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ON THE ZEROS OF POLYNOMIALS ORTHOGONAL ON THE SEMICIRCLE*

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Abstract. It is shown that the polynomials $\pi_n(\cdot; w)$ orthogonal in the sense of [W. Gautschi, H. J. Landau, and G. V. Milovanović, *Constr. Approx.*, 3 (1987), pp. 389-404] on the unit upper semicircle need not necessarily have all their zeros in the interior of the unit upper semidisc, not even for weight functions w that are symmetric, $w(-z)=w(z)$. A symmetric weight function w_a (depending on a parameter a) is exhibited, which has the property that $\pi_n(\cdot; w_a)$ for any fixed even n has a zero on the imaginary axis with imaginary part greater than one, provided a is large enough. Similarly, a weight function w^a is constructed for which the analogous property holds for $\pi_n(\cdot; w^a)$, n odd.

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1. In [3], [4] we introduced polynomials that are orthogonal on the semicircle with respect to the (non-Hermitian) inner product

$$(p, q)=\int_0^\pi p(e^{i\theta})q(e^{i\theta})w(e^{i\theta})\,d\theta. \tag{1.1}$$

Here, w is a "weight function" analytic on the semidisc $D_+=\{z\in\mathbb{C}: |z|<1, \operatorname{Im} z>0\}$, nonnegative on $(-1,1)$ and integrable over ∂D_+. We have shown that under the assumption

$$\operatorname{Re}\int_0^\pi w(e^{i\theta})\,d\theta\neq 0 \tag{1.2}$$

there exists a unique system $\{\pi_n\}_{n=0}^\infty$ of monic polynomials $\pi_n(\cdot)=\pi_n(\cdot; w)$ such that

$$\deg \pi_n=n,\quad n=0,1,2,\cdots,\qquad (\pi_k,\pi_l)\begin{cases}=0, & k\neq l,\\ \neq 0, & k=l.\end{cases} \tag{1.3}$$

They possess many of the properties familiar from orthogonal polynomials on the real line, such as satisfying a three-term recurrence relation and a second-order linear differential equation (for special weight functions), and in fact can be expressed as (complex) linear combinations of two successive polynomials orthogonal on the interval $(-1,1)$ with respect to the same weight function w. They give rise to Gauss-type quadrature rules for integration over the semicircle and to new, possibly more stable, quadrature formulae for evaluating Cauchy principal value integrals (see [3, §§ 7, 8]). Since the nodes of these quadrature rules involve the zeros of the polynomials π_n in (1.3), a study of the qualitative properties of these zeros is of interest.

In [4] we have shown that for weight functions analytic in $D=\{z\in\mathbb{C}: |z|<1\}$, symmetric in the sense

$$w(-z)=w(z)\quad\text{for all } z\in D, \tag{1.4}$$

and satisfying

$$w(x)\geqq 0\quad\text{on } (-1,1),\quad w(0)>0, \tag{1.5}$$

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all zeros of π_n are contained in D_+ with the possible exception of a single (simple) zero iy, $y \geqq 1$. For the Gegenbauer weight $w(z)=(1-z^2)^{\lambda-1/2}$, the exceptional case can only arise if $n=1$ and $-\frac{1}{2}<\lambda \leqq 0$. Likewise, no exceptional cases seem to occur for Jacobi weights $w(z)=(1-z)^\alpha(1+z)^\beta$, $\alpha>-1$, $\beta>-1$, if $n \geqq 2$, as was observed by numerical computation. We might be led to believe that this absence of exceptional cases prevails for arbitrary weight functions w. In this note we show, however, that this is not so, not even for symmetric weight functions. We exhibit symmetric functions w for which $\pi_n(\cdot; w)$, for arbitrary fixed n, has a zero iy with $y \geqq 1$.

