1. Collaborative efforts, and joint publications resulting therefrom, are much more prevalent in the physical sciences than they are in mathematics. The reason is that research in the physical sciences usually requires teamwork involving a number of scientists with specialized skills, whereas research in mathematics is a much more individual and solitary enterprise. Nevertheless, even in mathematics, collaboration between different mathematicians may come about through a variety of circumstances. In my own experience, most of my collaboration originated in my attending mathematical conferences, visiting other institutions, or entertaining guests at my own institution. Another not insignificant group of collaborators comes from Ph.D. or postdoctoral students. In all these cases, an important aspect is interpersonal communication and oral exchange of ideas. Not so in the case of Gradimir! Here, collaboration started anonymously, almost ghostlike, during a process of refereeing, exactly 25 years ago. (I may be permitted to divulge information that normally is held confidential!) That is when I received a manuscript from the editor of Mathematics of Computation authored by some Gradimir Milovanović, a name I had never heard of before. I was asked to referee it for the journal, whose editor-in-chief I was to become shortly thereafter.

2. The topic of the manuscript looked interesting enough: It was a matter of computing integrals that frequently occur in solid state physics, e.g. the total energy of thermal vibration of a crystal lattice, which is expressible as an integral

$$\int_0^\infty f(t) \frac{t}{e^t - 1} \, dt,$$

(1)
where \( f(t) \) is related to the phonon density of states, or the crystal lattice heat capacity at constant volume, which is

\[
\int_0^\infty g(t) \left( \frac{t}{e^t - 1} \right)^2 dt,
\]

with \( g(t) = e^t f(t) \). Gradimir's idea was to compute these integrals, and similar ones with \( t/(e^t - 1) \) replaced by \( 1/(e^t + 1) \), by Gaussian quadrature, treating \( t/(e^t - 1) \), or its square, as a weight function. This is a neat way of dealing with the poles of this function at \( \pm 2k\pi i, k = 0, 1, 2, \ldots \), which otherwise would adversely interfere with more standard integration techniques.

Another simple, but interesting observation of Gradimir was this: Integrals of the type (1) can be used to sum infinite series,

\[
\sum_{k=1}^\infty a_k = \int_0^\infty h(t) \frac{t}{e^t - 1} dt,
\]

if the general term of the series, \( a_k = -F'(k) \), is the (negative) derivative of the Laplace transform \( F(p) = \int_0^\infty e^{-pt} h(t) dt \) evaluated at \( p = k \) of some known function \( h \). Since series of this kind are typically slowly convergent, the representation (2) offers a useful summation procedure, the sequence of \( n \)-point Gaussian quadrature rules, \( n = 1, 2, 3, \ldots \), applied to the integral on the right converging rapidly if \( h \) is sufficiently smooth.

This is all very nice, but how do we generate Gaussian quadrature rules with such unusual weight functions? Classically, there is an approach via orthogonal polynomials and the moments of the weight function,

\[
\mu_k = \int_0^\infty t^k \frac{t}{e^t - 1} dt, \quad k = 0, 1, 2, \ldots
\]

In fact, this is the road Gradimir took in his manuscript, noting that the moments are expressible in terms of the Riemann zeta function,

\[
\mu_k = (k + 1)! \zeta(k + 2), \quad k = 0, 1, 2, \ldots
\]

It was at this point where I felt I had to exercise my prerogatives as a referee: I criticized the highly ill-conditioned nature of this approach and proposed more stable alternative methods that I developed just a year or two earlier. In the process, I rewrote a good portion of the manuscript and informed the editor that the manuscript so revised would be an appropriate and interesting contribution to computational mathematics. I suggested, subject to the author's approval, to publish the work as a joint paper. The approval was forthcoming, and that is how our first joint publication [6] came about.

