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On mean convergence of extended Lagrange interpolation

Walter Gautschi

Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, United States

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Abstract

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Lagrange interpolation to any continuous function on $[-1, 1]$ at the zeros of orthogonal polynomials is known to converge in the mean. Here, following Bellen, we study mean convergence of Lagrange interpolation on an extended set of nodes that includes, in addition to the n zeros of the orthogonal (relative to some positive weight function w) polynomial π_n of degree n, other $n+1$ nodes, which in turn are zeros of an orthogonal polynomial $\hat{\pi}_{n+1}$ of degree $n+1$ corresponding to the weight function $\hat{w}_n = \pi_n^2 w$. A sufficient criterion of Bellen for mean convergence (as $n \to \infty$) of such extended Lagrange interpolation, for arbitrary continuous functions, is shown to fail for Chebyshev weight functions of the first, third and fourth kind. (It holds trivially for Chebyshev weights of the second kind.) Based on extensive computations, it is conjectured, on the other hand, that the criterion is satisfied for certain Jacobi weights with parameters α and β suitably restricted. Necessary conditions for mean convergence, due to Erdős and Turán, are shown to be violated for the three kinds of Chebyshev weights mentioned above. For smooth functions, a comparison is made of the speed of convergence of simple vs. extended Lagrange interpolation.

Keywords: Extended Lagrange interpolation; convergence in the mean; orthogonal polynomials.

1. Introduction

Let $\pi_n(\cdot; w)$, $n \geqslant 1$, denote the nth-degree orthogonal polynomial on $(-1, 1)$ with respect to a positive weight function w. It is well known [6] that the Lagrange polynomial $(L_n f)(\cdot)$ of degree $\leqslant n-1$ interpolating f at the n zeros $\tau_i = \tau_i^{(n)}$ of π_n converges to f in the mean whenever f is a continuous function on $[-1, 1]$,

$$\lim_{n \to \infty} \| f - L_n f \|_w = 0, \quad \text{all } f \in C[-1, 1]. \tag{1.1}$$

Correspondence to: Prof. W. Gautschi, Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, United States. e-mail: wxg@cs.purdue.edu.

Here the norm is the weighted L_2-norm,

$$\| u \|_w = \left\{ \int_{-1}^{1} u^2(t) w(t) \, \mathrm{d}t \right\}^{1/2}.$$

Suppose we adjoin to the n zeros τ_i an additional $n+1$ nodes $\hat{\tau}_j = \hat{\tau}_j^{(n)}$, $j = 1, 2, \ldots, n+1$, distinct among themselves and from the τ_i, and form the Lagrange polynomial $(\hat{L}_{2n+1} f)(\cdot)$ of degree $\leqslant 2n$ interpolating f on the union of the nodes, $\{\tau_i\} \cup \{\hat{\tau}_j\}$. Is it still true that

$$\lim_{n \to \infty} \| f - \hat{L}_{2n+1} f \|_w = 0, \quad \text{all } f \in C[-1, 1]? \tag{1.2}$$

The answer can no longer be expected to be an unqualified "yes", since the behavior of $\hat{L}_{2n+1} f$ will strongly depend on the kind of additional nodes introduced. A natural choice for these nodes would be the zeros of $\pi_{n+1}(\cdot; w)$. Unfortunately, no criteria are known that would be applicable to prove mean convergence for all $C[-1, 1]$ in this case. An interesting choice, however, has recently been discussed by Bellen [2], who takes the nodes $\hat{\tau}_j$ to be the zeros of the polynomial $\hat{\pi}_{n+1}(\cdot) = \pi_{n+1}(\cdot; \pi_n^2 w)$ of degree $n+1$ orthogonal to all lower-degree polynomials with respect to the (positive) weight function $\hat{w}_n = \pi_n^2 w$. He proves, in this case, that (1.2) indeed is true provided that the ratio

$$M(n; w) := \frac{\| \pi_n \|_w^2}{\min_{1 \leqslant j \leqslant n+1} \pi_n^2(\hat{\tau}_j)} \tag{1.3}$$

remains uniformly bounded,

$$M(n; w) = \mathrm{O}(1), \quad \text{as } n \to \infty. \tag{1.4}$$

Note that (1.4) precludes any of the $\hat{\tau}_j$ from becoming equal, or too close, to any of the τ_i, and like the τ_i, they, too, are distinct from one another. (Clearly, it is irrelevant how one normalizes the polynomial π_n in (1.3).) Other types of extended Lagrange interpolation are studied in [1] for Lipschitz-continuous functions $f \in \mathrm{Lip}\, \gamma$, $\gamma > \frac{1}{2}$, and in [3–5] with a view toward uniform convergence.

We say that the nodes $\hat{\tau}_j$ interlace with the nodes τ_i, if they satisfy, when ordered decreasingly,

$$\hat{\tau}_1 > \tau_1 > \hat{\tau}_2 > \tau_2 > \cdots > \hat{\tau}_n > \tau_n > \hat{\tau}_{n+1}. \tag{1.5}$$

If the leading coefficient of $\hat{\pi}_{n+1}$ is positive, then (1.5) is equivalent to

$$\operatorname{sgn} \hat{\pi}_{n+1}(\tau_i) = (-1)^i, \quad i = 1, 2, \ldots, n. \tag{1.5'}$$

The only weight function w for which (1.4) (and also (1.5)) is known to be true is

$$w(t) = (1 - t^2)^{1/2}, \quad -1 < t < 1. \tag{1.6}$$

In this case, $\pi_n = U_n$ is the Chebyshev polynomial of the second kind, and $\hat{\pi}_{n+1} = T_{n+1}$ the $(n+1)$st-degree Chebyshev polynomial of the first (cf. [2]). In Section 2 we show that (1.4) is false for any of the other three Chebyshev weights, even though the respective nodes τ_i and $\hat{\tau}_j$ interlace. In Section 5 we discuss the validity of (1.4) numerically, based on methods described in Section 4, when w is a Jacobi weight function,

$$w^{(\alpha,\beta)}(t) = (1 - t)^\alpha (1 + t)^\beta, \quad -1 < t < 1, \tag{1.7}$$

in particular, a Gegenbauer weight $w^{(\alpha,\alpha)}$, with the parameters α, β suitably restricted. It suffices, in (1.7), to consider $\beta \geqslant \alpha$, since

$$M(n;\ w^{(\alpha,\beta)}) = M(n;\ w^{(\beta,\alpha)}). \tag{1.8}$$

This follows easily from the identity $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$ for Jacobi polynomials. We conjecture that (1.4) is valid for Jacobi weights $w^{(\alpha,\beta)}$ with $0 \leqslant \alpha \leqslant 1.6$, $\alpha \leqslant \beta < \beta_0$, where $1.55 < \beta_0 < 1.65$, and for Jacobi–Gegenbauer weights $w^{(\alpha,\alpha)}$ in the sharper form $\lim_{n\to\infty} M(n,\ w^{(\alpha,\alpha)}) = \frac{1}{2}\pi$, provided $0 \leqslant \alpha < \alpha^0$, where $1.6 < \alpha^0 < 1.7$. The case of negative α seems more subtle, and we dare not conjecture (1.4) except for Gegenbauer weights $w^{(\alpha,\alpha)}$ with $-\alpha_0 < \alpha \leqslant 0$ for some α_0 near and slightly larger than 0.31.

