Advances in Chebyshev Quadrature*

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1. Introduction

Let $d\mu(x)$ be a positive measure on the interval (a,b) admitting finite moments of all orders,

$$(1) \qquad \mu_r = \int_a^b x^r d\mu(x) < \infty, \qquad r = 0,1,2,\ldots \; .$$

We consider quadrature rules of the type

$$(2) \qquad \int_a^b f(x)d\mu(x) = \sum_{k=1}^{n} \gamma_k^{(n)} f(x_k^{(n)}) + R_n(f)$$

having equal weights

$$(3) \qquad \gamma_1^{(n)} = \gamma_2^{(n)} = \cdots = \gamma_n^{(n)} \; .$$

Equally-weighted quadrature sums have the property of minimizing the effect of random errors in the function values $f(x_k^{(n)})$, which may be a useful feature if these errors are considerably larger than the truncation error $|R_n(f)|$. Another, though minor, advantage of equal coefficients results from the fact that only one multiplication is required, as opposed to n, to evaluate the quadrature sum in (2) (not counting the work in evaluating f).

To be widely useful, quadrature rules of the type (2), (3) should have real distinct nodes $x_k^{(n)}$, preferably all located in (a,b). In addition, they should be reasonably accurate. We say that (2), (3) is a Chebyshev quadrature rule, if all nodes are real and if the formula has algebraic degree of exactness n, i.e.,

$$(4) \qquad R_n(f) = 0, \quad \text{all } f \in \mathbb{P}_n \, .$$

($\mathbb{P}_n$ denotes the class of polynomials of degree $\leq n$.) Letting $f \equiv 1$ in (2) then gives immediately

$$(5) \qquad \gamma_k^{(n)} = \frac{\mu_0}{n}, \qquad k = 1,2,\ldots,n \; .$$

We call (2), (5) a Chebyshev quadrature rule in the strict sense, if (4) holds and the nodes $x_k^{(n)}$ are not only real, but pairwise distinct and all contained in (a,b).

*This work was supported in part by the National Science Foundation under grant GP-36557.

Any quadrature rule (2), (3), on the other hand, with only real nodes, will be referred to as a Chebyshev-type quadrature formula. Such a quadrature rule, therefore, need not have algebraic degree of exactness n, in fact, need not even integrate constants exactly, and is permitted to have repeated nodes, i.e., $x_k^{(n)} = x_l^{(n)}$ for some $k \neq l$.

The monic polynomial of degree n whose zeros are $x_k^{(n)}$, $k = 1,2,\ldots,n$, will be denoted, throughout, by $p_n(x; d\mu)$,

$$(6) \qquad p_n(x; d\mu) = \prod_{k=1}^{n} (x-x_k^{(n)}) = x^n + a_1 x^{n-1} + \cdots + a_n .$$

Requirements (4) and (5) uniquely determine the polynomial $p_n(x; d\mu)$ (cf. §3.2). We may say, therefore, alternatively, that (2), (3) constitutes a Chebyshev quadrature rule if and only if the polynomial $p_n(x; d\mu)$ has only real zeros. We shall have occasion to consider also the polynomials which are orthogonal with respect to the measure $d\mu(x)$; these will be denoted by $\pi_n(x; d\mu)$, $n = 0,1,2,\ldots$.

The study of Chebyshev quadratures began in 1874 with a classical memoir of Chebyshev (Chebyshev [1874]). Important progress has subsequently been made by Bernstein [1937], [1938], Ullman [1966], Geronimus [1946], [1969], and others. Apart from a brief review in Wilf [1967], and traditional treatments in textbooks, no comprehensive account seems to be available. In the following we attempt to review recent advances in this field, covering roughly the period 1945-1975.

2. The classical Chebyshev quadrature formula

2.1 Bernstein's result. The quadrature formula §1(2), (4), (5), in which $d\mu(x) = dx$ on $[-1,1]$, will be referred to as the classical Chebyshev quadrature formula. It was computed by Chebyshev for $n = 2,3,\ldots,7$ and found, in these cases, to have only real nodes, all contained in $[-1,1]$. Radau [1880b] adds to this the case $n=9$, which also yields real nodes, but notes that the formula with $n=8$ involves complex nodes. It took some fifty years after that, until Bernstein, by extremely ingenious arguments, succeeded in proving that the formulas found by Chebyshev and Radau are in fact the only ones which have all nodes real. If $n > 9$, and $n=8$, the polynomial $p_n(x; d\mu)$ necessarily has complex zeros. A simplified version of Bernstein's proof is given by Krylov [1957] (and is also reproduced in Krylov [1962, p.192ff]). Kahaner [1969] interprets the presence of complex nodes as the result of a conflict between the equal coefficients requirement and the polynomial exactness requirement, the former tending to impose a uniform distribution on the real zeros, the latter a non-uniform distribution (as the zeros of orthogonal polynomials), when $n \to \infty$.

There is a fair amount of numerical information available on classical

Chebyshev quadrature. Salzer [1947] exhibits the polynomials $p_n(x; d\mu)$ in exact rational form for $n = 1(1)12$, and also gives the zeros to 10 decimal places for $n = 2(1)7$ and $n = 9$. (The latter are reprinted in Abramowitz and Stegun [1964, p.920].) On the microfiche addendum to Kahaner [1971] the zeros of $p_n(x; d\mu)$, including the complex ones, are tabulated to 14 decimals for $2 \leq n \leq 47$.

2.2 Geometry of the zeros of $p_n(x; d\mu)$. The distribution of the zeros of $p_n(x; d\mu)$ in the complex plane, when $d\mu(x) = dx$ on $[-1,1]$, is studied in detail by Kuzmin [1938]. It turns out that for large n, all zeros (except the zero at the origin, when n is odd) accumulate near the curve

$$(1) \qquad \omega(z) = \omega(1) , \qquad \omega(z) = \int_{-1}^{1} \ln|z-t|dt ,$$

familiar from potential theory. (This is an eye-shaped curve, centred at the origin, which intersects the real axis at ± 1, and the imaginary axis at about $\pm .52$.) More precisely, Kuzmin shows that for n sufficiently large and $h = \sqrt{\ln n}/n$, all zeros of $p_n(z; d\mu)$ (with the exception noted) are either located inside the circles about ± 1, with radii $3h$, or in the narrow band bounded by the curves $\omega(z) = \omega(1)$ and $\omega(z) = \omega(1-12h)$. The zeros thus approach the logarithmic potential curve (1) from the inside. The case $n = 20$ is depicted in Kahaner [1971]. Kuzmin also proves that the number of real zeros of $p_n(z; d\mu)$ is $O(\ln n)$ as $n \to \infty$. Additional properties of the zeros can be found in Mayot [1950] and Kahaner [1971].

