BIT
2002, Vol. 42, No. 1, pp. 110–118
0006-3835/02/4201-0110 $16.00

COMPUTATION OF BESSEL AND AIRY FUNCTIONS AND OF RELATED GAUSSIAN QUADRATURE FORMULAE *

WALTER GAUTSCHI

Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, USA. email: wxg@cs.purdue.edu

Abstract.

Procedures are described for the high-precision calculation of the modified Bessel function $K_\nu(x)$, $0 < \nu < 1$, and the Airy function $\mathrm{Ai}(x)$, for positive arguments x, as prerequisites for generating Gaussian quadrature rules having these functions as weight function.

AMS subject classification: 33C10, 33C45, 65D20, 65D32.

Key words: Modified Bessel function, Airy function, Gaussian quadrature.

1 Introduction.

Integrals involving modified Bessel functions K_ν for $\nu = \frac{1}{3}$ and the Airy function Ai occur naturally in some physics applications (see, e.g., Gordon [5, 6]); the weight function K_0 has also found use in the asymptotic estimation of oscillatory integral transforms (see Wong [12], Gautschi [4, Example 6.1, p. 94]). Efficient evaluation of such integrals calls for Gaussian quadrature rules having these functions as weight function. Such rules (involving up to 10 points) were already given in [6] for the modified Bessel function $K_{\frac{1}{3}}$ and in [5, 9] for the Airy function. It appears, however, that the latter are in error because of an incorrect calculation of the relevant moments. Here we develop Gaussian quadrature formulae for both weight functions with up to $n = 40$ points. The main task is to find the first n recursion coefficients in the three-term recurrence relation for the relevant orthogonal polynomials. These can be obtained to arbitrary precision from the known moments by symbolic computation, or else, to standard machine precisions, by general procedures developed earlier in [3]. To apply these procedures, it is important to have routines that calculate modified Bessel functions and the Airy function to high accuracy. Such routines are described in Section 2 for Bessel functions, and in Section 3 for the Airy function. Section 4 discusses the computation of the respective Gaussian quadrature rules. An appendix contains the first 40 of the requisite recursion coefficients to 28 decimals.

*Received December 2000. Communicated by Kaj Madsen.

2 Modified Bessel functions.

Our interest is in generating values to high accuracy of the modified Bessel function $K_\nu(x)$, where $0 < \nu < 1$ and $x > 0$, as a preparation for generating the Gaussian quadrature rules in Sections 4.1 and 4.2. A case of particular interest is $\nu = \frac{1}{3}$, but the procedure we develop is applicable also for other values of ν except those close to 0 and 1. It is not our intention, here, to develop a general-purpose production code. There are a number of such codes available in the literature (see, e.g., [8, Sections 4.1, 5.1]), that—like ours—are suitable also for high-accuracy work. What we find worth observing is the apparently novel use of integral representations and related generalized Gauss–Laguerre quadratures to compute modified Bessel functions for moderately large, and large, real arguments. The approach, in fact, is potentially useful also for complex arguments.

When x is relatively small, say $0 < x \le 2$, we use, as others have done before, the representation (cf. [1, Eq. 9.6.2])

$$K_\nu(x) = \tfrac{1}{2}\pi \, \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}\,, \tag{2.1}$$

and evaluate $I_{\pm\nu}(x)$ by Taylor expansion ([1, Eq. 9.6.10])

$$I_{\pm\nu}(x) = (\tfrac{1}{2}x)^{\pm\nu} \sum_{k=0}^{\infty} \frac{(\frac{1}{4}x^2)^k}{k!\Gamma(k+1\pm\nu)}\,. \tag{2.2}$$

If ν is close to 0 or 1, considerable cancellation occurs in the numerator of (2.1). This could be dealt with, if deemed necessary, by special additional procedures (cf., e.g., [10, Section II]). Here we simply assume $0.05 \le \nu \le 0.95$, which limits the loss of accuracy to at most two (or three, if x is near 2) decimal digits.

For $x > 2$, we use the integral representation ([1, Eq. 9.6.23])

$$K_\nu(x) = \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+\frac{1}{2})} \, \frac{e^{-x}}{\sqrt{x}} \int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t} dt, \tag{2.3}$$

where the integral is conveniently evaluated by generalized Gauss–Laguerre quadrature with parameter $\alpha = \nu - \frac{1}{2}$,

$$\int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t} dt \simeq \sum_{k=1}^{n} w_k^L \left(2 + \frac{t_k^L}{x}\right)^{\nu-\frac{1}{2}}, \quad x > 2. \tag{2.4}$$

Here, t_k^L, w_k^L are the nodes and weights of the generalized Gauss–Laguerre quadrature rule. (The dependence on n is suppressed in the notation.) These, for $\nu = \frac{1}{3}$, were generated by double-precision resp. quadruple-precision analogues of the procedures `recur` and `gauss` of [3]. While it would be unreasonable to expect convergence to full machine precision as $n \to \infty$, it was found that in double and quadruple precision, "numerical convergence" occurs to an accuracy of $10\times\varepsilon_{\text{dble}}$ resp. $1000\times\varepsilon_{\text{quad}}$, where $\varepsilon_{\text{dble}} \simeq .111\times10^{-15}$, $\varepsilon_{\text{quad}} \simeq .963\times10^{-34}$ are respectively the IEEE double- and quadruple-precision machine precisions. In other words, the approximants stabilized to these accuracies at certain values of n, which are shown in Table 2.1.