It should be borne in mind, however, that (1.4) is merely a sufficient condition for convergence in the mean (cf. (1.2)), and its failure to be satisfied does not necessarily invalidate (1.2). A condition that *does* invalidate (1.2) has been given by Erdős and Turán [6, Theorem III] in terms of the function

$$L(n;\ w) := \int_{-1}^{1} \left(\sum_{i=1}^{n} l_i^2(t) + \sum_{j=1}^{n+1} \hat{l}_j^2(t) \right) w(t)\ \mathrm{d}t, \tag{1.9}$$

where l_i and $\hat{l}_j$ are the elementary Lagrange interpolation polynomials for the point set $\{\tau_i\} \cup \{\hat{\tau}_j\}$,

$$\begin{aligned} l_i(t) &= \frac{\pi_n(t)\hat{\pi}_{n+1}(t)}{(t-\tau_i)\pi_n'(\tau_i)\hat{\pi}_{n+1}(\tau_i)}, \quad i = 1, 2, \ldots, n, \\ \hat{l}_j(t) &= \frac{\hat{\pi}_{n+1}(t)\pi_n(t)}{(t-\hat{\tau}_j)\hat{\pi}_{n+1}'(\hat{\tau}_j)\pi_n(\hat{\tau}_j)}, \quad j = 1, 2, \ldots, n+1. \end{aligned} \tag{1.10}$$

Indeed, if

$$\overline{\lim_{n\to\infty}}\ L(n;\ w) = +\infty, \tag{1.11}$$

it was shown [1] in [6] that there exists an $f \in C[-1, 1]$ such that

$$\overline{\lim_{n\to\infty}}\ \| f - \hat{L}_{2n+1} f \|_w = +\infty. \tag{1.12}$$

In Section 3 it will be shown that (1.11) is true for all Chebyshev weight functions other than the one of second kind, which establishes that mean convergence (1.2) does no longer hold in these cases. It should be stressed, nevertheless, that this negative result is not so much a critique of the special choice of interpolation nodes, as it is a reflection of the very large class of functions considered. Adding only a slight amount of regularity, for example, Lipschitz continuity with parameter larger than $\frac{1}{2}$, would already restore convergence. Indeed, for Chebyshev weights w, the referee has kindly pointed out that $\| f - \hat{L}_{2n+1} f \|_w \leqslant \text{const.} \cdot n^{1/2} E_n(f)$, where $E_n(f)$ is the error of best uniform approximation of f by polynomials of degree $\leqslant n$. For still more regularity, in particular analyticity, see also Section 6.

[1] [6, Theorem III], valid for an arbitrary triangular matrix of interpolation nodes, assumes $w(t) = 1$, but the proof goes through for an arbitrary weight function w.

2. The ratio $M(n; w)$ for Chebyshev weights

2.1. First-kind Chebyshev weight function

In this subsection we take the weight function w to be

$$w = w_1, \qquad w_1(t) = (1-t^2)^{-1/2}, \quad -1 < t < 1. \tag{2.1}$$

The corresponding orthogonal polynomial is the Chebyshev polynomial of the first kind,

$$\pi_n(t) = T_n(t), \qquad T_n(\cos\theta) = \cos n\theta. \tag{2.2}$$

As in Section 1, we assume $n \geqslant 1$. We claim that

$$\hat{\pi}_{n+1}(t) = T_{n+1}(t) - \tfrac{1}{2}T_{n-1}(t). \tag{2.3}$$

Indeed, using repeatedly the well-known identity

$$T_n T_m = \tfrac{1}{2}(T_{n+m} + T_{|n-m|}), \tag{2.4}$$

we have for any $p \in \mathbb{P}_n$,

$$\begin{aligned}\int_{-1}^{1}(T_{n+1} - \tfrac{1}{2}T_{n-1})pT_n^2 w_1\, dt &= \frac{1}{2}\int_{-1}^{1}\left[(T_{2n+1} + T_1) - \tfrac{1}{2}(T_{2n-1} + T_1)\right] pT_n w_1\, dt\\ &= \frac{1}{2}\int_{-1}^{1}(T_{2n+1} - \tfrac{1}{2}T_{2n-1} + \tfrac{1}{2}T_1)pT_n w_1\, dt\\ &= \frac{1}{4}\int_{-1}^{1}(T_{3n+1} - \tfrac{1}{2}T_{3n-1} + \tfrac{3}{2}T_{n+1})pw_1\, dt = 0,\end{aligned}$$

the last equality, since $p \in \mathbb{P}_n$, on account of the orthogonality of the Chebyshev polynomials. This proves (2.3).

With

$$\tau_i = \cos\theta_i, \quad \theta_i = \cos\left(\frac{2i-1}{2n}\pi\right), \quad i = 1, 2, \ldots, n, \tag{2.5}$$

denoting the zeros of T_n, it follows easily from (2.3) and (2.2) that

$$\hat{\pi}_{n+1}(\tau_i) = \tfrac{3}{2}(-1)^i \sin\theta_i, \quad i = 1, 2, \ldots, n,$$

so that (1.5′), and hence the interlacing property (1.5), holds for the zeros $\hat{\tau}_j$ of $\hat{\pi}_{n+1}$.

Letting $t = \cos\theta$ in (2.3), we can write the equation $\hat{\pi}_{n+1}(t) = 0$ in trigonometric form $\cos(n+1)\theta - \frac{1}{2}\cos(n-1)\theta = 0$, or, with the help of the addition theorem for the cosine, in the form

$$\tan n\theta \tan\theta = \tfrac{1}{3}, \quad 0 < \theta < \pi. \tag{2.6}$$

Since with π_n also $\hat{\pi}_{n+1}$ is an even polynomial, its zeros $\hat{\tau}_j$ are symmetric with respect to the origin; it suffices therefore to consider (2.6) in $0 < \theta < \frac{1}{2}\pi$. Using

$$\tan^2\theta = \frac{1}{\cos^2\theta} - 1 \quad \text{and} \quad t = \cos\theta,$$

we can write (2.6), when squared, in the form

$$\left(\frac{1}{T_n^2(t)}-1\right)\left(\frac{1}{t^2}-1\right)=\frac{1}{9},$$

or, equivalently, in the form

$$T_n^2(t)=\frac{9(1-t^2)}{9-8t^2}. \tag{2.7}$$

The rational function on the right decreases monotonically on (0, 1); therefore, by the symmetry of the $\hat{\tau}_j$,

$$\min_{1\leqslant j\leqslant n+1} T_n^2(\hat{\tau}_j)=\frac{9(1-\hat{\tau}_1^2)}{9-8\hat{\tau}_1^2}.$$

Since $\| T_n \|_{w_1}^2=\frac{1}{2}\pi$ for $n\geqslant 1$, we get from the definition in (1.3) that

$$M(n;\, w_1)=\tfrac{1}{18}\pi\frac{9-8\hat{\tau}_1^2}{1-\hat{\tau}_1^2}. \tag{2.8}$$

Writing (2.6) in the form

$$\tan n\theta=\frac{1}{3\tan\theta}, \tag{2.6$'$}$$

and examining the graphs of the two functions on the left and right, immediately yields

$$0<\hat{\theta}_1<\frac{\pi}{2n}$$

for the smallest positive root, $\theta=\hat{\theta}_1$, of (2.6′). Consequently,

$$\cos\frac{\pi}{2n}<\hat{\tau}_1=\cos\hat{\theta}_1<1,$$

and by (2.8),

$$M(n;\, w_1)>\tfrac{1}{18}\pi\frac{1}{\sin^2(\pi/2n)}. \tag{2.9}$$

This shows that $M(n;\, w_1)$ grows to ∞ as $n\to\infty$, at least like $O(n^2)$.