3. Mathematical techniques

A number of analytic tools have been developed to deal with the construction of Chebyshev quadratures, or with proofs of nonexistence. We briefly review four of them, and illustrate some by examples.

3.1 Chebyshev's method. This is the method used by Chebyshev in his original memoir (Chebyshev [1874]). The polynomial $p_n(z; d\mu)$ is represented explicitly in the form

$$(1) \qquad \begin{aligned} p_n(z; d\mu) &= E\{\exp(\frac{n}{\mu_0} \int_a^b \ln(z-x)d\mu(x))\} \\ &= E\{z^n \exp(-\frac{n}{\mu_0} \sum_{k=1}^{\infty} \frac{\mu_k}{kz^k})\} , \end{aligned}$$

where $E\{\cdot\}$ denotes the polynomial part of $\{\ \}$. Based on this formula, Chebyshev computes his original quadrature rules (with $d\mu(x) = dx$) for $n = 2,3,\ldots,7$.

In the case of $d\mu(x) = (1-x^2)^{-\frac{1}{2}}$ on $[-1,1]$, formula (1) gives

$$p_n(z;\, d\mu) = E\{(\frac{z+\sqrt{z^2-1}}{2})^n\}$$
$$= E\{(\frac{z+\sqrt{z^2-1}}{2})^n + (\frac{z-\sqrt{z^2-1}}{2})^n\} = \frac{1}{2^{n-1}} T_n(z) ,$$

where $T_n(z)$ is (what is now called) the Chebyshev polynomial of the first kind. In this way, Chebyshev recovers the classical Gauss-type quadrature rule

$$(2) \qquad \int_{-1}^{1} f(x)(1-x^2)^{-\frac{1}{2}}dx = \frac{\pi}{n} \sum_{k=1}^{n} f(x_k^{(n)}) + R_n(f), \qquad x_k^{(n)} = \cos(\frac{2k-1}{2n}\pi) ,$$

which he ascribes to Hermite.

An interesting recent extension of (2) is due to Ullman [1966a,b], who considers

$$(3) \qquad d\mu(x) = (1-x^2)^{-\frac{1}{2}}(1+ax)(1+a^2+2ax)^{-1} \quad \text{on } [-1,1], \qquad -1 < a < 1 ,$$

and finds that

$$z \exp(-\frac{1}{\mu_0} \sum_{k=1}^{\infty} \frac{\mu_k}{kz^k}) = \tfrac{1}{2}(z + \sqrt{z^2-1} + a), \qquad |z| > 1 .$$

Application of (1) thus gives

$$p_n(z;\, d\mu) = E\{(\frac{z+\sqrt{z^2-1}+a}{2})^n\}$$
$$= E\{(\frac{z+\sqrt{z^2-1}+a}{2})^n + (\frac{z-\sqrt{z^2-1}+a}{2})^n - (\frac{a}{2})^n\}$$
$$= \frac{1}{2^{n-1}} T_n^{(a)}(z) ,$$

where $T_n^{(a)}(z)$ is a polynomial of degree n generalizing the Chebyshev polynomial $T_n(z) = T_n^{(0)}(z)$. Ullman shows that $T_n^{(a)}(z)$, for each n, has only real zeros, whenever $-\frac{1}{2} < a < \frac{1}{2}$, thus exhibiting his celebrated example of a weight function, other than the classical one in (2), which admits Chebyshev quadrature for each n. Work along this line is continued by Geronimus [1969] (cf. §5.1).

3.2 Method based on Newton's identity. If the quadrature formula

$$(4) \qquad \int_a^b f(x)d\mu(x) = \frac{\mu_0}{n} \sum_{k=1}^{n} f(x_k^{(n)}) + R_n(f)$$

is to have algebraic degree of exactness n, then the nodes $x_k = x_k^{(n)}$ must satisfy

$$\sum_{k=1}^{n} x_k^r = \frac{n}{\mu_0}\mu_r , \qquad r = 1,2,\ldots,n .$$

Since the moments μ_r are known, these equations determine the first n power sums $s_r = \sum_{k=1}^{n} x_k^r$ of the nodes,

$$(5) \qquad s_r = \frac{n}{\mu_0}\mu_r , \qquad r = 1,2,\ldots,n .$$

These in turn determine the coefficients a_k in the polynomial $p_n(x; d\mu)$ by virtue of Newton's identities,

$$(6) \qquad \begin{cases} s_1 + a_1 = 0 , \\ s_2 + a_1 s_1 + 2a_2 = 0 , \\ \cdots\cdots\cdots\cdots \\ s_n + a_1 s_{n-1} + \cdots + a_{n-1}s_1 + na_n = 0 . \end{cases}$$

The desired nodes x_k in (4) can thus be computed by finding the roots of the algebraic equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where the coefficients a_k are obtained recursively from (6), the s_r being given by (5). This method, first used by Radau [1880a,b], is the one generally applied to compute Chebyshev quadratures. It is extended in Gautschi and Yanagiwara [1974] and Anderson and Gautschi [to appear] to deal with Chebyshev-type quadratures involving repeated nodes.

3.3 Bernstein's method. This is a powerful method for proving nonexistence results. Although Bernstein [1937] deals only with the case of a constant weight function, his method extends immediately to arbitrary measures $d\mu(x)$.