In retrospect, Gradimir's original approach via moments has regained some viability since software has become available in the last few years that allows generating
the required orthogonal polynomials in variable-precision arithmetic. One such program is the Matlab symbolic Chebyshev algorithm schebyshev.m (downloadable from http://www.cs.purdue.edu/archives/2002/wxg/codes/SOPQ.html),

which generates the required recurrence coefficients directly from the moments. Table 1 in [6], and similarly Tables 2–4 (cf. 4 [Sects. 4–5]), can thus be produced very simply using the following Matlab script:

```matlab
syms mom ab
digits(65); dig=65;
for k=1:80
    mom(k)=vpa(gamma(vpa(k+1))*zeta(vpa(k+1)));
end
ab=schebyshev(dig,40,mom);
ab=vpa(ab,25)
```

True, it takes 65-decimal-digit arithmetic to overcome the severe ill-conditioning and obtain the first 40 recursion coefficients (in the array ab) of the orthogonal polynomials to 5 decimal digits. But this is a one-time shot; once these coefficients are available, one can revert to ordinary arithmetic to compute the desired Gaussian quadratures and the integrals in question.

3. In March of 1984, on a visit to Niš, I had the opportunity to finally meet my collaborator in person. He invited me to dinner at his home (my compliments to Dobrila for her culinary art!), after which Gradimir and I retired to his study, where we engaged in a most lively brainstorming session. I was astonished how well he knew earlier work of mine. He must have read my short 1984 paper on spherically symmetric distributions and their approximation by step functions matching as many moments of the distribution as possible. Because he obviously had thought about extending this type of approximation to more general spline approximations. Another idea that surfaced during this discussion was orthogonality on the semicircle and related (complex-valued) orthogonal polynomials. We agreed to pursue these topics further, which provided enough material to keep us busy for several years to come. It so happened that it was the second of these problems that received our attention first, but soon enough we worked on both problems concurrently.

4. Polynomials that are orthogonal on curves \( \Gamma \) in the complex plane have a long history in the case where the underlying inner product is Hermitian, i.e., of the form \( (u,v) = \int_{\Gamma} u(z)\overline{v(z)} \, d\sigma(z), \) \( d\sigma \) being a positive measure on \( \Gamma; \) see, e.g., [14, Chaps. 11 and 16]. The case most studied, by far, is the unit circle, \( \Gamma = \{e^{i\theta}, 0 \leq \theta < 2\pi\}, \) which gives rise to Szegő’s theory of orthogonal polynomials on the unit circle. The question we asked ourselves is this: what happens if the second factor in the inner product is not conjugated? We decided to begin our study with a prototype inner product, namely, \( d\sigma \) the Lebesgue measure, and \( \Gamma \) the upper half of the unit
circle (the whole unit circle being ruled out by Cauchy’s theorem). Thus, we began looking at
\[ (u,v) = \int_0^\pi u(e^{i\theta})v(e^{i\theta}) \, d\theta, \]
postponing for later the study of more general weight functions.

The moment functional associated with (3) is \( \mathcal{L}_x \cdot (1, x^k), k = 0, 1, 2, \ldots \); it is well known that a sequence of monic polynomials \( \{\pi_n\} \) orthogonal with respect to the inner product (3) exists uniquely if the moment sequence \( \{\mu_k\}, \mu_k = \mathcal{L}_x \cdot (1, x^k) \), is quasi-definite, i.e., \( \Delta_n \neq 0 \) for all \( n \geq 1 \), where
\[
\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix}.
\]

We were able to prove quasi-definiteness by explicit computation of the moments and the determinant in (4).

Since \( (zu, v) = (u, zv) \), there must exist a three-term recurrence relation to the polynomials \( \{\pi_n\} \); we found it to be of the form
\[
\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k \pi_{k-1}(z), \quad k = 0, 1, 2, \ldots,
\]
\[
\pi_{-1}(z) = 0, \quad \pi_0(z) = 1,
\]
where
\[
\alpha_0 = \bar{\alpha}_0, \quad \alpha_k = \bar{\alpha}_k - \rho_{k-1}, \quad \beta_k = \rho_k^2, \quad k = 1, 2, \ldots,
\]
and
\[
\rho_k = \frac{2}{2k+1} \left[ \frac{\Gamma((k+2)/2)}{\Gamma((k+1)/2)} \right]^2, \quad k \geq 0.
\]
As \( k \to \infty \), one finds \( \alpha_k \to 0, \beta_k \to \frac{1}{4} \), familiar from Szegő’s class of polynomials orthogonal on the interval \([-1, 1]\).