2.2. Second-kind Chebyshev weight function

For completeness, we include here the Chebyshev weight function of the second kind,

$$w=w_2,\qquad w_2(t)=(1-t^2)^{1/2},\quad -1<t<1, \tag{2.10}$$

for which

$$\pi_n(t)=U_n(t),\qquad U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}. \tag{2.11}$$

Since $T_{n+1}U_n = \frac{1}{2}U_{2n+1}$, we have, for any $p \in \mathbb{P}_n$,

$$\int_{-1}^{1} T_{n+1} p U_n^2 w_2 \, \mathrm{d}t = \frac{1}{2}\int_{-1}^{1} U_{2n+1} p U_n w_2 \, \mathrm{d}t = 0,$$

by orthogonality, since $pU_n \in \mathbb{P}_{2n}$. Therefore (as already observed in [2]),

$$\hat{\pi}_{n+1}(t) = T_{n+1}(t). \tag{2.12}$$

The zeros $\hat{\tau}_j$ of T_{n+1} clearly interlace with those of $\pi_n = U_n$. Furthermore, with

$$\hat{\tau}_j = \cos\hat{\theta}_j, \quad \hat{\theta}_j = \frac{2j-1}{2n+2}\pi,$$

we have

$$U_n(\hat{\tau}_j) = \frac{\sin(n+1)\hat{\theta}_j}{\sin\hat{\theta}_j} = \frac{(-1)^{j-1}}{\sin((2j-1)\pi/(2n+2))},$$

from which

$$\min_{1 \leqslant j \leqslant n+1} U_n^2(\hat{\tau}_j) = \begin{cases} 1, & n \text{ even}, \\ \dfrac{1}{\sin^2(\frac{1}{2}\pi n/(n+1))} = \dfrac{1}{\cos^2(\pi/(2n+2))}, & n \text{ odd}. \end{cases}$$

Therefore, since $\| U_n \|_{w_2}^2 = \frac{1}{2}\pi$,

$$M(n; w_2) = \begin{cases} \frac{1}{2}\pi, & n \text{ even}, \\ \frac{1}{2}\pi \cos^2 \dfrac{\pi}{2n+2}, & n \text{ odd}. \end{cases} \tag{2.13}$$

We see that $M(n; w_2)$ now indeed satisfies (1.4); specifically, $M(n; w_2) \leqslant \frac{1}{2}\pi$ for all $n \geqslant 1$, and

$$\lim_{n\to\infty} M(n; w_2) = \tfrac{1}{2}\pi. \tag{2.14}$$

2.3. *Third- and fourth-kind Chebyshev weight functions*

These are the Jacobi weights (1.7) with $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, respectively. As remarked in (1.8), it suffices to consider the first of these,

$$w = w_3, \qquad w_3(t) = (1-t)^{-1/2}(1+t)^{1/2}, \quad -1 < t < 1. \tag{2.15}$$

The corresponding orthogonal polynomial is

$$\pi_n(t) = V_n(t), \qquad V_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}. \tag{2.16}$$

Here we have

$$\hat{\pi}_{n+1}(t) = T_{n+1}(t) - \tfrac{1}{2}T_n(t). \tag{2.17}$$

Indeed, noting that

$$T_n V_n = \tfrac{1}{2}(V_{2n} + 1), \qquad T_{n+1} V_n = \tfrac{1}{2}(V_{2n+1} + 1),$$

we compute, for any $p \in \mathbb{P}_n$,

$$\begin{aligned}\int_{-1}^{1} (T_{n+1} - \tfrac{1}{2}T_n) p V_n^2 w_3 \, dt &= \frac{1}{2}\int_{-1}^{1} \left[(V_{2n+1} + 1) - \tfrac{1}{2}(V_{2n} + 1) \right] p V_n w_3 \, dt \\ &= \frac{1}{2}\int_{-1}^{1} \left[V_{2n+1} - \tfrac{1}{2}(V_{2n} - 1) \right] p V_n w_3 \, dt \\ &= \frac{1}{4}\int_{-1}^{1} (1 - V_{2n}) p V_n w_3 \, dt. \end{aligned} \tag{2.18}$$

An elementary calculation shows that

$$\int_{-1}^{1} V_n^2(t) w_3(t) \, dt = \pi, \quad \text{for } n \geqslant 1. \tag{2.19}$$

Therefore, letting $p = V_n$ in (2.18) gives

$$\begin{aligned}\int_{-1}^{1} (1 - V_{2n}) V_n^2 w_3 \, dt &= \int_{-1}^{1} V_n^2 w_3 \, dt - \int_{-1}^{1} V_{2n} V_n^2 w_3 \, dt \\ &= \int_{-1}^{1} V_n^2 w_3 \, dt - \int_{-1}^{1} V_{2n}^2 w_3 \, dt = 0, \end{aligned}$$

since V_n^2 differs from V_{2n} by a polynomial of degree $< 2n$ and the last two integrals are the same by (2.19). Letting $p = V_m$, $m < n$, in (2.18) gives

$$\int_{-1}^{1} (1 - V_{2n}) V_m V_n w_3 \, dt = \int_{-1}^{1} V_m V_n w_3 \, dt - \int_{-1}^{1} V_{2n} V_m V_n w_3 \, dt = 0,$$

by orthogonality. Thus, the integral on the far right of (2.18), hence the one on the left, vanishes for every $p \in \mathbb{P}_n$, which proves (2.17).

It is again a simple matter to compute

$$\hat{\pi}_{n+1}(\tau_i) = \tfrac{3}{2}(-1)^i \sin\left(\frac{2i-1}{2n+1} \tfrac{1}{2}\pi \right), \quad i = 1, 2, \ldots, n, \tag{2.20}$$

for the zeros τ_i of V_n, and hence to verify the interlacing property (1.5).