Bernstein's idea is to confront Chebyshev's n-point quadrature formula

$$(7) \qquad \int_a^b f(x)d\mu(x) = \frac{\mu_0}{n}\sum_{k=1}^{n} f(x_k^{(n)}) + R_n^{Ch}(f)$$

with the m-point Gauss formula

$$(8) \qquad \int_a^b f(x)d\mu(x) = \sum_{r=1}^{m} \lambda_r^{(m)} f(\xi_r^{(m)}) + R_m^G(f) ,$$

where $\xi_r^{(m)}$ are the zeros of $\pi_m(x; d\mu)$, assumed in decreasing order,

$$a < \xi_m^{(m)} < \xi_{m-1}^{(m)} < \cdots < \xi_1^{(m)} < b ,$$

and $\lambda_r^{(m)}$ are the corresponding Christoffel numbers. Assuming distinct nodes $x_k^{(n)}$

in (7), all contained in (a,b), and assuming that (7) has polynomial degree of exactness $2m-1$, $m < n$, then Bernstein shows that, necessarily,

$$\frac{\mu_0}{n} \leq \min(\lambda_1^{(m)}, \lambda_m^{(m)}) . \tag{9}$$

This inequality remains valid under the weaker assumption of mere reality of the nodes $x_k^{(n)}$ (Gautschi [1975]).

The road from Bernstein's inequality (9) to nonexistence results is still fraught with considerable technical difficulties, particularly in the case of finite intervals. For measures on infinite intervals, the method appears to apply more easily, as we illustrate with the example of the Laguerre measure, $d\mu(x) = x^\alpha e^{-x}dx$. Here, (4) takes the form

$$\int_0^\infty f(x)x^\alpha e^{-x}dx = \frac{\Gamma(\alpha+1)}{n}\sum_{k=1}^{n} f(x_k^{(n)}) + R_n(f), \qquad \alpha > -1 . \tag{10}$$

The orthonormal polynomials are the normalised Laguerre polynomials,

$$\pi_0(x) = [\Gamma(\alpha+1)]^{-\frac{1}{2}} , \quad \pi_1(x) = [\Gamma(\alpha+2)]^{-\frac{1}{2}}(\alpha+1-x),\ldots, \tag{11}$$

and

$$\lambda_1^{(m)} = \Big(\sum_{k=0}^{m-1} [\pi_k(\xi_1^{(m)})]^2\Big)^{-1} \leq (\pi_0^2 + [\pi_1(\xi_1^{(m)})]^2)^{-1} , \qquad m \geq 2 .$$

Using the known inequality (Krylov [1958])

$$\xi_1^{(m)} > 2m + \alpha - 1 \qquad (m \geq 2,\ \alpha > -1) ,$$

and the explicit expressions in (11), we can further estimate

$$\lambda_1^{(m)} < \frac{\Gamma(\alpha+1)}{1 + 4(m-1)^2/(\alpha+1)} , \qquad m \geq 2 . \tag{12}$$

Now suppose n is even, $n = 2m$ $(m \geq 2)$, and the quadrature rule (10) has degree of exactness n. Then, a fortiori, it has degree of exactness $2m-1$, and hence by Bernstein's inequality,

$$\frac{\Gamma(\alpha+1)}{n} \leq \lambda_1^{(m)} . \tag{13}$$

If (13) is violated, then Chebyshev quadrature in (10) is impossible. By virtue of (12), this will be the case if

$$\frac{1}{n} \geq \frac{1}{1 + 4(m-1)^2/(\alpha+1)} , \qquad m \geq 2 .$$

Since $n = 2m$, the last inequality amounts to $n^2 - (\alpha+5)n + \alpha + 5 \geq 0$, $n \geq 4$, that is, to

(14) $$n \geq \tfrac{1}{2}\{\alpha + 5 + \sqrt{(\alpha+1)(\alpha+5)}\} \quad \text{and} \quad n \geq 4 .$$

For all even values of n satisfying (14), therefore, the Chebyshev formula (10) does not exist. A similar argument applies for n odd, and also for Chebyshev-type quadratures (10) of given degree of exactness $< n$ (Gautschi [1975]).

3.4 Methods based on moment sequences. We already observed in (5) that for (4) to be a Chebyshev quadrature formula, it is necessary and sufficient that the nodes be real and

$$s_r = \frac{n}{\mu_0}\,\mu_r \quad , \quad r = 1,2,\ldots,n,$$

where $s_r = \sum_{k=0}^{n} x_k^r$ are the power sums in the nodes $x_k = x_k^{(n)}$. Any general property valid for power sums s_r in real variables thus immediately translates into a property for the moments μ_r , $r = 1,2,\ldots,n$, which in turn represents a necessary condition for (4) to be a Chebyshev quadrature rule. Violation of this property implies nonexistence of (4).

One such property, used by Wilf [1961], is Jensen's inequality, which states that for nonnegative numbers, $\xi_k \geq 0$, the quantities $\sigma_r = (\sum_{k=1}^{n} \xi_k^r)^{1/r}$ are nonincreasing in r for $r > 0$, i.e., $\sigma_r \geq \sigma_s$ whenever $0 < r < s$ (Hardy, Littlewood and Pólya [1952, p.28]). Consequently, if all $x_k \geq 0$, then

(15) $$\tau_r = (\frac{n}{\mu_0}\,\mu_r)^{\frac{1}{r}} \text{ is nonincreasing for } r = 1,2,\ldots,n,$$

and if all x_k are arbitrary real,

(15*) $$\tau_r^* = (\frac{n}{\mu_0}\,\mu_{2r})^{\frac{1}{r}} \text{ is nonincreasing for } r = 1,2,\ldots,[\tfrac{n}{2}] .$$

Tureckiĭ [1962] and, subsequently, Janovič [1971] and Nutfullin and Janovič [1972] use the more obvious inequalities

$$s_n \leq s_{n-2r}s_{2r} \qquad (n \text{ even}; \quad r = 1,2,\ldots,\tfrac{n}{2}),$$
$$s_{n-1} \leq s_{n-1-2r}s_{2r} \quad (n \text{ odd} ; \quad r = 1,2,\ldots,\tfrac{n-1}{2}) ,$$

valid for arbitrary real x_k, to obtain the necessary conditions

(16) $$\begin{cases} \dfrac{\mu_n}{n\mu_{n-2r}} \leq \dfrac{\mu_{2r}}{\mu_0} & (n \text{ even}; \; r = 1,2,\ldots,\tfrac{n}{2}) , \\[2ex] \dfrac{\mu_{n-1}}{n\mu_{n-1-2r}} \leq \dfrac{\mu_{2r}}{\mu_0} & (n \text{ odd} ; \; r = 1,2,\ldots,\tfrac{n-1}{2}) . \end{cases}$$