Table 2.1: Number of Gauss points in (2.4), with $\nu = \frac{1}{3}$, required for double- and quadruple-precision accuracy.

ν	0	1/6	1/3	1/2	2/3	5/6	1
n double	23	23	22	1	21	22	22
n quadruple	87	85	83	1	82	82	82

(The data is for $x = 2$; as x increases, n decreases.)

In many applications (including the one in Section 4.1; cf. (4.7)), it is better to compute $e^x K_\nu(x)$. This is also the function tabulated (for $\nu = \frac{1}{3}$) in [11, Table III].

We remark that our evaluation procedure is also applicable for x in the complex plane cut along the negative real axis, but we will not pursue this here any further.

3 The Airy function.

The Airy function is related to the modified Bessel function $K_{\frac{1}{3}}$ as follows (cf. [1, Eq. 10.4.14]):

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\, K_{\frac{1}{3}}(\zeta), \quad \zeta = \tfrac{2}{3}x^{\frac{3}{2}}. \tag{3.1}$$

Using (2.1) with $\nu = \frac{1}{3}$, one gets

$$\mathrm{Ai}(x) = \frac{1}{3}\sqrt{x}\,[I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)]. \tag{3.2}$$

Here, for $0 < \zeta \le 2$, i.e., $0 < x \le 3^{2/3} = 2.08008\ldots$, both Bessel functions can be evaluated by Taylor expansion as in (2.2) with $\nu = \frac{1}{3}$.

For $\zeta > 2$, we use the integral representation (2.3) for $K_{\frac{1}{3}}$ in conjunction with (3.1) to obtain

$$\mathrm{Ai}(x) = \frac{1}{\sqrt{\pi}}\,\frac{\zeta^{-\frac{1}{6}}e^{-\zeta}}{(48)^{\frac{1}{6}}\Gamma(\frac{5}{6})}\int_0^\infty \left(2 + \frac{t}{\zeta}\right)^{-\frac{1}{6}} \cdot t^{-\frac{1}{6}}e^{-t}dt, \tag{3.3}$$

with ζ as defined in (3.1). Now generalized Gauss–Laguerre quadrature is appropriate with Laguerre parameter $\alpha = -\frac{1}{6}$. According to Table 2.1 ($\nu = \frac{1}{3}$), a 22-point formula yields double-precision accuracy and a 83-point formula quadruple-precision accuracy.

This procedure can be used also for complex x, at least in the sector $|\arg x| < \frac{2}{3}\pi$.

4 Gauss quadratures.

4.1 Gauss quadrature with Bessel weight function.

We define the weight function

$$w(x) = \frac{2}{\pi}\cos(\tfrac{1}{2}\nu\pi)K_\nu(x), \quad 0 < x < \infty. \tag{4.1}$$

Its moments can be calculated explicitly by (cf. [7, 6.561.16])

$$\mu_k = \int_0^\infty x^k w(x)dx = \frac{1}{\pi}\cos(\tfrac{1}{2}\nu\pi)\cdot 2^k\Gamma(\tfrac{1}{2}(k+1+\nu))\Gamma(\tfrac{1}{2}(k+1-\nu)). \tag{4.2}$$

In particular, for $k = 0$,

$$\mu_0 = \frac{1}{\pi}\cos(\tfrac{1}{2}\nu\pi)\cdot\Gamma(\tfrac{1}{2}(1+\nu))\Gamma(\tfrac{1}{2}(1-\nu)) = 1 \tag{4.3}$$

by virtue of the reflection formula ([1, 6.1.17]) for $\Gamma(z)\Gamma(1-z)$, $z = \frac{1}{2}(1+\nu)$. Thus, (4.1) is a normalized weight function.

In principle, the first $2n$ moments μ_k of w can be used to generate the n-point Gauss formula for w. It is well known, however, that this becomes quickly unstable as n increases. A way around this problem is to employ a symbolic computation package combined with extended-precision arithmetic. A Maple script, named `cheb.mws`[1], has been developed for this purpose and has been used to produce the required recursion coefficients α_k, β_k (cf. (4.4) below) for $0 \le k \le 39$ to as many as 100 decimal digits[2]. It can be accessed at
`http://www.cs.purdue.edu/archives/2001/wxg/codes/cheb.mws`.
(A text file `cheb.txt` can be found at the same URL.)