The equation $\hat{\pi}_{n+1}(t) = 0$ in trigonometric form ($t = \cos\theta$) is $\cos(n+1)\theta - \frac{1}{2}\cos n\theta = 0$, which, by writing $(n+1)\theta = (n+\frac{1}{2})\theta + \frac{1}{2}\theta$ and $n\theta = (n+\frac{1}{2})\theta - \frac{1}{2}\theta$ and using the addition theorem for the cosine, becomes

$$\tan(n + \tfrac{1}{2})\theta \tan \tfrac{1}{2}\theta = \tfrac{1}{3}, \quad 0 < \theta < \pi. \tag{2.21}$$

By manipulations similar to those in Section 2.1, this can be written as

$$V_n^2(t) = \frac{9(1-t)}{(1+t)(5-4t)}, \quad t = \cos\theta. \tag{2.22}$$

The rational function on the right decreases monotonically on $(-1, 1)$, implying that

$$\min_{1 \leqslant j \leqslant n+1} V_n^2(\hat{\tau}_j) = \frac{9(1-\hat{\tau}_1)}{(1+\hat{\tau}_1)(5-4\hat{\tau}_1)}.$$

Combining this with $\| V_n \|_{w_3}^2 = \pi$ (cf. (2.19)), we get

$$M(n; w_3) = \tfrac{1}{9}\pi \frac{(1+\hat{\tau}_1)(5-4\hat{\tau}_1)}{1-\hat{\tau}_1}. \tag{2.23}$$

Similarly as in Section 2.1, we find that $0 < \hat{\theta}_1 < \pi/(2n+1)$, hence $\hat{\tau}_1 = \cos \hat{\theta}_1 > \cos(\pi/(2n+1))$, and thus, again by the monotonicity of the rational function in (2.22),

$$M(n; w_3) > \tfrac{1}{9}\pi \frac{[1+\cos(\pi/(2n+1))][5-4\cos(\pi/(2n+1))]}{1-\cos(\pi/(2n+1))}.$$

Thus, finally,

$$M(n; w_3) > \tfrac{1}{9}\pi \cot^2 \frac{\pi}{2(2n+1)}. \tag{2.24}$$

Again, $M(n; w_3)$ grows to ∞ as $n \to \infty$, at least like $O(n^2)$.

3. The function $L(n; w)$ for Chebyshev weights

3.1. First-kind Chebyshev weight function

We begin with the case $w_1(t) = (1-t^2)^{-1/2}$. Since

$$L(n; w_1) > \int_{-1}^{1} \sum_{i=1}^{n} l_i^2(t) w_1(t) \, dt, \tag{3.1}$$

it suffices, for showing (1.11), that the right-hand side of (3.1) is unbounded as $n \to \infty$. We may choose in (1.10)

$$\pi_n(t) = T_n(t), \qquad \hat{\pi}_{n+1}(t) = T_{n+1}(t) - \tfrac{1}{2}T_{n-1}(t) \tag{3.2}$$

(cf. (2.3)). Letting, as in (2.5), $\tau_i = \cos \theta_i$, one easily computes

$$\pi_n'(\tau_i) = \frac{(-1)^{i-1} n}{\sin \theta_i}, \qquad \hat{\pi}_{n+1}(\tau_i) = \tfrac{3}{2}(-1)^i \sin \theta_i, \tag{3.3}$$

so that

$$\pi_n'(\tau_i)\hat{\pi}_{n+1}(\tau_i) = -\tfrac{3}{2}n. \tag{3.4}$$

Furthermore, using (2.4),

$$\begin{aligned}
\hat{\pi}_{n+1}^2(t) &= T_{n+1}^2 - T_{n+1}T_{n-1} + \tfrac{1}{4}T_{n-1}^2 \\
&= \tfrac{1}{2}\left[(T_{2n+2}+1) - (T_{2n}+T_2) + \tfrac{1}{4}(T_{2n-2}+1)\right] \\
&= \tfrac{1}{2}\left(T_{2n+2} - T_{2n} + \tfrac{1}{4}T_{2n-2} - T_2 + \tfrac{5}{4}\right),
\end{aligned}$$

hence, by orthogonality,

$$\int_{-1}^{1}\left(\frac{\pi_n(t)}{t-\tau_i}\right)^2 \hat{\pi}_{n+1}^2(t)w_1(t)\,\mathrm{d}t = \frac{1}{8}\int_{-1}^{1}\left(\frac{T_n}{t-\tau_i}\right)^2 [T_{2n-2}-4T_2+5]w_1\,\mathrm{d}t. \tag{3.5}$$

Now for the first term on the right we use the fact that $(T_n/(t-\tau_i))^2 = 2T_{2n-2}$ modulo $\mathbb{P}_{2n-3}$, if $n>1$, so that, again by orthogonality,

$$\int_{-1}^{1}\left(\frac{T_n}{t-\tau_i}\right)^2 T_{2n-2}w_1\,\mathrm{d}t = 2\int_{-1}^{1} T_{2n-2}^2 w_1\,\mathrm{d}t = \pi, \quad n>1. \tag{3.6}$$

The remaining part of the integral on the right of (3.5) can be evaluated by the n-point Gauss–Chebyshev quadrature rule with remainder term. Since

$$\frac{1}{(2n)!}\left\{\left(\frac{T_n}{t-\tau_i}\right)^2(-4T_2+5)\right\}^{(2n)} = -2^{2n}$$

and $T_n(\tau_k)=0$, $k=1, 2,\dots, n$, we get, upon using (3.3),

$$\begin{aligned}\int_{-1}^{1}&\left(\frac{T_n}{t-\tau_i}\right)^2(-4T_2+5)w_1\,\mathrm{d}t\\ &= \frac{\pi}{n}[T_n'(\tau_i)]^2[-4T_2(\tau_i)+5] - 2^{2n}\int_{-1}^{1}\left[\frac{1}{2^{n-1}}T_n(t)\right]^2 w_1(t)\,\mathrm{d}t\\ &= \frac{\pi}{n}\frac{n^2}{\sin^2\theta_i}[5-4\cos 2\theta_i] - 4\cdot\tfrac{1}{2}\pi > \frac{\pi n}{\sin^2\theta_i} - 2\pi.\end{aligned} \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) gives

$$\int_{-1}^{1}\left(\frac{\pi_n}{t-\tau_i}\right)^2 \hat{\pi}_{n+1}^2 w_1\,\mathrm{d}t > \tfrac{1}{8}\pi\left(\frac{n}{\sin^2\theta_i}-1\right).$$

Therefore, by (1.10) and (3.4),

$$\begin{aligned}\int_{-1}^{1}\sum_{i=1}^{n} l_i^2(t)w_1\,\mathrm{d}t &> \frac{\pi}{8\cdot\frac{9}{4}n^2}\left(n\sum_{i=1}^{n}\frac{1}{\sin^2\theta_i}-n\right) = \frac{\pi}{18n}\left(\sum_{i=1}^{n}\frac{1}{\sin^2\theta_i}-1\right)\\ &> \frac{\pi}{18n}\left(\frac{1}{\sin^2\theta_1}-1\right) > \tfrac{1}{18}\pi\left(\frac{4}{\pi^2}n-\frac{1}{n}\right),\end{aligned}$$

since $\sin\theta_1 = \sin(\pi/2n) < \pi/2n$. This shows that the right-hand side of (3.1), hence also the left-hand side, is unbounded as $n\to\infty$.

3.2. Second-kind Chebyshev weight function

Since here we know that $M(n; w_2)$ satisfies (1.4) (cf. (2.14)), we must necessarily have uniform boundedness of $L(n; w_2)$. We show, in fact, that

$$L(n; w_2) = \tfrac{1}{2}\pi, \quad \text{for all } n \geqslant 1. \tag{3.8}$$

Indeed, since $\pi_n = U_n$, $\hat{\pi}_{n+1} = T_{n+1}$ (cf. (2.11), (2.12)), and the set $\{\tau_i\} \cup \{\hat{\tau}_j\}$ consists precisely of the zeros of U_{2n+1}, we have

$$L(n;\, w_2) = \sum_{i=1}^{2n+1} \gamma_i^{(2n+1)}(w_2) = \int_{-1}^{1} w_2(t)\, \mathrm{d}t,$$

where $\gamma_i^{(2n+1)}(w_2)$ are the weights in the $(2n+1)$-point Gauss formula for w_2. From this, (3.8) follows immediately.