2. Let $b_k = b_k(w)$, $k=1, 2, 3, \cdots$, be the coefficients in the recurrence formula

$$y_{k+1} = xy_k - b_k y_{k-1}, \quad k=0,1,2,\cdots, \qquad y_{-1}=0, \quad y_0=1 \tag{2.1}$$

satisfied by the polynomials $p_n(x; w)$ orthogonal on the interval $(-1, 1)$ relative to the symmetric weight function w. We recall from the proof of Theorem 6.5 and equations (5.2), (5.4) of [4] that iy is a zero of $\pi_n(\cdot; w)$ if and only if

$$\omega_n(y) - \theta_{n-1} = 0, \tag{2.2}$$

where

$$\omega_1(y) = y, \quad \omega_k(y) = y + \frac{b_{k-1}}{\omega_{k-1}(y)}, \quad k=2,3,\cdots, \tag{2.3}$$

$$\theta_{n-1} = \begin{cases} \dfrac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_{n-2}} \dfrac{\pi}{m_0}, & n \text{ even}, \\[2ex] \dfrac{b_2 b_4 \cdots b_{n-1}}{b_1 b_3 \cdots b_{n-2}} \dfrac{m_0}{\pi}, & n \text{ odd}, \end{cases} \tag{2.4}$$

and

$$m_0 = \int_{-1}^{1} w(x)\, dx = 2\int_0^1 w(x)\, dx, \tag{2.5}$$

the weight function w having been normalized to satisfy

$$w(0) = 1. \tag{2.6}$$

If $n=1$ or $n=2$, empty products in (2.4) are assumed to be one. Equation (2.2) holds for some $y \geqq 1$ if and only if

$$\omega_n(1) - \theta_{n-1} \leqq 0. \tag{2.7}$$

Indeed, since $\omega_n(y) \to \infty$ as $y \to \infty$, inequality (2.7) trivially implies (2.2) for some $y \geqq 1$. Conversely, if (2.2) holds for some $y \geqq 1$, but (2.7) (if $n \geqq 2$) does not, the left-hand side of (2.2), hence $\pi_n(iy)$, would have either two distinct zeros >1, or a double zero >1, which is impossible by Theorem 6.2 of [4]. By (2.3), we can write (2.7) in the form

$$1 + \frac{b_{n-1}}{1+}\,\frac{b_{n-2}}{1+} \cdots \frac{b_1}{1} \leqq \theta_{n-1}. \tag{2.8}$$

We now show that (2.8), for any fixed $n \geqq 1$, can always be achieved for some suitable weight function w.

3. It is necessary to distinguish the cases n even and n odd. In the former case, (2.8) becomes

$$1 + \frac{b_{n-1}}{1+}\,\frac{b_{n-2}}{1+} \cdots \frac{b_1}{1} \leqq \frac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_{n-2}} \frac{\pi}{m_0}. \tag{3.1}$$

It is clear that we can enforce (3.1) to hold if we can find a family of weight functions w for which m_0 tends to zero and the b_k remain bounded and bounded away from zero. Such a family of weight functions (keeping in mind that they should be analytic in D, satisfy (1.4), and be normalized by (2.6)) is given by

$$w(z) = w_a(z) = \frac{1+\sqrt{a/\pi}\, e^{-az^2}}{1+\sqrt{a/\pi}}, \qquad a>0. \tag{3.2}$$

The fact that w_a also satisfies (1.2) follows from Theorem 5.1 of [4]. We note that the second term in the numerator of (3.2), for real $z=x$, is an approximation to the Dirac delta function $\delta(x)$, to which it converges as $a\to\infty$. It follows that, for any polynomial p,

$$\begin{aligned}\left(1+\sqrt{\frac{a}{\pi}}\right)\int_{-1}^{1} w_a(x)p(x)\,dx &\to \int_{-1}^{1}[1+\delta(x)]p(x)\,dx\\ &= \int_{-1}^{1} p(x)\,dx + p(0) \quad \text{as } a\to\infty.\end{aligned} \tag{3.3}$$

