\sqrt{k}}{1+\mathrm{i}/\sqrt{k+\alpha-1}}\right)\\
&= \sum_{k=1}^{\infty}(1+\mathrm{i}/\sqrt{k+\alpha-1})\frac{\mathrm{i}}{2}\frac{(k+\alpha-1)^{-3/2}}{(1+\mathrm{i}/\sqrt{k+\alpha-1})^2}\\
&= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)(\sqrt{k+\alpha-1}+\mathrm{i})}\\
&= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\frac{\sqrt{k+\alpha-1}-\mathrm{i}}{(k+\alpha-1)(k+\alpha)}.
\end{aligned}$$

Decomposing the last series into its real and imaginary parts yields

$$\begin{aligned}\frac{T'(\alpha)}{T(\alpha)} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)(k+\alpha)} + \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)^{3/2}+(k+\alpha-1)^{1/2}}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k+\alpha-1}-\frac{1}{k+\alpha}\right)+\frac{\mathrm{i}}{2}U(\alpha)\\ &= \frac{1}{2\alpha}+\frac{\mathrm{i}}{2}U(\alpha),\end{aligned}$$

where

$$U(\alpha) = \sum_{k=1}^{\infty}\frac{1}{(k+\alpha-1)^{3/2}+(k+\alpha-1)^{1/2}}. \tag{11}$$

Now integrating from 1 to α gives

$$\ln T(\alpha) = \ln(\alpha^{1/2}) + \frac{\mathrm{i}}{2}\int_1^{\alpha} U(\alpha)\mathrm{d}\alpha,$$

which by exponentiation yields the desired representation,

$$T(\alpha) = \sqrt{\alpha}\exp\left(\frac{\mathrm{i}}{2}\int_1^{\alpha}U(\alpha)\mathrm{d}\alpha\right), \quad \alpha > 0. \tag{12}$$

Thus, in polar coordinates (r,φ), the analytic spiral of Theodorus has the parametric representation

$$r = r(\alpha), \qquad \varphi = \varphi(\alpha) \quad \text{where } r(\alpha) = \sqrt{\alpha}, \qquad \varphi(\alpha) = \frac{1}{2}\int_1^{\alpha}U(\alpha)\mathrm{d}\alpha. \tag{13}$$

In terms of this representation, we can rewrite (10) (multiplied by 2) as follows:

$$\int_1^{\alpha+n}U(\alpha)\mathrm{d}\alpha - \int_1^{\alpha}U(\alpha)\mathrm{d}\alpha = 2\sum_{k=1}^{n}\sin^{-1}\frac{1}{\sqrt{k+\alpha}}.$$

We know this to be true for $\alpha = 1$. To prove it for general α, it suffices to prove that the derivatives with respect to α of the two sides are equal,

$$U(\alpha+n) - U(\alpha) = -\sum_{k=1}^{n}\frac{1}{(k+\alpha)\sqrt{k+\alpha-1}}.$$

This, however, follows readily from the definition of U in (11).

We note that the tangent vector to the spiral at $\alpha = 1$ is $T'(1) = \frac{1}{2} + \frac{\mathrm{i}}{2}U(1)$, so

$$U(1) = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}+k^{1/2}}$$

is precisely the slope of the tangent vector to the spiral at $\alpha = 1$ where it crosses the real axis for the first time.

We have come halfway to the mysterious series introduced at the beginning of this lecture. As a universal constant, like π, with a solid geometric meaning, it deserves to be given a name, and to be calculated to high precision; we name it, as Davis already did in [1], the "Theodorus constant", and denote it by

$$\theta = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}+k^{1/2}} \tag{14}$$

("θ" for "$\theta\epsilon\omega\delta o\rho o\sigma$").

There is, of course, another number-theoretic problem awaiting attention: the arithmetic nature of the number θ. A solution, however, seems far beyond sight at this time.