To illustrate (15), consider again the Laguerre weight $x^{\alpha} e^{-x}$, $\alpha > -1$, for which $\mu_r = \Gamma(\alpha+r+1)$, $r = 0,1,2,\dots$. Requiring nonnegative nodes x_k, we can apply (15), i.e., $\tau_{r-1} \geq \tau_r$ for $2 \leq r \leq n$, which, for $r = n$, gives

$$\left(\frac{n}{\mu_0}\, \mu_{n-1}\right)^{\frac{1}{n-1}} \geq \left(\frac{n}{\mu_0}\, \mu_n\right)^{\frac{1}{n}} ,$$

or, equivalently,

$$\frac{n}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+n+1)}{(\alpha+n)^n} \geq 1 . \tag{17}$$

Since the left-hand side is asymptotically equal to $\sqrt{2\pi}e^{-\alpha}\,[\Gamma(\alpha+1)]^{-1} n^{\alpha+3/2} e^{-n}$ as $n \to \infty$, it is clear that (17) will be false for all n sufficiently large, hence Chebyshev quadrature (in the strict sense) not possible.

Similarly, putting $r = 1$ in the first of (16), we find the necessary condition (Tureckiĭ [1962])

$$n^2 - (\alpha^2+\alpha+3)n + \alpha(\alpha-1) \leq 0 , \qquad n(\text{even}) \geq 4 ,$$

which leads to a nonexistence result similar to, but not as sharp as, the one obtained in (14) by Bernstein's method.

4. Chebyshev quadrature and Gaussian quadrature

We noted before in §3(2) that the Gauss quadrature formula for $d\mu(x) = (1-x^2)^{-\frac{1}{2}}dx$ on $[-1,1]$ is also a Chebyshev formula. One naturally wonders whether there are other Gauss-type quadrature formulas whose coefficients are all equal. The question was settled negatively at a surprisingly early stage (Posse [1875], Sonin [1887], Krawtchouk [1935], Bailey [1936]). An elegant proof of a rather more far-reaching result is due to Geronimus [1944], [1946] (and also reproduced in Krylov [1962, p.183ff] and Natanson [1965, p.150 ff]).

Let $d\mu(x)$ be a measure which admits a set of orthogonal polynomials, $\{\pi_n(x;\, d\mu)\}_{n=0}^{\infty}$. Positivity of the measure neednot be assumed. Let $\{\xi_k^{(n)}\}_{k=1}^{n}$ be the zeros of $\pi_n(x;\, d\mu)$, and consider

$$\int_a^b f(x)d\mu(x) = \frac{\mu_0}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) + R_n(f) . \tag{1}$$

Then Geronimus proves the following: *If for each* $n = 1,2,3,\dots$ *we have* $R_n(f) = 0$ *whenever* $f(x) = x$ *and* $f(x) = x^2$ *(if* $n > 1$*), then* $d\mu$ *is the Chebyshev measure* $d\mu(x) = (1-x^2)^{-\frac{1}{2}}dx$ *, except for a linear transformation.*

The proof can be sketched in a few lines. Introducing the power means in the nodes $\xi_k^{(n)}$,

$$m_r^{(n)} = \left(\frac{1}{n}\sum_{k=1}^{n} [\xi_k^{(n)}]^r\right)^{\frac{1}{r}} ,$$

the hypothesis implies

$$m_1^{(n)} = \frac{\mu_1}{\mu_0} = m_1 , \quad m_2^{(n)} = \left(\frac{\mu_2}{\mu_0}\right)^{\frac{1}{2}} = m_2 ,$$

that is, $m_1^{(n)}$ and $m_2^{(n)}$ are <u>independent of</u> n. Assuming the polynomials $\pi_n(x) = \pi_n(x;\, d\mu)$ normalised to have leading coefficients 1, we have on the one hand, by Newton's identities, that

(2) $$\pi_n(x) = x^n - nm_1x^{n-1} + \frac{n}{2}(nm_1^2 - m_2^2)x^{n-2} - \dots ,$$

and on the other, that

(3) $$\begin{cases} \pi_n(x) = (x-\alpha_n)\pi_{n-1}(x) - \beta_n\pi_{n-2}(x), & n = 1,2,3,\dots, \\ \pi_{-1} = 0 , \quad \pi_0 = 1 \end{cases}$$

for some constants α_n, β_n. Inserting (2) into (3), and comparing coefficients of x^{n-1} and x^{n-2} on either side, gives

$$\alpha_n = \alpha , \qquad n = 1,2,3,\dots,$$

$$\beta_n = \beta , \qquad n = 3,4,5,\dots, \quad \beta_2 = 2\beta ,$$

where

$$\alpha = m_1 , \quad \beta = \tfrac{1}{2}(m_2^2 - m_1^2) .$$

It then follows from (3) that

$$\pi_n(x) = \left(\frac{x-\alpha+\sqrt{(x-\alpha)^2-4\beta}}{2}\right)^n + \left(\frac{x-\alpha-\sqrt{(x-\alpha)^2-4\beta}}{2}\right)^n ,$$

which is essentially the Chebyshev polynomial of the first kind, $T_n(x) = \frac{1}{2}\{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\}$, except for a linear transformation in the independent variable and a numerical factor (cf. Rivlin [1974, p.5]).

If the measure $d\mu$ is positive, then all $\xi_k^{(n)}$ are real and $|m_1| < m_2$ by the well-known monotonicity of the power mean $m_r^{(n)}$ as a function of r. It then follows that $\beta > 0$.

5. <u>Existence and nonexistence results</u>

Given a positive measure $d\mu(x)$ on (a,b), we say that Chebyshev quadrature is possible for n [in the strict sense] if there exist n real numbers $x_k^{(n)}$ [pairwise distinct in (a,b)] such that

$$(1) \qquad \int_a^b f(x)d\mu(x) = \frac{\mu_0}{n} \sum_{k=1}^{n} f(x_k^{(n)}) + R_n(f)$$

has algebraic degree of exactness n. The finite or infinite sequence $\{n_j\}$ of all those integers $n_j \geq 1$ for which Chebyshev quadrature is possible will be called the T-sequence of $d\mu(x)$. It will be denoted by $T(d\mu)$, or simply by T. We say that the measure $d\mu(x)$ has property T if its T-sequence consists of all natural numbers, property T^∞, if its T-sequence is infinite, and property T^0, if it is finite. In this terminology, Bernstein's result may be rephrased by saying that the uniform measure $d\mu(x) = dx$ on $[-1,1]$ has the T-sequence $T = \{1,2,3,4,5,6,7,9\}$, hence property T^0. The Chebyshev measure $d\mu(x) = (1-x^2)^{-\frac{1}{2}}dx$, on the other hand, has property T.