Interestingly, the polynomials \( \pi_n \) are closely connected to Legendre polynomials,
\[
\pi_n(z) = \hat{P}_n(z) - i \hat{\rho}_{n-1} \hat{P}_{n-1}(z), \quad n \geq 1,
\]
where \( \hat{P}_k \) is the monic Legendre polynomial of degree \( k \). This allowed us to derive a linear second-order differential equation for \( \pi_n \), which, like the differential equation for Legendre polynomials, has regular singular points at \( 1, -1, \) and \( \infty \), but unlike Legendre polynomials, an additional singular point on the negative imaginary axis, which depends on \( n \) and approaches the origin monotonically as \( n \uparrow \infty \).

All zeros of \( \pi_n \) are contained in the half disk \( D_+ = \{z \in \mathbb{C} : |z| < 1, \, \text{Im} z > 0\} \) and located symmetrically with respect to the imaginary axis. They are all simple and can be computed as the eigenvalues of the real, nonsymmetric, tridiagonal matrix.
having the first \( n \) of the coefficients \( a_k \) on the diagonal, the first \( n - 1 \) of the \( \vartheta_k \) on the upper side diagonal and their negatives on the lower side diagonal.

There is a Gaussian quadrature formula for integrals over the semicircle,

\[
\int_0^\pi g(e^{i\theta}) \, d\theta = \sum_{v=1}^n \sigma_v g(\zeta_v), \quad g \in \mathbb{P}_{2n-1},
\]

where \( \zeta_v \) are the zeros of \( \pi_n \) and \( \sigma_v \) the (complex) Christoffel numbers. The latter can be computed by an adaptation of the well-known Golub/Welsch procedure.

All these results are briefly announced in [7] and fully developed in [8], where one also finds applications of the Gauss formula (9) to numerical differentiation and the evaluation of Cauchy principal value integrals.

Partial results for Gegenbauer weight functions had already been obtained, when new impulses were received through collaboration with Henry J. Landau, cf. [12]. This resulted in a considerable simplification of the existence and uniqueness theory. Indeed, if the inner product is

\[
(u, v) = \int_0^\pi u(e^{i\theta}) v(e^{i\theta}) w(e^{i\theta}) \, d\theta,
\]

where \( w \) is positive on \((-1, 1)\) and holomorphic in \( D_+ \), then the (monic) polynomials \( \{\pi_n\} \) orthogonal with respect to (10) exist uniquely if

\[
\text{Re} (1, 1) = \text{Re} \int_0^\pi w(e^{i\theta}) \, d\theta \neq 0.
\]

This is always true for symmetric weight functions,

\[
w(-z) = w(z) \quad \text{and} \quad w(0) > 0,
\]

for example, the Gegenbauer weight \( w(z) = (1 - z^2)^{\lambda-1/2} \), \( \lambda > -1/2 \), and also for the Jacobi weight function \( w(z) = (1 - z)^{\alpha} (1 + z)^{\beta} \), \( \alpha > -1, \beta > -1 \).

There are interesting interrelations between the (monic) complex polynomials \( \{\pi_n\} \) orthogonal with respect to the inner product (10), the (monic) real polynomials \( \{p_n\} \) orthogonal with respect to the inner product \([u, v] = \int_{-1}^{1} u(x)v(x)w(x) \, dx\), and the associated polynomials of the second kind,

\[
q_n(z) = \int_{-1}^{1} \frac{p_n(z) - p_n(x)}{z-x} w(x) \, dx, \quad n = 0, 1, 2, \ldots; \quad q_{-1}(z) = -1.
\]