We now proceed to the next topic on our agenda, the computation and identification of the function $U(\alpha)$ in (11) and its integral $\int_1^{\alpha}U(\alpha)\mathrm{d}\alpha$ for $1 < \alpha < 2$. This requires two digressions, one on an appropriate summation procedure, the other on Gaussian quadrature.

4. Two digressions

4.1. Summation by integration

There are several ways to convert a problem of summation, especially the summation of slowly convergent series, to a problem of integration. Here we consider a procedure proposed in 1985 in a joint paper with Milovanović [9] that applies to a special class of series in which the generic term is the Laplace transform of some known function f,

$$s=\sum_{k=1}^{\infty}a_k,\quad a_k=(\mathcal{L}f)(k). \tag{15}$$

Then

$$s=\sum_{k=1}^{\infty}(\mathcal{L}f)(k)=\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{e}^{-kt}f(t)\mathrm{d}t,$$

and interchanging summation and integration yields

$$s=\int_0^{\infty}\left(\sum_{k=1}^{\infty}\mathrm{e}^{-kt}\right)f(t)\mathrm{d}t=\int_0^{\infty}\frac{t}{\mathrm{e}^t-1}\frac{f(t)}{t}\mathrm{d}t.$$

Thus

$$\sum_{k=1}^{\infty}a_k=\int_0^{\infty}\frac{f(t)}{t}\varepsilon(t)\mathrm{d}t,\quad f=\mathcal{L}^{-1}a, \tag{16}$$

where

$$\varepsilon(t)=\frac{t}{\mathrm{e}^t-1},\quad t\in\mathbb{R}_+. \tag{17}$$

In statistical mechanics, (17) is known as the Bose–Einstein distribution; it is also the generating function of the Bernoulli numbers.

Changing the minus sign in the denominator of (17) to a plus sign and replacing t in the numerator by 1 gives another distribution important in statistical mechanics: the Fermi–Dirac distribution. In our context it arises when the series in (15) contains alternating sign factors.

How does this apply to the Theodorus constant? Here,

$$a_k=\frac{1}{k^{3/2}+k^{1/2}}=\frac{k^{-1/2}}{k+1}.$$

Since

$$k^{-1/2}=\left(\mathcal{L}\frac{t^{-1/2}}{\sqrt{\pi}}\right)(k),\quad \frac{1}{k+1}=\left(\mathcal{L}\mathrm{e}^{-t}\right)(k),$$

the convolution theorem for Laplace transforms yields

$$a_k=\left(\mathcal{L}\frac{t^{-1/2}}{\sqrt{\pi}}\right)(k)\cdot\left(\mathcal{L}\mathrm{e}^{-t}\right)(k)=\left(\mathcal{L}\frac{1}{\sqrt{\pi}}\int_0^t\tau^{-1/2}\mathrm{e}^{-(t-\tau)}\mathrm{d}\tau\right)(k), \tag{18}$$

where the integral on the right is the convolution of $t^{-1/2}$ and e^{-t}. Thus,

$$f(t)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-t}\int_0^t\tau^{-1/2}\mathrm{e}^{\tau}\mathrm{d}\tau=\frac{2}{\sqrt{\pi}}\mathrm{e}^{-t}\int_0^{\sqrt{t}}\mathrm{e}^{x^2}\mathrm{d}x=\frac{2}{\sqrt{\pi}}F(\sqrt{t}),$$

where $F(x)$ is Dawson's integral (3). There follows, from (16),

$$\sum_{k=1}^{\infty}\frac{1}{k^{3/2}+k^{1/2}}=\int_0^{\infty}\frac{f(t)}{t}\varepsilon(t)\mathrm{d}t.$$