Bernstein's method, as well as the methods based on moment sequences (cf. §3.3, 3.4) yield necessary conditions for $d\mu(x)$ to have property T^∞, hence, by default, also proofs for property T^0.

5.1 Measures with property T or T^∞. Measures $d\mu(x)$ with property T are rare; in fact, they occur with probability zero, if viewed as moment sequences in appropriate moment spaces (Salkauskas [1975]). Up until Ullman's discovery (cf. §3.1), Chebyshev's measure indeed was the only known measure with property T. Geronimus [1969] continues Ullman's work by first establishing an interesting sufficient condition for Chebyshev quadrature to be possible for n. To describe it, let $d\mu(x) = \omega(x)dx$ on $[-1,1]$, and assume

$$\omega(\cos\theta) = \frac{1}{\pi \sin\theta} \sum_{k=0}^{\infty} a_k \cos k\theta, \quad 0 \leq \theta \leq \pi, \quad a_0 = 1.$$

Define the constants $\{A_m^{(n)}\}_{m=0}^{\infty}$ by

$$\exp\left(-n \sum_{k=1}^{\infty} \frac{a_k}{kz^k}\right) = \sum_{m=0}^{\infty} \frac{A_m^{(n)}}{z^m}, \quad A_0^{(m)} = 1, \quad |z| > 1.$$

Then Chebyshev quadrature is possible for n if the polynomial $\sum_{m=0}^{n-1} A_m^{(n)} z^m + \frac{1}{2}A_n^{(n)} z^n$ has all its zeros in $|z| > 1$. In this case, moreover,

$$2^{n-1} p_n(x;d\mu) = \sum_{m=0}^{n-1} A_m^{(n)} \cos(n-m)\theta + \tfrac{1}{2}A_n^{(n)}, \quad x = \cos\theta.$$

Ullman's measure with property T falls out as a simple example, by taking

$a_k = (-a)^k$. Geronimus also gives several examples of even weight functions $\omega(x)$ admitting Chebyshev quadratures for all even integers $n = 2\nu$. (These automatically have degree of exactness $2\nu + 1$.)

A measure $d\mu(x)$ on $(-\infty, \infty)$ with infinite support (i.e., with positive mass outside of every finite interval) cannot have property T^∞ unless its T-sequence contains very large gaps. For example, if $\{2\nu_j\}$ is the even subsequence of $T(d\mu)$, and m any fixed integer, then one has $\nu_j > \nu_{j-1}^{\ m}$ for infinitely many j (Wilf [1961]). Similarly for the odd subsequence. It follows, in particular, that a measure with property T necessarily has finite support. Wilf in fact conjectures that property T^∞ already implies finite support. This, however, is disproved by Ullman [1962], [1963], who in turn poses the question (still open) of formulating criteria in terms of the gaps of an infinite T-sequence, which would allow to discriminate between measures with infinite, and measures with finite, support.

Kahaner and Ullman [1971] establish conditions on the measure $d\mu(x)$ on $(-\infty, \infty)$ which either imply the absence of property T, or property T^o. The conditions involve the limit behaviour (as $n \to \infty$) of certain discrete measures concentrated at the zeros of the orthogonal polynomials $\pi_n(x;d\mu)$.

5.2 Chebyshev quadrature on finite intervals. Soon after Bernstein obtained his classical result, Akhiezer [1937], in a little-known paper, proved that the Jacobi measure $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$ on $[-1,1]$ has property T^o whenever $-\frac{1}{2} \le \alpha \le \frac{1}{2}$, $-\frac{1}{2} \le \beta \le \frac{1}{2}$ (excepting $\alpha = \beta = -\frac{1}{2}$). More recently, using Bernstein's method, Gatteschi [1963/64] proves property T^o for all $\alpha = \beta > -\frac{1}{2}$, while Ossicini [1966] extends it to $\alpha > -\frac{1}{2}$, $\beta > -1$, hence, by symmetry, also to $\alpha > -1$, $\beta > -\frac{1}{2}$. In the remaining square $-1 < \alpha \le -\frac{1}{2}$, $-1 < \beta \le -\frac{1}{2}$ (with $\alpha = \beta = -\frac{1}{2}$ deleted), the matter appears to be still unsettled.

Greenwood and Danford [1949] consider the integral $\int_0^1 xf(x)dx$ (which amounts to Jacobi's case $\alpha = 0$, $\beta = 1$) and find by computation that Chebyshev quadrature is possible (in the strict sense) if $n = 1,2,3$, but not if $4 \le n \le 10$. A similar result is stated in Greenwood, Carnahan and Nolly [1959] for the integral $\int_{-1}^1 x^2 f(x)dx$ (which can be reduced to the case $\alpha = 0$, $\beta = \frac{1}{2}$). The exact T-sequence has not been established in either case.

5.3 Chebyshev quadrature on infinite intervals. Computational results of Salzer [1955] suggested that the T-sequence for the Laguerre measure $d\mu(x) = e^{-x}dx$ on $(0,\infty)$, as well as the one for the Hermite measure $d\mu(x) = e^{-x^2}$ on $(-\infty,\infty)$, must be rather short, in fact $T = \{1,2\}$ in the former, and $T = \{1,2,3\}$ in the latter case. This was first proved by Krylov [1958], by an application of Bernstein's method, and again later, independently, by Gatteschi [1964/65]. Burgoyne [1963], unaware of Krylov's result, confirms it up to $n = 50$ by computing the maximum number of nonnegative, resp. real, nodes. For more general Laguerre measures $d\mu(x) = x^\alpha e^{-x}$, $\alpha > -1$, property T^0 is proved by Wilf [1961], Tureckiĭ [1962] and Gautschi [1975], using methods already illustrated in §3.3, 3.4.