3.3. Third- and fourth-kind Chebyshev weight functions

As in (1.8), one easily shows that

$$L(n;\, w^{(\alpha,\beta)}) = L(n;\, w^{(\beta,\alpha)}). \tag{3.9}$$

It suffices therefore to assume $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, that is,

$$w(t) = w_3(t), \qquad w_3(t) = (1-t)^{-1/2}(1+t)^{1/2}.$$

We will show that $\int_{-1}^{1}\sum_{i=1}^{n} l_i^2(t) w_3(t)\, \mathrm{d}t$ is unbounded as $n \to \infty$, which, as in (3.1), will prove the same for $L(n;\, w_3)$. Since $\pi_n = V_n$ and $\hat{\pi}_{n+1} = T_{n+1} - \frac{1}{2}T_n$ (cf. (2.16), (2.17)), we have

$$\begin{aligned}
\int_{-1}^{1}\left(\frac{\pi_n}{t-\tau_i}\right)^2 \hat{\pi}_{n+1}^2 w_3\, \mathrm{d}t &= \int_{-1}^{1}(1+t)\left(\frac{V_n}{t-\tau_i}\right)^2 \left(T_{n+1} - \tfrac{1}{2}T_n\right)^2 \frac{w_3}{1+t}\, \mathrm{d}t \\
&= \int_{-1}^{1}(1+t)\left(\frac{V_n}{t-\tau_i}\right)^2 \left(T_{n+1}^2 - T_{n+1}T_n + \tfrac{1}{4}T_n^2\right) w_1\, \mathrm{d}t \\
&= \frac{1}{2}\int_{-1}^{1}(1+t)\left(\frac{V_n}{t-\tau_i}\right)^2 \left[(T_{2n+2}+1) - (T_{2n+1}+T_1)\right. \\
&\qquad\qquad \left. + \tfrac{1}{4}(T_{2n}+1)\right] w_1\, \mathrm{d}t.
\end{aligned}$$

By orthogonality, this simplifies to

$$\frac{1}{8}\int_{-1}^{1}(1+t)\left(\frac{V_n}{t-\tau_i}\right)^2 [-4T_1 + 5] w_1\, \mathrm{d}t = \frac{1}{8}\int_{-1}^{1}\left(\frac{V_n}{t-\tau_i}\right)^2 [-4T_1+5] w_3\, \mathrm{d}t.$$

Now, with $\gamma_i = \gamma_i^{(n)}(w_3)$ denoting the Christoffel numbers for the n-point Gauss formula for $w = w_3$, we get, noting that the integrand in the last integral is a polynomial of degree $2n-1$, and $V_n(\tau_k) = 0$, that

$$\int_{-1}^{1}\left(\frac{\pi_n}{t-\tau_i}\right)^2 \hat{\pi}_{n+1}^2 w_3\, \mathrm{d}t = \tfrac{1}{8}\gamma_i\left[\pi_n'(\tau_i)\right]^2 [5 - 4\tau_i].$$

Since furthermore, by (2.20),

$$\hat{\pi}_{n+1}(\tau_i) = \tfrac{3}{2}(-1)^i \sin \tfrac{1}{2}\theta_i, \quad \tau_i = \cos\theta_i,\ \theta_i = \frac{2i-1}{2n+1}\pi,$$

we obtain from (1.10)

$$\int_{-1}^{1} \sum_{i=1}^{n} l_i^2(t) w_3(t)\, dt = \frac{1}{8} \sum_{i=1}^{n} \gamma_i \frac{5-4\tau_i}{\frac{9}{4}\sin^2\frac{1}{2}\theta_i} = \frac{1}{9} \sum_{i=1}^{n} \gamma_i \frac{5-4\tau_i}{1-\tau_i} > \frac{1}{9} \frac{\gamma_1}{1-\tau_1}. \tag{3.10}$$

Noting that $1-\tau_1 = 2\sin^2\frac{1}{2}\theta_1$ and, as is well known,

$$\gamma_1 = \frac{4\pi}{2n+1} \cos^2\tfrac{1}{2}\theta_1,$$

we see that the lower bound in (3.10) is

$$\frac{2\pi}{9(2n+1)\tan^2\frac{1}{2}\theta_1} = \frac{2\pi}{9(2n+1)\tan^2(\pi/(2(2n+1)))} = O(n), \quad n \to \infty,$$

which proves the assertion.

4. Computational methods

For Chebyshev weight functions w, the polynomials $\hat{\pi}_{n+1}$ (cf. (2.3), (2.12) and (2.17)) are easily computed, as they are all orthogonal with respect to either the Chebyshev weight function of the first kind, as in the case of (2.12), or with respect to this same weight function divided by a quadratic or linear polynomial, as in the other two cases (cf., e.g., [8, §5]). In particular, the three-term recurrence relation for these polynomials is explicitly known. This is no longer true for other weight functions, where computational methods have to be employed to generate the desired polynomials.

The key constituents of any computation involving orthogonal polynomials relative to a weight function $\omega \geqslant 0$ are the recursion coefficients $\alpha_k = \alpha_k(\omega)$, $\beta_k = \beta_k(\omega)$ in the basic three-term recurrence relation satisfied by the (monic) polynomials $\pi_k(\cdot) = \pi_k(\cdot;\, \omega)$,

$$\begin{aligned} &\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2, \ldots, \\ &\pi_{-1}(t) = 0, \qquad \pi_0(t) = 1 \end{aligned} \tag{4.1}$$

(cf. [7]). The Jacobi matrix $J(\omega)$ of the weight function ω is the infinite symmetric tridiagonal matrix

$$J(\omega) = \mathrm{tri}\left(\alpha_0, \alpha_1, \alpha_2, \ldots;\ \sqrt{\beta_1}, \sqrt{\beta_2}, \sqrt{\beta_3}, \ldots\right) \tag{4.2}$$

having the α's on the main diagonal, and the square roots of the β's on the two side diagonals. Its leading principal minor matrix of order m will be denoted by

$$J_m = J_m(\omega) = J(\omega)_{m\times m}. \tag{4.3}$$

In terms of the matrix J_m, the zeros $\tau_\mu = \tau_\mu^{(m)}$ of $\pi_m(\cdot;\, \omega)$ are best computed as eigenvalues of J_m,

$$\det(J_m - \tau_\mu I_m) = 0, \quad \mu = 1, 2, \ldots, m, \tag{4.4}$$

using the QR algorithm with judiciously selected shifts.