By writing $t=\sqrt{t}\cdot\sqrt{t}$ in the denominator of the integrand and associating one square root with f and the other with ε, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{k^{3/2}+k^{1/2}}=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{F(\sqrt{t})}{\sqrt{t}}w(t)\mathrm{d}t, \tag{19}$$

where

$$w(t) = t^{-1/2}\varepsilon(t) = \frac{t^{1/2}}{e^t - 1}. \tag{20}$$

We recall that $F(x)$ is an entire, odd function of x; hence $F(\sqrt{t})/\sqrt{t}$ in (19) is a power series in t that converges in the whole complex plane, and hence in turn an entire function. For the purpose of numerical integration, entire functions are usually conducive to rapid convergence; hence the first factor, $F(\sqrt{t})/\sqrt{t}$, in the integrand of (19) is a nice, benign function. The second factor, $w(t)$, though positive on $\mathbb{R}_+$, is difficult: for one thing, it blows up like $t^{-1/2}$ at $t = 0$, and for another, it has an infinite string of poles on the imaginary axis at the integer multiples of 2πi. Both are troublesome for numerical integration. But in numerical analysis one knows of an effective approach for integrating such a product: one treats the difficult factor as a weight function and applies weighted numerical integration, for example, Gaussian quadrature.

4.2. Gaussian quadrature

An n-point Gaussian quadrature formula for an integral as in (19) is a relation

$$\int_0^\infty g(t)w(t)\mathrm{d}t = \sum_{k=1}^{n} \lambda_k^{(n)} g(\tau_k^{(n)}), \quad g \in \mathbb{P}_{2n-1}, \tag{21}$$

which expresses the integral exactly as a linear combination of n function values provided the function is a polynomial of degree $\le 2n-1$. It is known that such a representation exists uniquely, and that the "weights" $\lambda_k^{(n)}$ are positive (if w is positive) and the "nodes" $\tau_k^{(n)}$ are mutually distinct and contained in the open interval $(0, \infty)$. If g is not a polynomial, but is polynomial-like, for example an entire function as in (19), then (21) will no longer be an exact equality but very likely a good approximation, especially if n is large.

But how do we find the weights $\lambda_k^{(n)}$ and nodes $\tau_k^{(n)}$ for any given n? The answer is well known in principle: we need the orthogonal polynomials with respect to the weight function w, that is, the (monic) polynomials π_k of degree k, $k = 0, 1, 2, \ldots$, satisfying

$$(\pi_k, \pi_\ell) = 0, \quad k \ne \ell, \quad \text{where } (u, v) = \int_0^\infty u(t)v(t)w(t)\mathrm{d}t.$$

It is known that they exist uniquely and satisfy a three-term recurrence relation

$$\begin{aligned}&\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, 2, \ldots,\\ &\pi_{-1}(t) = 0, \qquad \pi_0(t) = 1,\end{aligned}$$

where the coefficients $\alpha_k = \alpha_k(w)$ and $\beta_k = \beta_k(w)$ are respectively real and positive numbers depending on w. Although β_0 is arbitrary, it is convenient to define $\beta_0 = \int_0^\infty w(t)\mathrm{d}t$. The nth-order Jacobi matrix

$$\boldsymbol{J}_n(w) = \begin{bmatrix} \alpha_0 & \beta_1 & & & \mathbf{0} \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \beta_2 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ \mathbf{0} & & & \beta_{n-1} & \alpha_{n-1} \end{bmatrix} \tag{22}$$

is formed by placing the first n coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ on the diagonal, the $n-1$ coefficients $\beta_1, \beta_2, \ldots, \beta_{n-1}$ on the two side diagonals, and filling the rest of the matrix with zeros. It is the eigenvalues and eigenvectors of this symmetric, tridiagonal matrix that yield the Gaussian nodes and weights: the nodes $\tau_k^{(n)}$ are the eigenvalues of $\boldsymbol{J}_n$, and the weights $\lambda_k^{(n)}$ expressible as $\lambda_k^{(n)} = \beta_0 \boldsymbol{v}_{k,1}^2$ in terms of the first components $\boldsymbol{v}_{k,1}$ of the corresponding (normalized) eigenvectors $\boldsymbol{v}_k$ [10].