Nutfullin and Janovič [1972], using the method of Tureckiĭ, prove property T^0 for the measures

$$d\mu(x) = (x^{2p+1}/\sinh \pi x)dx , \quad p = 0,1,2,\ldots,$$

$$d\mu(x) = (x^{2p}/\cosh \pi x)dx , \quad p = 0,1,2,\ldots,$$

and

$$d\mu(x) = |x|^\alpha e^{-x^2} dx , \quad \alpha > -1 ,$$

all on $(-\infty,\infty)$, and for each give an upper bound for $\max_{n_j \in T(d\mu)} n_j$. They also determine the T-sequence for some of these measures. For example, the first, when $p = 0$, has $T = \{1,2,3\}$, while the last has $T = \{1,2,3\}$ for $-1 < \alpha < 1/3$, $T = \{1,2,3,5\}$ for $1/3 \leq \alpha < 1$, $T = \{1,2,3,4,5\}$ for $1 \leq \alpha \leq 7$, $T = \{1,2,3,4\}$ for $7 < \alpha < 15$. For $\alpha \geq 15$, Chebyshev quadrature is possible when $n = 1,2,3,4,6$ but the exact T-sequence is not known.

Janovič [1971] previously used Tureckiĭ's method to show that a certain measure $d\mu(x)$ on $(0,\infty)$, of interest in the theory of Wiener integrals, has $T = \{1,2\}$.

5.4 Chebyshev-type quadrature. If a measure $d\mu(x)$ has property T^0, and $p_n = p_n(d\mu)$ is the maximum degree of exactness of (1), subject to the reality of all nodes, it becomes of interest to determine upper bounds for p_n as $n \to \infty$. In the classical case $d\mu(x) = dx$, Bernstein [1937] already showed that $p_n < 4\sqrt{n}$. For Jacobi measures $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$, Costabile [1974] establishes $p_n < c(\alpha,\beta)n^{1/(2\alpha+2)}$, as has previously been found by Meir and Sharma [1967] in the ultraspherical case $\alpha = \beta > -\frac{1}{2}$. In this latter case, Costabile further expresses the constant c explicitly in terms of gamma and Bessel functions. For more general weight functions on $[-1,1]$, having branch point and other singularities at the endpoints, the problem is studied extensively by Geronimus [1969], [1970]. For the Laguerre measure $d\mu(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, one finds by Bernstein's method that $p_n < 2 + \sqrt{(\alpha+1)(n-1)}$ if $p_n \geq 3$. Similar bounds hold for symmetric Hermite quadrature rules (Gautschi [1975]; see also Tureckiĭ [1962]).

Chebyshev-type quadratures having degree of exactness 1 always exist. The most familiar example is the composite midpoint rule on $[-1,1]$, with $d\mu(x) = dx$. Another example is the nontrivial extension of the midpoint rule to integrals with arbitrary positive measure, due to Stetter [1968b], which improves upon an earlier extension of Jagerman [1966].

6. Optimal Chebyshev-type quadrature formulas

Only relatively recently have attempts been made to develop Chebyshev-type quadrature formulas in cases where true Chebyshev formulas do not exist. The approach generally consists in replacing the algebraic exactness condition by some optimality condition, unconstrained or constrained. This yields new formulas even in cases where ordinary ones exist.

6.1 Optimal formulas in the sense of Sard. For the classical weight $d\mu(x) = dx$ on $[-1,1]$, consider a Chebyshev-type quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{2}{n}\sum_{k=1}^{n} f(x_k) + R_n(f) . \tag{1}$$

We require that (1) has polynomial degree of exactness $p < n$,

$$R_n(f) = 0 , \quad \text{all } f \in \mathbb{P}_p , \tag{2}$$

and assume $f \in AC^p[-1,1]$. The remainder $R_n(f)$, as is well known (see, e.g., Sard [1963, p.25]), can then be represented in the form

$$R_n(f) = \int_{-1}^{1} K_p(t) f^{(p+1)}(t)dt ,$$

where $K_p(t) = K_p(t;\ x_1,x_2,\ldots,x_n)$ is the Peano kernel of R_n [cf. §7(3)]. By the Schwarz inequality, therefore,

$$|R_n(f)| \leqslant \gamma_p \|f^{(p+1)}\|_{L_2} , \qquad \gamma_p = \|K_p\|_{L_2} , \tag{3}$$

where $\|u\|_{L_2} = \left(\int_{-1}^{1}[u(t)]^2dt\right)^{\frac{1}{2}}$. An optimal Chebyshev-type formula in the sense of Sard is a formula (1), satisfying (2), which minimizes γ_p as a function of $x_1,x_2,\ldots,x_n$. Franke [1971] studies such formulas in the cases $p = 0$ and $p = 1$, under the additional assumption of symmetry,

$$x_{n+1-k} + x_k = 0 , \quad k = 1,2,\ldots,n. \tag{4}$$

The condition (2) is then automatically satisfied, so that the problem reduces to an unconstrained optimization problem. The solution for $p = 0$, as has been noted previously (Krylov [1962, pp.138-140]), is the composite midpoint rule, for which $\gamma_0 = 2/3n^2$. In the case $p = 1$, numerical answers are given for $2 \leqslant n \leqslant 11$. A similar problem, without the symmetry assumption (4), is considered in Coman [1970].

6.2 Least squares criteria. Instead of minimizing γ_p in (3), we may wish to minimize the errors of (1) which result if the formula is applied to successive monomials. More precisely, given an integer p , with $0 \le p < n$, and an integer q , with $q \ge n$, or $q = \infty$, we determine the nodes x_k in (1) such that

$$\sum_{j=p+1}^{q} [R_n(x^j)]^2 = \min , \tag{5}$$

subject to

$$R_n(x^j) = 0 , \quad j = 1,2,\dots,p . \tag{6}$$

Symmetry, as in (4), may or may not be imposed.