The following observation, due to [9], will be the basis for our computational method. Given the Jacobi matrix $J_{m+1}(\omega)$ of order $m+1$ for the weight function $\omega(t)$, one can obtain the Jacobi matrix of order m, $J_m((\cdot-\tau)^2\omega)$, for the weight function $(t-\tau)^2\omega(t)$, $\tau\in\mathbb{R}$, by one step of the QR algorithm with shift τ: from the QR decomposition

$$J_{m+1}(\omega)-\tau I_{m+1}=QR,\quad Q \text{ orthogonal, } R \text{ upper triangular, diag } R\geqslant 0, \tag{4.5}$$

one obtains

$$J_m\big((\cdot-\tau)^2\omega\big)=(RQ+\tau I_{m+1})_{m\times m}, \tag{4.6}$$

discarding, as indicated by the subscript, the last row and last column of the transformed matrix. An efficient and stable algorithm for carrying out this transformation can be found, e.g., in [10, p.567].

If we denote $\beta_0(\omega)=\int\omega(t)\,\mathrm{d}t$, where the integral extends over the support of ω, then we can also get $\beta_0((\cdot-\tau)^2\omega)$ by the simple formula

$$\beta_0\big((\cdot-\tau)^2\omega\big)=\beta_0(\omega)\big[\beta_1(\omega)+(\tau-\alpha_0(\omega))^2\big]. \tag{4.7}$$

Indeed, if α_0, β_0, β_1 are the quantities appearing on the right of (4.7), we have, as is well known,

$$\int t\omega(t)\,\mathrm{d}t=\alpha_0\beta_0,\qquad \int(t-\alpha_0)^2\omega(t)\,\mathrm{d}t=\beta_0\beta_1.$$

Expanding the square in the second formula, and using the first, we find

$$\int t^2\omega(t)\,\mathrm{d}t=2\alpha_0\cdot\alpha_0\beta_0-\alpha_0^2\beta_0+\beta_0\beta_1=\beta_0(\beta_1+\alpha_0^2),$$

and hence

$$\int(t-\tau)^2\omega(t)\,\mathrm{d}t=\beta_0(\beta_1+\alpha_0^2)-2\tau\alpha_0\beta_0+\tau^2\beta_0=\beta_0(\beta_1+\alpha_0^2-2\tau\alpha_0+\tau^2),$$

which is (4.7).

Now let $\omega=w$, and write $\hat{w}_n$ in the form

$$\hat{w}_n(t)=w(t)\pi_n^2(t;w)=w(t)\prod_{i=1}^{n}(t-\tau_i)^2, \tag{4.8}$$

where $\tau_i=\tau_i^{(n)}$ are the zeros of $\pi_n(\cdot;w)$. Define

$$\hat{w}(t;k)=w(t)\prod_{i=1}^{k}(t-\tau_i)^2, \tag{4.9}$$

so that

$$\hat{w}(t;0)=w(t),\qquad \hat{w}(t;n)=\hat{w}_n(t).$$

Let $J_{m,k}$ denote the Jacobi matrix of order m of the "intermediate" weight function $w(\cdot;k)$,

$$J_{m,k}=J_m(\hat{w}(\cdot;k)),\quad k=0,1,\ldots,n. \tag{4.10}$$

Since

$$\hat{w}(t;k)=(t-\tau_k)^2\hat{w}(t;k-1),\quad k=1,2,\ldots,n, \tag{4.11}$$

it is clear that $J_{n+1,n}$ can be obtained from $J_{2n+1,0}$ by n successive transformations of the type (4.5), (4.6) with shifts τ_k:

$$J_{2n-k+2,k-1} - \tau_k I_{2n-k+2} = QR, \qquad J_{2n-k+1,k} = (RQ + \tau_k I_{2n-k+2})_{2n-k+1 \times 2n-k+1}. \tag{4.12}$$

Upon completion of (4.12) for $k = 1, 2, \ldots, n$, we will have the desired Jacobi matrix

$$J_{n+1,n} = J_{n+1}(\hat{w}_n). \tag{4.13}$$

The zeros τ_k of π_n required in (4.12), as well as those of $\hat{\pi}_{n+1}$, are computed as eigenvalues of $J_n(w)$ and $J_{n+1}(\hat{w}_n)$, respectively (cf. (4.4)). Also, since in our implementation we successively update β_0 by means of (4.7), the numerator in (1.3) will simply be

$$\| \pi_n \|_w^2 = \int_{-1}^{1} \pi_n^2(t) w(t) \, \mathrm{d}t = \int_{-1}^{1} \hat{w}_n(t) \, \mathrm{d}t = \hat{\beta}_0,$$

the final β_0-coefficient.

Our experience has indicated that this procedure is quite stable, even for values of n as large as $n = 320$. The results reported in the next section are all obtained in this manner.

5. Numerical study for Jacobi weight functions

The reason why in Sections 2.1–2.3 the polynomials $\hat{\pi}_{n+1}$ for the four Chebyshev weights could be obtained analytically is the fact that in all these cases, $\hat{\pi}_{k,n}(\cdot) = \pi_k(\cdot;\, \pi_n^2 w)$ is one of the Chebyshev polynomials for each $k \leqslant n$. This is no longer true for general Jacobi weights, not even in the simple case $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$. Here, the Jacobi matrix $J_{n+1}(\hat{w}_n^{(1/2,3/2)})$, computed by the methods of Section 4, turns out to be a nontrivial perturbation of the Jacobi matrix $J_{n+1}(\hat{w}_n^{(1/2,1/2)})$ $[= J_{n+1}(w^{(-1/2,-1/2)})]$, showing that the $\hat{\pi}_{k,n}$ are no longer pure Chebyshev polynomials in the range $k \leqslant n$. This also suggests expanding $\hat{\pi}_{n+1}$ in Chebyshev polynomials of the first kind. In doing so, one finds by computation that all expansion coefficients are different from zero (and alternating in sign). It appears unlikely, therefore, that analytic methods will be successful in this case, let alone in the case of general Jacobi weight functions. We therefore undertake to explore the problem computationally.

It is, of course, intrinsically impossible to demonstrate the validity of (1.4) by (a finite number of) computations. Nevertheless, extensive and well-targeted computations can be carried out to come up with certain conjectures, which we now formulate. Each conjecture will be followed by numerical (and other) evidence supporting it.

Conjecture 5.1. *For the Jacobi–Gegenbauer weight function $w = w^{(\alpha,\alpha)}$, there holds*

$$\lim_{n \to \infty} M(n;\, w^{(\alpha,\alpha)}) = \tfrac{1}{2}\pi, \quad \text{if } 0 \leqslant \alpha < \alpha^0, \tag{5.1}$$

where α^0 is some number between 1.6 *and* 1.7. (*For $\alpha = 0$, this was conjectured in* [2].)

Numerical evidence. Let the minimum in the denominator of (1.3) be attained for $j = j_n$,

$$\min_{1 \leqslant j \leqslant n+1} \pi_n^2(\hat{\tau}_j) = \pi_n^2(\hat{\tau}_{j_n}), \quad 1 \leqslant j_n \leqslant n+1. \tag{5.2}$$

We say that the minimum is attained "in the middle", if

$$j_n = \begin{cases} \frac{1}{2}(n+2), & n \text{ even}, \\ \frac{1}{2}(n+1) \text{ or } \frac{1}{2}(n+3), & n \text{ odd}. \end{cases} \tag{5.3}$$

This terminology is justified by virtue of the $n+1$ nodes $\hat{\tau}_j$ being symmetric with respect to the origin. Note, in particular, that (5.3), for n even, implies $\hat{\tau}_{j_n} = 0$.