We are done, once we are in possession of the recurrence coefficients α_k, β_k. There are various numerical techniques for computing them (cf., for example, [11, Sections 2.1, 2.2]). For our purposes here, the classical approach based on moments

$$\mu_k = \int_0^\infty t^k w(t)\mathrm{d}t, \quad k = 0, 1, 2, \ldots, \tag{23}$$

suffices. An algorithm due to Chebyshev takes the first $2n$ moments (23), and from them generates the first n coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ and $\beta_0, \beta_1, \ldots, \beta_{n-1}$ by a simple nonlinear recursion. The algorithm is elegant but highly unstable, the more so the larger n. This drawback, however, can be overcome by running the algorithm in sufficiently high precision. Relevant software is available; see, e.g., [12].

To show how this works for the Theodorus constant, we first note that the moments of the weight function w in (20) are

$$\mu_k = \int_0^\infty \frac{t^{k+1/2}}{e^t - 1}\mathrm{d}t = \Gamma(k + 3/2)\zeta(k + 3/2).$$

Table 1
Gaussian quadrature approximations to the Theodorus constant.

n	s_n
5	1.85997...
15	1.86002507922117...
25	1.86002507922119030718068 9...
35	1.86002507922119030718069591571714 1...
45	1.860025079221190307180695915717143324666523 5...
55	1.86002507922119030718069591571714332466652412152345 13...
65	1.86002507922119030718069591571714332466652412152345149304919944...
75	1.86002507922119030718069591571714332466652412152345149304919950 3...

Both the gamma function Γ and the Riemann zeta function ζ are computable by variable-precision calculation. Applying the Chebyshev algorithm in sufficiently high precision to get the Jacobi matrix (22), and then well-known eigenvalue/eigenvector techniques to get the Gaussian quadrature formula, we can now approximate

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+k^{1/2}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} [F(\sqrt{t})/\sqrt{t}]w(t)\mathrm{d}t \tag{24}$$

by

$$s_n = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{n} \lambda_k^{(n)} F\left(\sqrt{\tau_k^{(n)}}\right) \Big/ \sqrt{\tau_k^{(n)}}.$$

Numerical results for $n = 5 : 10 : 75$ are shown in Table 1. We see now how the answer given in (2) comes about. Allowing for a sufficient amount of computer time, we could obtain it to an arbitrary number of decimal digits. Faster high-precision computational techniques, however, can be found in [13].

5. Computation and identification

We are now in a position to deal with the computation of $U(\alpha)$ (cf. (11)) and $\int_1^{\alpha} U(\alpha)\mathrm{d}\alpha$ for $1 < \alpha < 2$. The series in (11) is the same as the series (14) for the Theodorus constant except that k in the latter has to be replaced by $k + \alpha - 1$. From the computation in (18), we thus find that

$$\frac{(k+\alpha-1)^{-1/2}}{(k+\alpha-1)+1} = \frac{1}{\sqrt{\pi}} \left(\mathcal{L} \int_0^t \tau^{-1/2} \mathrm{e}^{-(t-\tau)} \mathrm{d}\tau\right)(k+\alpha-1).$$

It is now a matter of applying the shift property of the Laplace transform to obtain

$$\frac{(k+\alpha-1)^{-1/2}}{(k+\alpha-1)+1} = \frac{1}{\sqrt{\pi}} \mathcal{L}\left(\mathrm{e}^{-\alpha t} \int_0^t \tau^{-1/2} \mathrm{e}^{\tau} \mathrm{d}\tau\right)(k);$$

hence, the function f in (15) is

$$f(t) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\alpha t} \int_0^t \tau^{-1/2} \mathrm{e}^{\tau} \mathrm{d}\tau = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-(\alpha-1)t} F(\sqrt{t}).$$

We find, analogously to (19),

$$U(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \mathrm{e}^{-(\alpha-1)t} \frac{F(\sqrt{t})}{\sqrt{t}} w(t)\mathrm{d}t, \quad 1 < \alpha < 2. \tag{25}$$