If $n \le 7$, or $n = 9$, and $q = n$, Problem (5), (6) is trivially solved by the classical Chebyshev formulas, which drive the objective function in (5) to zero. In the case $p = 0$, and for various choices of q , including $q = \infty$, numerical answers are given by Barnhill, Dennis and Nielson [1969] for $n = 8,10,11$. Kahaner [1970] has analogous results for $q = n$ and $p = n - 1$ or $n - 2$. An interesting (although somewhat counterproductive) feature of this work is the apparent necessity of assuming repeated nodes for the minimization procedures to converge. It is shown in Gautschi and Yanagiwara [1974] that repeated nodes are indeed unavoidable, if $q = n$, whenever the constraints in (6) admit real solutions. The same is proved in Salkauskas [1973] for the case $p = 0$, all nodes being constrained to the interval $[-1,1]$. We conjecture that the same situation prevails for arbitrary $q > n$.

There is computational evidence that the optimal formulas are indeed symmetric, but the question remains open. If we knew that Problem (5), (6) had a unique solution, modulo permutations, symmetry would follow (Gautschi and Yanagiwara [1974]).

6.3 Minimum norm quadratures. A quadrature rule, such as (1), which minimizes the norm of the error functional $R_n(f)$ in some appropriate function space is called a minimum norm quadrature formula. For Chebyshev quadratures, such formulas are studied by Rabinowitz and Richter [1970]. They consider two families of Hilbert spaces. Each space consists of functions which are analytic in an ellipse $\mathcal{E}_\rho$, $\rho > 1$, having foci at ± 1 and semiaxes summing up to ρ . ($\{\mathcal{E}_\rho\}$ is a family of confocal ellipses, which as $\rho \to 1$ shrink to the interval $[-1,1]$, and as $\rho \to \infty$ inflate into progressively more circle-like regions invading the whole complex plane.) The first space, $L^2[\mathcal{E}_\rho]$, contains functions f for which $\iint_{\mathcal{E}_\rho} |f(z)|^2 dxdy < \infty$, and is equipped with the inner product $(f,g) = \iint_{\mathcal{E}_\rho} f(z)\overline{g(z)}dxdy$. The second, $H^2[\mathcal{E}_\rho]$, consists of functions f with $\int_{\partial\mathcal{E}_\rho} |f(z)|^2 |1-z^2|^{-\frac{1}{2}} |dz| < \infty$ and carries the inner product $\int_{\partial\mathcal{E}_\rho} f(z)\overline{g(z)}|1-z^2|^{-\frac{1}{2}} |dz|$.

The norm of $R_n(f)$, in each of these spaces, can be expressed explicitly in terms of the respective orthonormal bases. Thus, in $L^2[\mathcal{E}_\rho]$,

$$\|R_n\| = \frac{4}{\pi} \sum_{j=0}^{\infty} \left[\frac{j+1}{\rho^{2j+2} - \rho^{-2j-2}} R_n(U_j)\right]^2 , \tag{7}$$

where U_j are the Chebyshev polynomials of the second kind, and in $H^2[\mathcal{E}_\rho]$,

$$\|R_n\| = \frac{2}{\pi} \sum_{j=0}^{\infty}{}' \left[\frac{1}{\rho^{2j} + \rho^{-2j}} R_n(T_j)\right]^2 , \tag{8}$$

where T_j are the Chebyshev polynomials of the first kind. (The prime indicates that the term with $j = 0$ is to be halved.) It is shown by Rabinowitz and Richter that there exists a set of nodes x_k in $[-1,1]$ for which (7), and one for which (8), is a minimum, regardless of whether the weight in the quadrature rule is fixed to be $2/n$, as in (1), or whether it is treated as a free parameter. Numerical results given by Rabinowitz and Richter suggest that the optimal nodes are mutually distinct for each $\rho > 1$, but this remains a conjecture.

Rabinowitz and Richter also investigate the behaviour of the optimal Chebyshev-type rules in the limit cases $\rho \to 1$ and $\rho \to \infty$. In the former case, the limit behaviour is somewhat bizarre, and we shall not attempt to describe it here. In the latter case, it follows from (7), (8) that, both in $L^2[\mathcal{E}_\rho]$ and $H^2[\mathcal{E}_\rho]$, the optimal rule must be such that it integrates exactly as many monomials as possible, and gives minimum error for the first monomial which cannot be integrated exactly. Thus,

$$\begin{cases} R_n(x^j) = 0 , \quad j = 0,1,2,\dots,p, \quad p = \max(= p_n) , \\ |R_n(x^{p+1})| = \min . \end{cases} \tag{9}$$

We call the corresponding quadrature rules briefly E-optimal. Numerical results given by Rabinowitz and Richter for $n = 8,10,11,12,13$ show again the presence of repeated nodes.

6.4 E-optimal quadratures. An algebraic study of E-optimal Chebyshev-type quadrature rules is made in Gautschi and Yanagiwara [1974] for $n = 8,10,11,13$, and in Anderson and Gautschi [to appear] for general n. One of the key results of this work reveals that an E-optimal n-point Chebyshev-type formula can have at most p_n distinct nodes, whenever $p_n < n$. It follows from this immediately that some of the nodes must be repeated. The (generally distinct) p_n optimal nodes are found among the real solutions of systems of algebraic equations of the type

$$\sum_{r=1}^{p} \nu_r x_r^{\,j} = s_j , \qquad j = 1,2,\dots,p, \tag{10}$$

where ν_r are integers with $\nu_1 + \nu_2 + \dots + \nu_p = n$ and $p = p_n$ an integer generally not known a priori (cf. Eq.(9)). Finding all real solutions of such systems is a challenging computational problem. It is solved in the cited references for $n \leq 17$ by a reduction to single algebraic equations. For other techniques, see also Yanagiwara and Shibata [1974] and Yanagiwara, Fukutake and Shibata [1975]. A summary of results is given below in Table 1, where crosses indicate the availability of E-optimal Chebyshev formulas, zeros the nonexistence of Chebyshev-type quadrature formulas, and question marks unsettled cases.

p \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23...
2[n/2]+1	X	X	X	X	X	X	X	0	X	0	0	0	0	0	0	0	0	0	0	0	0	0	0...
2[n/2]-1								X		X	X	0	X	0	0	0	0	0	0	0	0	0	0...
2[n/2]-3												X		X	X	0	X	0	0	0	0	0	0...
2[n/2]-5																X		?	?	?	?	0	0...