In particular, putting $p(x)\equiv 1$,

$$m_0 \sim 3\sqrt{\pi/a}, \qquad a\to\infty. \tag{3.4}$$

Furthermore,

$$\lim_{a\to\infty} b_k(w_a) = b_{k,\infty} > 0, \qquad k=1,2,3,\cdots, \tag{3.5}$$

where $b_{k,\infty}$ are the recursion coefficients of the monic polynomials orthogonal with respect to the weight function $1+\delta(x)$ on $[-1,1]$ (Legendre weight plus Dirac function centered at the origin). It follows from (3.4) and (3.5) that for a sufficiently large, (3.1) will be true (even with strict inequality). The proof of (3.5) is deferred to § 4.

Assume next that n is odd. Then, (2.8) becomes

$$1+\frac{b_{n-1}}{1+}\,\frac{b_{n-2}}{1+}\cdots\frac{b_1}{1} \leqq \frac{b_2b_4\cdots b_{n-1}}{b_1b_3\cdots b_{n-2}}\,\frac{m_0}{\pi}. \tag{3.6}$$

We now want m_0 to be large and may choose, for example,

$$w(z) = w^a(z) = 1+az^2, \qquad a>0. \tag{3.7}$$

Then

$$m_0 = \tfrac{2}{3}a + 2 \tag{3.8}$$

and

$$\lim_{a\to\infty} b_k(w^a) = b_k^\infty > 0, \qquad k=1,2,3,\cdots, \tag{3.9}$$

where b_k^∞ are the recursion coefficients of the monic polynomials orthogonal with respect to the weight function x^2 on $[-1,1]$. Again, from (3.8) and (3.9) it follows that (3.6) will be true for a sufficiently large. It remains to prove (3.5) and (3.9).

4. We denote the moments of w by m_k,

$$m_{2r+1}=0, \qquad m_{2r} = 2\int_0^1 x^{2r}w(x)\,dx > 0. \tag{4.1}$$

The recursion coefficients $b_k(w)$ can be expressed in terms of Hankel determinants:

$$\Delta_n(m) = \det\,(m_{i+j})_{\substack{i=0,1,\cdots,n-1\\ j=0,1,\cdots,n-1}}, \qquad \Delta_0 = 1, \tag{4.2}$$

by means of [1, p. 19]

$$b_k(w)=\frac{\Delta_{k-1}(m)\Delta_{k+1}(m)}{[\Delta_k(m)]^2}, \qquad k=1,2,3,\cdots. \tag{4.3}$$

In the case of $w(x)=w_a(x)$ [cf. (3.2)], we have by (3.3)

$$m_r\sim\sqrt{\pi/a}\,m_{r,\infty}, \qquad a\to\infty, \quad r=0,1,2,\cdots, \tag{4.4}$$

where $m_{r,\infty}$ are the moments of the weight function $1+\delta(x)$ on $[-1,1]$. Therefore,

$$\Delta_n(m)\sim\left(\frac{\pi}{a}\right)^{n/2}\Delta_n(m_\infty), \qquad a\to\infty,$$

and, consequently, by (4.3),

$$b_k(w_a)\sim\frac{\Delta_{k-1}(m_\infty)\Delta_{k+1}(m_\infty)}{[\Delta_k(m_\infty)]^2}, \qquad a\to\infty,$$

that is,

$$b_k(w_a)\to b_{k,\infty} \quad \text{as } a\to\infty. \tag{4.5}$$

Likewise, for $w(x)=w^a(x)$ [cf. (3.7)],

$$m_r\sim am_r^\infty, \quad a\to\infty, \quad r=0,1,2,\cdots,$$

where m_r^∞ are the moments of the weight function x^2 on $[-1,1]$, and thus,

$$\Delta_n(m)\sim a^n\Delta_n(m^\infty), \qquad a\to\infty,$$

giving

$$b_k(w^a)\to b_k^\infty \quad \text{as } a\to\infty. \tag{4.6}$$

This proves the assertions in (3.5) and (3.9).