This again can be readily computed by Gauss quadrature, just like (24). We note, incidentally, that $U(\alpha)$ can be identified as a Laplace transform itself, namely

$$U(\alpha) = (\mathcal{L}u)(\alpha - 1),$$

where

$$u(t) = \frac{2}{\sqrt{\pi}} \frac{F(\sqrt{t})}{\sqrt{t}} w(t) = \frac{2}{\sqrt{\pi}} \frac{F(\sqrt{t})}{\mathrm{e}^t - 1}.$$

As far as the integral of $U(\alpha)$ is concerned, we only need to integrate under the integral sign in (25) to obtain

$$\int_1^{\alpha} U(\alpha)\mathrm{d}\alpha = \frac{2(\alpha-1)}{\sqrt{\pi}} \int_0^{\infty} \frac{1-\mathrm{e}^{-(\alpha-1)t}}{(\alpha-1)t} \frac{F(\sqrt{t})}{\sqrt{t}} w(t)\mathrm{d}t. \tag{26}$$

This, too, is amenable to Gauss quadrature but requires a little extra care in the evaluation near $t = 0$ of the first factor on the right.

6. Epilogue

6.1. The Theodorus constant to very high precision

Waldvogel [13] has calculated the Theodorus constant θ to over a thousand decimal places, using a line integral representation in the complex plane, the trapezoidal rule, and the computer algebra system PARI. With the same package, at the suggestion of N. A'Campo, he computed the continued fraction

$$\theta = 1 + \frac{1}{1+}\frac{1}{6+}\frac{1}{6+}\frac{1}{1+}\frac{1}{15+}\frac{1}{11+}\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{5+}\cdots$$

to some 300 partial denominators to look for patterns. None were found.

6.2. Summation by integration; extensions

The summation process of Section 4.1 can be generalized to series

$$s_+ = \sum_{k=1}^{\infty} k^{\nu-1}R(k), \qquad s_- = \sum_{k=1}^{\infty}(-1)^k k^{\nu-1}R(k), \quad 0 < \nu < 1,$$

where R is a rational function having all its poles in the left half of the complex plane (cf. [14]). The integration measures that arise are, as in Section 4.1, the Bose–Einstein distribution for the series s_+ and the Fermi–Dirac distribution for the series s_-. The special functions involved, however, are more elaborate, being based on Tricomi's form of the incomplete gamma function. Also, there are serious complications that arise when the poles of R are large in magnitude, in which case Gaussian quadrature converges very slowly. Satisfactory convergence can be restored by a process called "stratified summation" in [14].

6.3. The analytic spiral of Theodorus; an alternative approach

Heuvers et al. [15], apparently unaware of Davis's work, gave the following analytic interpolant of the discrete Theodorus spiral, expressed in polar coordinates:

$$\varphi = g(r), \qquad g(r) = \sum_{j=0}^{\infty}\left(\tan^{-1}\frac{1}{\sqrt{j+1}} - \tan^{-1}\frac{1}{\sqrt{j+r^2}}\right), \quad r \geq 1. \tag{27}$$

They proved that $g(r)$ in (27) is the unique monotonically increasing solution, satisfying $g(1) = 0$, of the functional equation

$$g\left(\sqrt{1+r^2}\right) - g(r) = \tan^{-1}\frac{1}{r}, \quad r \geq 1, \tag{28}$$

thus anticipating Gronau's uniqueness result.

The connection of (27) and (28) with Davis's spiral is as follows. The angle φ in Davis's spiral, as a function of r, can be seen from (13), since $\alpha = r^2$, to be

$$\varphi = \frac{1}{2}\int_1^{r^2} U(\alpha)\mathrm{d}\alpha, \tag{29}$$

which is identical to (27). In fact, when $r = 1$, this is obvious, and differentiating with respect to r we get $rU(r^2)$ from (29) and

$$-\sum_{j=0}^{\infty}\frac{1}{1+(j+r^2)^{-1}}\left(-\frac{1}{2}\right)(j+r^2)^{-3/2}2r = r\sum_{j=0}^{\infty}\frac{1}{(j+r^2)^{3/2}+(j+r^2)^{1/2}},$$