Thus, for example (cf. (8)),

\[
\pi_n(z) = p_n(z) - i \vartheta_{n-1} p_{n-1}(z), \quad n = 0, 1, 2, \ldots,
\]

where

\[
\vartheta_{n-1} = \frac{\mu_0 \, p_n(0) + i q_n(0)}{i \mu_0 \, p_{n-1}(0) - q_{n-1}(0)}, \quad \mu_0 = (1, 1),
\]
or, alternatively,

$$\vartheta_n = i\alpha_n + \frac{b_n}{\vartheta_{n-1}}, \quad n = 0, 1, 2, \ldots; \quad \vartheta_{-1} = \mu_0,$$

(12)

where \(\alpha_k, b_k\) are the recursion coefficients for the real orthogonal polynomials \(\{p_n\}\).

For symmetric weight functions (11), one can prove \(\mu_0 = \pi w(0) > 0\), so that by (12), since \(\alpha_n = 0\) and \(b_n > 0\), all \(\vartheta_n\) are positive.

Moreover, the three-term recurrence relation for the \(\pi_n\) again has the form (5), where now

$$\alpha_0 = \vartheta_0 - i\alpha_0, \quad \alpha_k = \vartheta_k - \vartheta_{k-1} - i\alpha_k\quad (k \geq 1),$$

$$\beta_0 = \mu_0, \quad \beta_k = \vartheta_{k-1}(\vartheta_{k-1} - i\alpha_{k-1})\quad (k \geq 1).$$

For symmetric weight functions \((\alpha_k = 0)\), this reduces to (6), and for Gegenbauer weight functions, one finds

$$\vartheta_0 = \frac{\Gamma(\lambda + 1/2)}{\sqrt{\pi} \Gamma(\lambda + 1)}, \quad \vartheta_k = \frac{1}{\lambda + k} \frac{\Gamma((k+2)/2)\Gamma(\lambda + (k+1)/2)}{\Gamma((k+1)/2)\Gamma(\lambda + (k+2)/2)}\quad k \geq 1,$$

generalizing (7).

With regard to the location of the zeros of \(\pi_n\), we showed for symmetric weight functions that they are contained in \(D_+\), with the possible exception of a single (simple) zero on the positive imaginary axis outside the unit disk. (For a related result, see also [5]). The exception cannot occur for Gegenbauer weights, at least not when \(n \geq 2\), and all zeros in this case can be shown to be simple. For Gegenbauer weights, one can also obtain the linear second-order differential equation for \(\pi_n\), which has properties analogous to those stated above for Legendre weight functions.

5. Our second joint venture deals with a problem of spline approximation on the half line \(\mathbb{R}_+ = \{t : t \geq 0\}\). Given a function \(f\) on \(\mathbb{R}_+\) having finite moments, we want to approximate \(f\) by a spline function \(s\) of degree \(m \geq 0\) that also has finite moments; in fact, we want \(f\) and \(s\) to have the same successive moments up to an order as high as possible.

Now any spline function of degree \(m\) is the sum of a polynomial of degree \(m\) and a linear combination of truncated \(m\)th powers. If this is to have finite moments on \(\mathbb{R}_+\), then the polynomial part must be identically zero, and the spline \(s\) therefore is of the form

$$s_{n,m}(t) = \sum_{v=1}^{n} a_v (\tau_v - t)^m_+,$$

(13)

where \(u^m_+\) are the truncated powers

$$u^m_+ = \begin{cases} 
  u^m & \text{if } u \geq 0, \\
  0 & \text{if } u < 0,
\end{cases} \quad m = 0, 1, 2, \ldots.$$
The coefficients $a_v$ are real and the "knots" $\tau_v$ mutually distinct and positive, say $0 < \tau_1 < \tau_2 < \cdots < \tau_n$, but otherwise can be freely chosen. Since there are $2n$ unknowns, we can impose $2n$ moment conditions,

$$
\int_{\mathbb{R}^+} t^j s_{n,m}(t) \, dt = \mu_j, \quad j = 0, 1, 2, \ldots, 2n - 1,
$$

where $\mu_j = \int_{\mathbb{R}^+} t^j f(t) \, dt$ are the (given) moments of $f$. The problem thus amounts to solving the system (14) of $2n$ nonlinear equations in the $2n$ unknowns $a_v, \tau_v, v = 1, 2, \ldots, n$.