We remark that instead of one in (2.7) we could have selected any number larger than one, which means that the zeros of $\pi_n(\cdot\,;w_a)$ and $\pi_n(\cdot\,;w^a)$ on the imaginary axis can be made to have arbitrarily large imaginary parts by choosing a sufficiently large.

5. We now confirm the validity of the construction in § 3 numerically by computing the zeros of $\pi_n(\cdot\,;w_a)$, n even, and of $\pi_n(\cdot\,;w^a)$, n odd, for the critical value $a=a_n^*$ (which should yield a zero at i) and a few selected values $a>a_n^*$. We compute these zeros in terms of eigenvalues of a real tridiagonal (nonsymmetric) matrix, as indicated in [4, § 6.1], the coefficients $b_k(w_a)$ and $b_k(w^a)$ being generated by the "discretized Stieltjes procedure" (cf. [2, § 2.2]).

Table 5.1 shows the values of a_n^* for $n=2(1)10$ obtained to eight significant decimal digits by using the bisection method on (2.2) where $y=1$. The zeros of $\pi_n(\cdot\,;w_a)$

TABLE 5.1
Values of a_n^* for $n=2(1)10$.

n	a_n^*	n	a_n^*
2	55.274946	3	17.009652
4	250.25427	5	46.413430
6	798.58573	7	89.537192
8	1951.2926	9	146.34390
10	4037.4957		

TABLE 5.2
Zeros of $\pi_n(\cdot; w_a)$, $a=(1+\kappa)a_n^*$, $\kappa=0, \frac{1}{2}, 1, \infty$, *where* $n=2(2)10$.

n	κ	Zeros					
2	0	$.225i$	$1.000i$				
	.5	$.177i$	$1.264i$				
	1.0	$.152i$	$1.472i$				
	∞	0	∞i				
4	0	$.065i$	$1.000i$	$\pm.797+.038i$			
	.5	$.053i$	$1.237i$	$\pm.791+.035i$			
	1.0	$.046i$	$1.435i$	$\pm.788+.032i$			
	∞	0	∞i	$\pm.775$			
6	0	$.031i$	$1.000i$	$\pm.912+.011i$	$\pm.559+.051i$		
	.5	$.025i$	$1.251i$	$\pm.911+.010i$	$\pm.553+.044i$		
	1.0	$.022i$	$1.461i$	$\pm.910+.009i$	$\pm.550+.039i$		
	∞	0	∞i	$\pm.906$	$\pm.538$		
8	0	$.018i$	$1.000i$	$\pm.951+.004i$	$\pm.751+.022i$	$\pm.421+.046i$	
	.5	$.015i$	$1.262i$	$\pm.951+.004i$	$\pm.749+.019i$	$\pm.416+.039i$	
	1.0	$.013i$	$1.479i$	$\pm.950+.004i$	$\pm.747+.017i$	$\pm.413+.034i$	
	∞	0	∞i	$\pm.949$	$\pm.742$	$\pm.406$	
10	0	$.012i$	$1.000i$	$\pm.969+.002i$	$\pm.841+.011i$	$\pm.623+.026i$	$\pm.335+.040i$
	.5	$.010i$	$1.269i$	$\pm.969+.002i$	$\pm.840+.010i$	$\pm.620+.022i$	$\pm.331+.033i$
	1.0	$.008i$	$1.492i$	$\pm.969+.002i$	$\pm.839+.009i$	$\pm.618+.019i$	$\pm.330+.029i$
	∞	0	∞i	$\pm.968$	$\pm.836$	$\pm.613$	$\pm.324$

TABLE 5.3
Zeros of $\pi_n(\cdot; w^a)$, $a=(1+\kappa)a_n^*$, $\kappa=0, \frac{1}{2}, 1, \infty$, *where* $n=3(2)9$.