from (27), which by (11) is indeed $rU(r^2)$. On writing

$$1 + \frac{\mathrm{i}}{\sqrt{\alpha}} = \sqrt{\frac{\alpha+1}{\alpha}}\mathrm{e}^{\mathrm{i}\theta(\alpha)}, \quad \theta(\alpha) = \tan^{-1}\frac{1}{\sqrt{\alpha}},$$

the difference equation (7) splits into

$$\frac{r(\alpha+1)}{r(\alpha)} = \sqrt{\frac{\alpha+1}{\alpha}}, \qquad \varphi(\alpha+1) = \varphi(\alpha) + \tan^{-1}\frac{1}{\sqrt{\alpha}},$$

the latter, on setting $\varphi(r^2) = g(r)$, becoming (28).

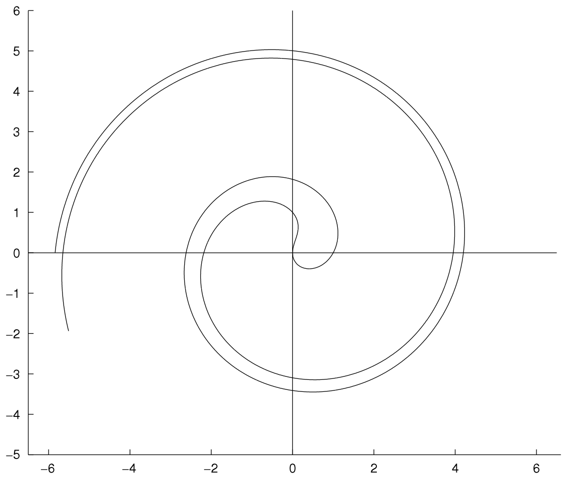

Fig. 3. Twin-spiral of Theodorus.

6.4. Analytic continuation of the spiral of Theodorus

For complex α, Davis's function $T(\alpha)$ in (5) is multivalued, owing to the square roots in the denominator. A useful "regularizing" transformation,

$$\alpha = r^2, \quad r \in \mathbb{R}, \tag{30}$$

is suggested by Waldvogel in [13]. It has the effect of transforming $T(\alpha)$ into a function

$$T(r^2) = \frac{1+\mathrm{i}}{1+\mathrm{i}/r} \prod_{k=2}^{\infty} \frac{1+\mathrm{i}/\sqrt{k}}{1+\mathrm{i}/\sqrt{r^2+k-1}} \tag{31}$$

that is regular analytic in the complex r-plane cut along the lines from i to $\mathrm{i}\infty$ and $-\mathrm{i}$ to $-\mathrm{i}\infty$ on the imaginary axis. The part of (31) corresponding to positive values of r coincides with the spiral shown in Fig. 1, whereas the part corresponding to negative values of r may be considered the analytic continuation of the spiral into the second sheet of the Riemann surface for the square root. Both parts together, shown in Fig. 3, constitute what may be called the "twin-spiral of Theodorus".

If $T(\alpha)$, $\alpha > 0$, is on the original spiral (5), then

$$S(\alpha) = \frac{1+\mathrm{i}/\sqrt{\alpha}}{1-\mathrm{i}/\sqrt{\alpha}} T(\alpha)$$

is the corresponding point on the twin branch of the spiral. Therefore, by (7),

$$S(\alpha) = \frac{1}{1-\mathrm{i}/\sqrt{\alpha}} T(\alpha+1), \tag{32}$$

whereas

$$T(\alpha) = \frac{1}{1+\mathrm{i}/\sqrt{\alpha}} T(\alpha+1), \tag{33}$$

showing that the two points in (32) and (33) are mirror images with respect to the line $T_0T(\alpha+1)$. In the special case $\alpha = n^2$, $n > 0$ an integer, i.e., in the case of the discrete Theodorus spiral, this was observed in [13].

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