The problem is reminiscent of Gaussian quadrature and in fact can be solved by constructing a suitable $n$-point Gaussian quadrature rule [9]. Indeed, if $f$ is such that

(a) $f \in C^{m+1}(\mathbb{R}^+)$

(b) The moments $\mu_j = \int_{\mathbb{R}^+} t^j f(t) \, dt, j = 0, 1, 2, \ldots, 2n - 1$ exist

(c) $f^{(m)}(t) = o(t^{-2m-\mu})$ as $t \to \infty, \mu = 0, 1, \ldots, m$

then the equations (14) have a unique solution if and only if the measure

$$
d\lambda_m(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f^{(m+1)}(t) \, dt \quad \text{on } \mathbb{R}^+
$$

admits an $n$-point Gaussian quadrature formula

$$
\int_{\mathbb{R}^+} g(t) \, d\lambda_m(t) = \sum_{v=1}^n \lambda_v^G g(t_v^G), \quad g \in \mathbb{P}_{2n-1},
$$

satisfying $0 < t_1^G < t_2^G < \cdots < t_n^G$. If so, then the solution to (14) is

$$
\tau_v = t_v^G; \quad a_v = \frac{\lambda_v^G}{|t_v^G|^{m+1}}, \quad v = 1, 2, \ldots, n.
$$

In general, of course, $d\lambda_m$ is not a positive measure, and therefore the existence of the Gauss formula (16) with positive nodes is by no means guaranteed. However, when $f$ is completely monotone, i.e., $(-1)^k f^{(k)}(t) > 0$ on $\mathbb{R}^+$ for $k = 0, 1, 2, \ldots$, then the measure (15) is obviously positive and under the assumptions (a)–(b) can be shown to have finite moments of orders up to $2n - 1$. In this case, the quadrature formula (16) exists uniquely and has distinct positive nodes $t_v^G$. Moreover, by (17), the coefficients $a_v$ are all positive, so that $s_{n,m}$ is also completely monotone, at least in the weak sense that $(-1)^k f^{(k)}(t) \geq 0$ for all $k \geq 0$ a.e. on $\mathbb{R}_+$.

In case the spline approximation $s_{n,m}(t)$ exists, its error $f(t) - s_{n,m}(t)$ at $t = x$ can be expressed in terms of the error of the Gauss formula (16) for a special spline function $g(t) = t^{-(m+1)}(t-x)^m$ (cf. [10, Theorem 2.3]).

Similar problems of moment-preserving spline approximation can be considered on a finite interval, say $[0, 1]$. In this case, we can add to (13) a polynomial of degree $m$, which increases the degree of freedom by $m + 1$. We may use this increased degree of freedom either to add $m + 1$ more moment conditions, or to impose $m + 1$
boundary conditions of the form \( s_{n,m}^{(k)}(1) = f^{(k)}(1), \ k = 0, 1, \ldots, m \). The relevant measure then becomes

\[
d\lambda_{m}(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t) \, dt \quad \text{on } [0,1],
\]

and the solution of the two problems can be given (if it exists) in terms of generalized Gauss–Lobatto formulae for the first problem, and generalized Gauss–Radau formulae for the other, both for integration with respect to the measure \((18)\); cf. [1]. The numerical construction of such formulae, however, is rather more complicated, and has been considered by the author only recently in [2, 3]. Gradimir, together with M.A. Kovačević [13], also studied moment-preserving approximation on \( IR^+ \) by defective splines, which gives rise to Gauss–Turán quadrature rules for the measure \((15)\). Undoubtedly, this led Gradimir to wonder about how to compute these quadratures effectively.