Now for $w = w^{(\alpha,\alpha)}$, $\alpha > -1$, one easily computes

$$\frac{\| \pi_n \|_w^2}{\pi_n^2(0)} = \sqrt{\pi} \frac{2^n(\frac{1}{2}n)!^2 \Gamma(\alpha + 1 + \frac{1}{2}n)}{n!(n+\alpha+\frac{1}{2})\Gamma(\alpha + \frac{1}{2}(n+1))} \sim \tfrac{1}{2}\pi \quad \text{as } n \to \infty,\ w = w^{(\alpha,\alpha)}. \tag{5.4}$$

Therefore, if (5.3) holds, then the limit relation in Conjecture 5.1 is valid when restricted to even n (and very likely for unrestricted n as well). By computation we have found that for $\alpha = 0$, 0.5, 1.0, 1.5, 1.6, the minimum (5.2) is consistently attained in the middle, whenever $n = 1,\dots,80$ and $n = 159$, 160, 239, 240, 319, 320. In each case computed, the interlacing property (1.5) also holds. Moreover, $M(240; w)$ and $M(320; w)$ agree with $\frac{1}{2}\pi$ to at least 5 decimal digits. When $\alpha = 1.7$, this pattern changes significantly: the minimum is still attained in the middle for $n = 1,\dots,117$, but no longer so for $n = 118,\dots,130$, in which cases it is attained "near the ends" ($j_n = 7$ or 8). The interlacing property, while true for $1 \leqslant n \leqslant 130$, breaks down at $n = 131$. As n is further increased, the ratio $M(n; w)$ takes on larger and larger values, for example, $M(160; w) = 45.964$ and $M(320; w) = 223.78$.

Figure 5.1 shows the behavior of $M(n; w^{(\alpha,\alpha)})$ for $1 \leqslant n \leqslant 160$; Fig. 5.1(a) is for $\alpha = 1.6$, Fig. 5.1(b) for $\alpha = 1.7$ (in a logarithmic scale!).

Thus, it appears that the proven behavior of $M(n; w^{(\alpha,\alpha)})$ in the case $\alpha = \frac{1}{2}$ (cf. (2.14)) is typical for all $0 \leqslant \alpha \leqslant 1.6$, but certainly not for $\alpha = 1.7$.

Conjecture 5.2. *The limit relation* (5.1) *holds for* $-\alpha_0 < \alpha \leqslant 0$, *where* α_0 *is some number between* 0.31 *and* 0.3125.

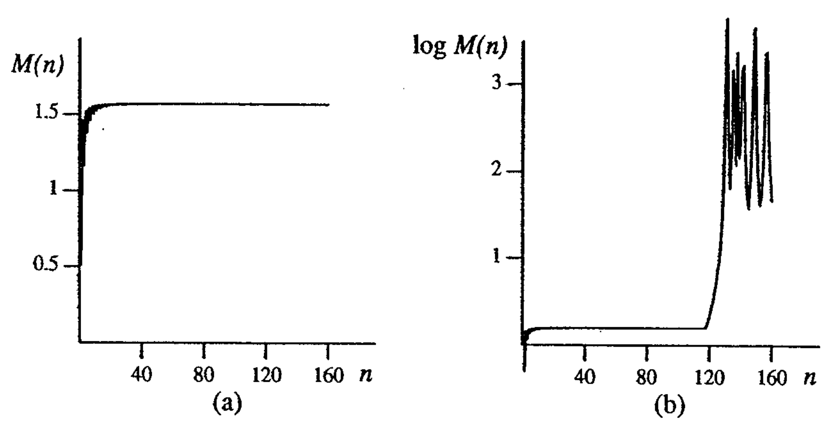

Fig. 5.1. $M(n; w^{(\alpha,\alpha)})$ for $1 \leqslant n \leqslant 160$; (a) $\alpha = 1.6$; (b) $\alpha = 1.7$.

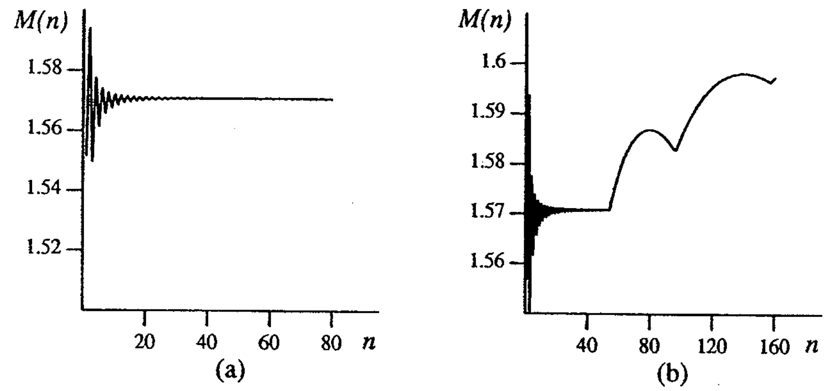

Fig. 5.2. $M(n;\, w^{(\alpha,\,\alpha)})$ for (a) $\alpha = -0.31$, $1 \leqslant n \leqslant 80$; (b) $\alpha = -0.3125$, $1 \leqslant n \leqslant 160$.

Numerical evidence. Verification of Conjecture 5.2 is made more difficult by the apparent fact that in the range under consideration, the interlacing property holds regardless of whether (1.4) is valid or not. We have found, however, that for $\alpha = -0.1, -0.2, -0.3, -0.31$, the minimum in (5.2) is consistently attained in the middle for the same values of n ($\leqslant 320$) as used in the discussion of Conjecture 5.1. This pattern changes when $\alpha = -0.3125$, in which case the minimum is attained in the middle for $1 \leqslant n \leqslant 53$, but near the ends ($j_n = 1$ or 2) for $54 \leqslant n \leqslant 80$. Along with this abrupt change in pattern goes a change in the behavior of the function $M(n;\, w)$; see Fig. 5.2. While the magnitude of $M(n;\, w)$ remains relatively small, even for n as larger as 320, the sudden development of distinct "vaults" raises some legitimate doubts as to uniform boundedness. (For still smaller values of $\alpha > -\frac{1}{2}$, the vaults seem to spread out over larger n-domains, making a determination of boundedness even more problematic.)

Conjecture 5.3. *For the Jacobi weight* $w = w^{(\alpha,\beta)}$, *the relation* (1.4) *is valid if* $0 \leqslant \alpha \leqslant 1.6$, $\alpha \leqslant \beta < \beta_0$, *where* β_0 *is some number (depending on* α*) between* 1.55 *and* 1.65.

Numerical evidence. For each $\alpha = 0(0.5)1.5$ we computed $M(n;\, w^{(\alpha,\beta)})$ for $\beta = 0(0.1)1.5$ and $n = 20, 41, 60, 79, 80$ and found the results to approach a limit to within 3–4 decimal digits. For the same values of α, we further scrutinized the case $\beta = 1.5$ by computing $M(n;\, w^{(\alpha,\beta)})$ for $n = 40, 81, 160, 241, 320$. The last two values (for $n = 241$ and $n = 320$) consistently agreed to 2–4 decimal digits. The same was observed for $\beta = 1.55$. In all cases, the interlacing property was found to hold. When $\beta = 1.6$, however, $M(n;\, w^{(\alpha,\beta)})$ takes on values of the order 10^3–10^5 when $n = 320$, and the interlacing property consistently breaks down for some $n \leqslant 320$, for each of the above α's, except $\alpha = 1.5$. In the cases $\alpha = 1.5(0.05)1.6$, $\beta = 1.6$, we still seem to have convergence as $n \to \infty$, but no longer so if $\beta = 1.65$ for the same three values of α.