n	κ	Zeros				
3	0	$1.000i$	$\pm.781+.046i$			
	.5	$1.356i$	$\pm.776+.038i$			
	1.0	$1.710i$	$\pm.774+.032i$			
	∞	∞i	$\pm.775$			
5	0	$1.000i$	$\pm.909+.012i$	$\pm.541+.057i$		
	.5	$1.407i$	$\pm.908+.010i$	$\pm.536+.044i$		
	1.0	$1.807i$	$\pm.907+.008i$	$\pm.535+.035i$		
	∞	∞i	$\pm.906$	$\pm.538$		
7	0	$1.000i$	$\pm.951+.005i$	$\pm.747+.023i$	$\pm.405+.051i$	
	.5	$1.429i$	$\pm.950+.004i$	$\pm.744+.018i$	$\pm.402+.038i$	
	1.0	$1.847i$	$\pm.950+.003i$	$\pm.743+.015i$	$\pm.401+.030i$	
	∞	∞i	$\pm.949$	$\pm.742$	$\pm.406$	
9	0	$1.000i$	$\pm.969+.002i$	$\pm.839+.012i$	$\pm.618+.027i$	$\pm.321+.044i$
	.5	$1.440i$	$\pm.969+.002i$	$\pm.838+.009i$	$\pm.615+.020i$	$\pm.320+.032i$
	1.0	$1.868i$	$\pm.968+.002i$	$\pm.837+.008i$	$\pm.614+.016i$	$\pm.320+.025i$
	∞	∞i	$\pm.968$	$\pm.836$	$\pm.613$	$\pm.324$

and $\pi_n(\cdot; w^a)$ for $n=2(2)10$ and $n=3(2)9$, respectively, where $a=(1+\kappa)a_n^*$, $\kappa=0, \frac{1}{2}, 1, \infty$, are listed in Tables 5.2 and 5.3. Although they were computed to eight significant digits, only three-digit values are shown because of space considerations. If $a \to \infty$ (i.e., $\kappa \to \infty$), it follows from $\pi_n = p_n - i\theta_{n-1}p_{n-1}$ (cf. [4, eq. (2.9)]) and $\theta_{n-1} \to \infty$ that the (finite) zeros of π_n tend to those of p_{n-1}, the orthogonal polynomial of degree $n-1$ relative to the limiting weight function $w_\infty(x) = 1+\delta(x)$ and $w^\infty(x) = x^2$, for n even and odd, respectively. These limiting weights are not as unrelated as we might think at first. We have, in fact,

$$p_{n-1}(x; w_\infty) = xp_{n-2}(x; w^\infty), \qquad n(\text{even}) \geqq 2,$$

since each side is easily seen to be orthogonal on $[-1, 1]$ to all powers of degree $\leqq n-2$ with respect to the constant weight function $w \equiv 1$, and hence equal to the monic Legendre polynomial of degree $n-1$. This is why the limiting zeros for $\kappa = \infty$ in Table 5.2 and Table 5.3 are the same.

It can be seen that for n even, there are two zeros on the imaginary axis moving in opposite directions as a increases from a_n^* to ∞, one up from i to $i\infty$ (cf. the remark at the end of § 4), the other down from some iy_n^*, $0<y_n^*<1$, to zero. For n odd, there is one zero on the imaginary axis moving up from i to $i\infty$.

It is also easy to compute the coefficients $b_{k,\infty}$ and b_k^∞ and to observe numerically the convergence in (3.5) and (3.9).

REFERENCES

[1] T. S. CHIHARA, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[2] W. GAUTSCHI, *On generating orthogonal polynomials*, SIAM J. Sci. Statist. Comput., 3 (1982), pp. 289-317.

[3] W. GAUTSCHI AND G. V. MILOVANOVIĆ, *Polynomials orthogonal on the semicircle*, J. Approx. Theory, 46 (1986), pp. 230-250.

[4] W. GAUTSCHI, H. J. LANDAU, AND G. V. MILOVANOVIĆ, *Polynomials orthogonal on the semicircle*, II, Constr. Approx., 3 (1987), pp. 389-404.