6. Gauss–Turán quadrature formulae are of Gaussian type, i.e., have maximum algebraic degree of exactness, and involve not only values of the integrand function, but also values of its successive derivatives up to an even order \(2s\), all evaluated at a common set of \(n\) nodes. Since there are \((2s + 1)n\) coefficients \((n\) for each derivative) and \(n\) nodes to be determined, the maximum degree of exactness is expected to be \(2(s + 1)n - 1\), and the formula thus has the form

\[
\int_{IR} f(t) \, d\lambda(t) = \sum_{i=0}^{2s} \sum_{v=1}^{n} \lambda_{v} v f^{(i)}(\tau_{v}), \quad f \in P_{2(s+1)n-1},
\]

where \(d\lambda\) is a given positive measure. It is known that the nodes \(\tau_{v}\) must be the zeros of the (monic) polynomial \(\pi_{n} = \pi_{n,s}\) of degree \(n\) whose \((2s + 1)\)st power is orthogonal (relative to the measure \(d\lambda\)) to all polynomials of degree \(<n\). In other words, \(\pi_{n}\) is the \(n\)-th-degree polynomial orthogonal with respect to the positive measure

\[
d\mu(t) = \pi_{n}^{2s}(t) \, d\lambda(t).
\]

We have here a case of implicit orthogonality—also called \(s\)-orthogonality—since the polynomial \(\pi_{n}\) to be determined appears also in the measure of orthogonality. The problem of computing Gauss–Turán quadrature rules \((19)\), considered in [10], thus will in some way come down to a problem of solving a system of nonlinear equations.

Gradimir’s idea was to embed \(\pi_{n}\) in a sequence of \(n + 1\) polynomials \(\pi_{0}, \pi_{1}, \ldots, \pi_{n}\), namely, the polynomials orthogonal with respect to the measure \((20)\). As such, they must satisfy a three-term recurrence relation

\[
\pi_{k+1}(t) = (t - \alpha_{k}) \pi_{k}(t) - \beta_{k} \pi_{k-1}(t), \quad k = 0, 1, \ldots, n - 1,
\]

\[
\pi_{-1}(t) = 0, \quad \pi_{0}(t) = 1.
\]
Although the coefficients \(a_0, a_1, \ldots, a_{n-1}; \beta_0, \beta_1, \ldots, \beta_{n-1}\) are not known, in fact, need to be determined, we know from the general theory of orthogonal polynomials that they are expressible in terms of inner products involving the polynomials \(\pi_0, \pi_1, \ldots, \pi_{n-1}\); specifically,

\[
\alpha_k = \frac{\int_{\mathbb{R}} t \pi_k^2(t) \, d\mu(t)}{\int_{\mathbb{R}} \pi_k^2(t) \, d\mu(t)}, \quad k = 0, 1, \ldots, n-1;
\]

\[
\beta_k = \frac{\int_{\mathbb{R}} \pi_k^2(t) \, d\mu(t)}{\int_{\mathbb{R}} \pi_{k-1}^2(t) \, d\mu(t)}, \quad k = 1, 2, \ldots, n-1,
\]

and \(\beta_0 = \int_{\mathbb{R}} d\mu(t)\) by convention. If we insert here the definition (20) of the measure \(d\mu\) and clear all fractions, we arrive at the following system of \(2n\) equations:

\[
\begin{align*}
\beta_0 - \int_{\mathbb{R}} \pi_n^{2s}(t) \, d\lambda(t) &= 0, \\
\int_{\mathbb{R}} (\alpha_k - t) \pi_k^2(t) \pi_n^{2s}(t) \, d\lambda(t) &= 0, \quad k = 0, 1, \ldots, n-1; \\
\int_{\mathbb{R}} (\beta_k \pi_{k-1}^2(t) - \pi_k^2(t)) \pi_n^{2s}(t) \, d\lambda(t) &= 0, \quad k = 1, 2, \ldots, n-1.
\end{align*}
\]

(22)