Table 1. Existence and nonexistence of n-point Chebyshev-type quadrature formulas of degree of exactness p

E-optimal formulas have been obtained also for infinite and semi-infinite intervals involving weight functions of the Hermite and Laguerre type (Anderson and Gautschi [to appear]). The confluence of nodes is rather more severe in these cases. For example, in the Laguerre case, when $3 \leq n \leq 6$, there are only two distinct nodes, one being simple, the other having multiplicity $n - 1$.

7. Error and convergence

7.1 The remainder term. Remainder terms in Chebyshev-type quadratures are generally ignored, except for the classical formulas

$$(1) \qquad \int_{-1}^{1} f(x)dx = \frac{2}{n}\sum_{k=1}^{n} f(x_k^{(n)}) + R_n(f), \qquad n = 1,2,\dots,7,9,$$

and for the Gauss-Chebyshev formula (with $d\mu(x) = (1-x^2)^{-\frac{1}{2}}dx$).

Each of the formulas (1) has polynomial degree of exactness $p = 2[n/2]+1$, that is, $p = n$ if n is odd, and $p = n+1$ if n is even. Assuming $f \in C^{p+1}[-1,1]$, we obtain from Peano's theorem (see, e.g., Davis [1963, p.70])

$$(2) \qquad R_n(f) = \int_{-1}^{1} K_p(t) f^{(p+1)}(t)\,dt ,$$

where $K_p(t)$ is the Peano kernel of the functional $R_n(f)$,

(3) $$K_p(t) = \frac{1}{p!}\left\{\frac{(1-t)^{p+1}}{p+1} - \frac{2}{n}\sum_{k=1}^{n}(x_k^{(n)} - t)_+^p\right\},$$

with

$$u_+^p = \begin{cases} u^p & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases} \quad p \geq 0 .$$

Ghizzetti and Ossicini [1970], and Kozlovskiĭ [1971], give different proofs of the fact that the Peano kernel is positive,

(4) $$K_p(t) \geq 0 \quad \text{on} \quad [-1,1] .$$

From (2), it then follows that

(5) $$R_n(f) = \kappa_n f^{(p+1)}(\tau) , \qquad -1 \leq \tau \leq 1 ,$$

where

(6) $$\kappa_n = \int_{-1}^{1} K_p(t)dt = R_n\left[\frac{x^{p+1}}{(p+1)!}\right] , \quad p = 2[\tfrac{n}{2}]+1 .$$

Numerical values of the constants κ_n for $n = 1(1)7$ and $n = 9$ can be found in Ghizzetti and Ossicini [1970, pp.129-130]. (They have previously been tabulated by Berezin and Židkov [1965, p.262], but with an incorrect value of κ_9.)

The remainder in the Gauss-Chebyshev quadrature formula has been estimated by a number of writers; see, e.g., Stetter [1968a], Chawla and Jain [1968], Chawla [1969], Riess and Johnson [1969], Chui [1972], Jayarajan [1974].

For E-optimal quadrature rules of the type (1), the remainder $R_n(f)$ is analysed by Anderson [1974].

7.2 Convergence of Chebyshev quadrature formulas. In order to study convergence of the classical Chebyshev quadrature formulas, one must, of course, allow for complex nodes. From the known distribution of the nodes in the complex plane (cf. §2.2) it follows easily from Runge's theorem that convergence is assured for functions which are analytic in a closed domain $\mathcal{D}$ containing the curve of logarithmic potential, §2.2(1), in its interior (Kahaner [1971]). Convergence, in fact, is geometric for $\mathcal{D}$ sufficiently large.

8. Miscellaneous extensions and generalizations of Chebyshev quadrature

There are many variations on the theme of Chebyshev quadrature. A natural thing to try, e.g., is to relax the rigid requirement of equal coefficients and merely seek to minimize some measure of the variance in the coefficients. The problem, first suggested by Ostrowski [1959], is discussed by Kahaner [1969] and

Salkauskas [1971].

A more substantial modification is made by Erdős and Sharma [1965], and Meir and Sharma [1967], who associate equal coefficients only with part of the nodes and leave the coefficients for the remaining nodes, as well as the nodes themselves, variable. Even with this modification, provided the number of variable coefficients is kept fixed, and the polynomial degree of exactness maximized, some of the nodes again turn complex as n, the total number of nodes, becomes large. Erdős and Sharma show this for the measure $d\mu(x) = dx$ on $[-1,1]$, and Meir and Sharma for the ultraspherical measure $d\mu(x) = (1-x^2)^{\alpha}dx$, $\alpha > -\frac{1}{2}$. The maximum polynomial degree of exactness, p_n, subject to the reality of all nodes, when $d\mu(x) = dx$, in fact obeys the law $p_n = O(\sqrt{n})$ familiar from Bernstein's theory of the classical Chebyshev quadratures. For Jacobi measures $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha > -\frac{1}{2}$, $\beta > -1$, Gatteschi, Monegato and Vinardi [to appear] associate variable coefficients with fixed nodes at ± 1, and equal coefficients with the remaining nodes, and for this case, too, establish the impossibility of n-point Chebyshev quadrature for n sufficiently large.

For quadrature sums involving derivative values as well as function values, the natural extension of Chebyshev's problem would be to require equal coefficients for all derivative terms involving the same order derivative. The problem, as far as we know, has not been treated in any great detail, although it is briefly mentioned by Ghizzetti [1954/55] (see also Ghizzetti and Ossicini [1970, p.43ff]).

Chebyshev quadrature rules integrating exactly trigonometric, rather than algebraic, polynomials are considered by Keda [1962] and Rosati [1968]. Rosati includes derivative terms in his quadrature sums.

Equally-weighted quadrature rules for integration in the complex plane are developed by Salzer [1947] in connection with the inversion of Laplace transforms.

An extension of Chebyshev quadrature to double and triple integrals is discussed by Georgiev [1953]. Coman [1970] derives optimal Chebyshev-type formulas in two dimensions.

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Note added in proof. In addition to the references Ghizzetti and Ossicini [1970], Kozlovskiĭ [1971] in §7.1, mention should be made of the paper T. Popoviciu, "La simplicité du reste dans certaines formules de quadrature", Mathematica (Cluj) 6 (29) (1964), 157-184 {MR32 #4848}, in which the remainder is studied not only of the classical Chebyshev quadrature rule, but also of the Chebyshev-Laguerre and Chebyshev-Hermite formulas obtained by Salzer [1955].