For $\alpha < 0$ and $\beta > \alpha$, the numerical results seem inconclusive, as they do not permit a distinction between (slow) divergence and convergence. We are not prepared to make any conjecture in this range.

6. Simple vs. extended interpolation for smooth functions

Extended interpolation can be used, in practice, to check the accuracy of (simple) interpolation at the zeros of orthogonal polynomials. Thus, one would compare the (simple) interpolant $L_n f$ with the extended interpolant $\hat{L}_{2n+1} f$ (as defined in Section 1) to see how well they agree. This can be done at a cost of $2n+1$ function values. If we were to do the same with simple interpolation alone, and insisted on equal cost, we would have to compare $L_n f$ with $L_{n+1} f$, since all nodes change going from n to $n+1$. This has the serious disadvantage that the reference interpolant we are comparing with, i.e., $L_{n+1} f$, is only modestly more accurate than the interpolant to be checked, $L_n f$. In extended interpolation, on the other hand, the reference interpolant can be expected to be substantially more accurate, at least when f is sufficiently smooth.

To analyze the matter further, assume that $f \in C^{2n+1}[-1, 1]$, and let the scaled kth derivative of f be bounded on $[-1, 1]$ by M_k,

$$\frac{1}{k!} \| f^{(k)} \|_\infty \leqslant M_k, \quad k = 0, 1, \ldots, 2n+1. \tag{6.1}$$

Then it follows from interpolation theory that

$$\| f - L_{n+1} f \|_w^2 \leqslant M_{n+1}^2 \int_{-1}^{1} \pi_{n+1}^2(t; w) w(t) \, \mathrm{d}t, \tag{6.2}$$

while

$$\left\| f - \hat{L}_{2n+1} f \right\|_w^2 \leqslant M_{2n+1}^2 \int_{-1}^{1} \hat{\pi}_{n+1}^2(t) \pi_n^2(t; w) w(t) \, \mathrm{d}t, \tag{6.3}$$

the polynomials π_n, π_{n+1} and $\hat{\pi}_{n+1}$ all being assumed monic. Since

$$\int_{-1}^{1} \pi_{n+1}^2 w \, \mathrm{d}t = \beta_0 \beta_1 \cdots \beta_n \beta_{n+1},$$

where the β's are the recursion coefficients $\beta_k(w)$ for the orthogonal polynomials $\{\pi_k(\cdot; w)\}$ (cf. (4.1)), and similarly

$$\int_{-1}^{1} \hat{\pi}_{n+1}^2 \pi_n^2 w \, \mathrm{d}t = \hat{\beta}_0 \hat{\beta}_1 \cdots \hat{\beta}_n \hat{\beta}_{n+1},$$

where $\hat{\beta}_k = \beta_k(\hat{w}_n)$ (cf. (4.13)), the ratio ρ_n of the upper bounds in (6.3) and (6.2) is

$$\rho_n = \frac{\hat{\beta}_0 \hat{\beta}_1 \cdots \hat{\beta}_n \hat{\beta}_{n+1}}{\beta_0 \beta_1 \cdots \beta_n \beta_{n+1}} \left(\frac{M_{2n+1}}{M_{n+1}} \right)^2.$$

Since $\hat{\beta}_0 = \int_{-1}^{1} \pi_n^2 w \, \mathrm{d}t = \beta_0 \beta_1 \cdots \beta_n$, this simplifies to

$$\rho_n = \omega_n \left(\frac{M_{2n+1}}{M_{n+1}} \right)^2, \quad \omega_n = \frac{\hat{\beta}_1 \hat{\beta}_2 \cdots \hat{\beta}_{n+1}}{\beta_{n+1}}. \tag{6.4}$$

Although conceivably M_{2n+1} is considerably larger than M_{n+1}, the quantity ω_n goes to zero rather quickly, so that the L_2-error of $\hat{L}_{2n+1} f$ is typically much smaller than that of $L_{n+1} f$.

Table 6.1
The quantities ω_n, $n=5(5)40$, for Gegenbauer weights $w^{(\alpha,\alpha)}$, $\alpha=0, 1, 2, 5, 10$

n	$\alpha=0$	$\alpha=1$	$\alpha=2$	$\alpha=5$	$\alpha=10$
5	$1.092\cdot10^{-3}$	$9.923\cdot10^{-4}$	$1.075\cdot10^{-3}$	$7.987\cdot10^{-4}$	$2.930\cdot10^{-4}$
10	$1.082\cdot10^{-6}$	$1.024\cdot10^{-6}$	$1.373\cdot10^{-6}$	$9.282\cdot10^{-7}$	$2.686\cdot10^{-7}$
15	$1.063\cdot10^{-9}$	$1.033\cdot10^{-9}$	$1.548\cdot10^{-9}$	$9.113\cdot10^{-10}$	$3.922\cdot10^{-10}$
20	$1.041\cdot10^{-12}$	$1.030\cdot10^{-12}$	$1.642\cdot10^{-12}$	$9.295\cdot10^{-13}$	$4.524\cdot10^{-13}$
25	$1.018\cdot10^{-15}$	$1.021\cdot10^{-15}$	$1.688\cdot10^{-15}$	$9.805\cdot10^{-16}$	$4.548\cdot10^{-16}$
30	$9.952\cdot10^{-19}$	$1.009\cdot10^{-18}$	$1.705\cdot10^{-18}$	$1.046\cdot10^{-18}$	$4.779\cdot10^{-19}$
35	$9.726\cdot10^{-22}$	$9.939\cdot10^{-22}$	$1.705\cdot10^{-21}$	$1.111\cdot10^{-21}$	$5.361\cdot10^{-22}$
40	$9.505\cdot10^{-25}$	$9.778\cdot10^{-25}$	$1.693\cdot10^{-24}$	$1.168\cdot10^{-24}$	$6.136\cdot10^{-25}$

Some numerical values of ω_n, for Gegenbauer weights $w^{(\alpha,\alpha)}$, $\alpha=0, 1, 2, 5, 10$, are shown in Table 6.1.

For $w=w_1$ (Chebyshev weight of the first kind) we know from [8] that $\hat{\beta}_1=\beta_1=\frac{1}{2}$, $\hat{\beta}_k=\beta_k=\frac{1}{4}$ for $2\leqslant k\leqslant n-1$, $\hat{\beta}_n=\frac{3}{8}$ and $\hat{\beta}_{n+1}=\frac{1}{8}$, so that $\omega_n=3\cdot2^{-(2n+1)}$. Similarly, for $w=w_2$, we get $\omega_n=2^{-2n}$. If, then, f is analytic in the disk $|z|\leqslant r$ with $r>1$, one finds $\rho_n=\mathrm{O}(2^{-2n}(r-1)^{-2n})$ as $n\to\infty$, hence $\rho_n=\mathrm{o}(1)$ if $r>\frac{3}{2}$.

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