We note that (22) represents a system of \(2n\) nonlinear equations in the \(2n\) unknowns \(\alpha_k, \beta_k\). Indeed, each of the polynomials \(\pi_r\) appearing in (22) can be thought of as a function of \(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}; \beta_0, \beta_1, \ldots, \beta_{n-1}\) by virtue of the relations in (21). Furthermore, each integrand in (22) is a polynomial of degree \(\leq 2(s+1)n-2\), and therefore all integrals in (22) can be evaluated exactly by an \(N\)-point Gaussian quadrature rule relative to the measure \(d\lambda\), where \(N = (s+1)n\). The same is true for the integrals appearing in the Jacobian matrix of the system (22). We were able, therefore, to apply to (22) the Newton–Kantorovich method, which could be made to converge by a careful choice of the initial approximations \(\alpha_k^{(0)}, \beta_k^{(0)}\) to the unknowns \(\alpha_k, \beta_k\). Once the \(\alpha_k\) and \(\beta_k\) have been obtained, the zeros \(\tau_v\) of \(\pi_n\), i.e., the nodes in the quadrature rule (19), can be computed by the well-known Golub/Welsch procedure.

All that remains is to compute the coefficients \(\lambda_{i,v}\) in (19). By inserting in (19) suitably selected polynomials of degree \(\leq 2(s+1)n-1\), the coefficients \(\lambda_{i,v}\) for each fixed \(v, 1 \leq v \leq n\), can be found by solving an upper triangular system of \(2s+1\) linear algebraic equations (cf. [10, Theorem 3.3]).

7. Conformal maps in fluid mechanics often require Cauchy principal value integrals of the form

\[
(I_{(\alpha, \beta)} \phi)(\xi) = \int_{\alpha}^{\beta} \phi(\tau) \coth \frac{\tau - \xi}{2} \, d\tau, \quad \alpha < \xi < \beta,
\]

(23)
which are notoriously difficult to evaluate numerically. Some possible approaches are discussed in [11]. If the interval \([\alpha, \beta]\) is finite, (23) can be transformed to

\[
(I_a f)(x) = \frac{1}{a} \int_{-1}^{1} f(t) w(t) \, dt, \quad -1 < t < 1,
\]

where \(x = \frac{2\xi - (\alpha + \beta)}{(\beta - \alpha)}, a = \frac{(\beta - \alpha)}{4}, \) and

\[
w(t) = \omega(a(t - x)), \quad \omega(u) = u \coth u.
\]

The difficulty here is caused by the poles of \(w(t)\) at \(t = x \pm ki\pi/a, k = 1, 2, \ldots\), which can be close to the interval \([-1, 1]\) if \(a\) is large. Following standard procedure, (24) is first written in the form

\[
(I_a f)(x) = \frac{1}{a} \left\{ f(x) \ln \frac{\sinh a(1 - x)}{\sinh a(1 + x)} + \int_{-1}^{1} g(t) w(t) \, dt \right\},
\]

where \(g(t) = [x, t] f\) is the divided difference of \(f\) at \(x\) and \(t\). The integral in (25) can then be computed either by Gauss quadrature relative to the (nonstandard) weight function \(w\), which gives excellent results but is somewhat expensive, or by the less expensive Gauss–Legendre quadrature, which, however, works well only for \(a\) relatively small. An alternative procedure is interpolatory quadrature of (24) on the zeros of the respective orthogonal polynomials, which circumvents the need of computing a divided difference. Error analyses are provided, either in real variable form, involving derivatives, or in terms of contour integration in the complex plane.

Cauchy principal value integrals (23) with infinite interval \([\alpha, \beta] = [-\infty, \infty]\) can be rendered accessible to the same approaches after suitable truncation of the interval.

8. In looking back on my collaboration with Gradimir, I can only marvel at the spontaneity and originality of his input, which often reduced my own role to one of implementor and organizer. It has been truly a pleasure to work together with Gradimir, and I am sure I am sharing this feeling with the many other individuals who have had the privilege of collaborating with Gradimir. I wish him many more years of good health and continued excellence and success in his research.

References