Here, we describe a stable numerical procedure—a four-pronged discretization procedure (cf. [4, Section 6])—that discretizes the inner product for the weight function w and generates the corresponding discrete orthogonal polynomials. If the discretization is chosen judiciously, the discrete orthogonal polynomials converge to the desired ones as the discretization is made increasingly finer. The first n recursion coefficients α_k, β_k in the three-term recurrence relation

$$\begin{gathered}\pi_{k+1}(x) = (x-\alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0, 1, \ldots, n-1,\\ \pi_0(x) = 1, \quad \pi_{-1}(x) = 0,\end{gathered} \tag{4.4}$$

for the monic orthogonal polynomials (where $\beta_0 = \int_0^\infty w(x)dx$) are computed by the Stieltjes procedure ([4, Section 6.3]).

The discretization we choose makes use of a composition of the positive real axis into three subintervals, $\mathbb{R}_+ = (0, x_0] \cup [x_0, x_1] \cup [x_1, \infty)$, with x_0, x_1 still to be selected such that $0 < x_0 \le 1$, $1 < x_1 < \infty$. In the first subinterval, the behavior of $K_\nu(x)$ for small x must be properly accounted for. One has [1, Eqs. 9.6.2 and 9.6.10]

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left\{\frac{(\frac{1}{2}x)^{-\nu}}{\Gamma(1-\nu)}S_{-\nu}(x) - \frac{(\frac{1}{2}x)^{\nu}}{\Gamma(1+\nu)}S_\nu(x)\right\}, \tag{4.5}$$

where

$$S_{\pm\nu}(x) = \sum_{k=0}^\infty \frac{(\frac{1}{4}x^2)^k\Gamma(1\pm\nu)}{k!\Gamma(k+1\pm\nu)}.$$

[1]This is written for Maple.Release 5.

[2]The author is indebted to Oscar Chinellato at the Institute for Scientific Computing of the ETH Zurich, Switzerland, for translating a slightly edited version of our ORTHPOL routine `cheb` (cf. [3]) into a Maple script.

The two distinct behaviors $x^{-\nu}$ and x^{ν} as $x \to 0$ need to be treated separately for purposes of integration. Indeed, the following composition for integrals over $(0, x_0]$ is suggested:

$$\int_0^{x_0} p(x)K_\nu(x)dx = \frac{2^{\nu-1}\pi}{\sin\nu\pi\cdot\Gamma(1-\nu)}\int_0^{x_0} p(x)S_{-\nu}(x)\cdot x^{-\nu}dx - \frac{\pi}{2^{\nu+1}\sin\nu\pi\cdot\Gamma(1+\nu)}\int_0^{x_0} p(x)S_\nu(x)\cdot x^{\nu}dx. \tag{4.6}$$

The first integral on the right is approximated by an N-point Gauss–Jacobi quadrature rule relative to the interval $(0, x_0]$ with Jacobi parameters $\alpha = 0$, $\beta = -\nu$, the second by a similar N-point Gauss–Jacobi rule with parameters $\alpha = 0$, $\beta = \nu$. In the second interval, we apply the ordinary N-point Gauss–Legendre rule transformed to the interval $[x_0, x_1]$. For the last interval, we write

$$\int_{x_1}^{\infty} p(x)K_\nu(x)dx = e^{-x_1}\int_0^{\infty} p(x_1+t)[e^{x_1+t}K_\nu(x_1+t)]\cdot e^{-t}dt \tag{4.7}$$

and approximate the integral on the right by an N-point Gauss–Laguerre quadrature rule, the function $e^x K_\nu(x)$, $x \geq x_1$, being computed from the integral representation (2.3) as discussed in the text following (2.3).

We call this a "four-pronged" discretization procedure since four different quadrature rules are employed to discretize the integral $\int_0^\infty p(x)K_\nu(x)dx$ (and with it the inner product relative to the weight function (4.1)): two Gauss–Jacobi rules for approximating the two integrals on the right of (4.6), the Gauss rule over $[x_0, x_1]$, and the Gauss–Laguerre rule to deal with the integral on the right of (4.7).

The parameters x_0, x_1 are chosen in an attempt to reduce the number N of quadrature terms required to achieve a given accuracy. Some limited experimentation suggested the choice $x_0 = 1$ and $x_1 = 10$. When $\nu = \frac{1}{3}$ and $n = 10$, then $N = 41$ will yield a relative accuracy of 1000 times the double machine precision (about 12 decimal-digit accuracy), and $N = 71$ a relative accuracy of 10^5 times the quadruple machine precision (about 29 decimal digits). For $n = 40$ the respective numbers are both $N = 81$.

The values of the recursion coefficients α_k, β_k (for $\nu = \frac{1}{3}$) to 28 decimal digits are given in the appendix, Table A1, for $k = 0, 1, \ldots, 39$. These allow us to generate the respective Gauss and Gauss–Radau quadrature rules for up to 40 points by well-known eigenvalue/eigenvector techniques [4, Section 4]. Double- and quadruple-precision fortran programs producing these coefficients are accessible at `http://www.cs.purdue.edu/archives/2001/wxg/codes/` in the files `dOPbess.f` and `qOPbess.f`. The coefficients themselves to 28 decimals can be found in the file `coeffbess` at the same URL. The file `ORTHPOLq` contains the quadruple-precision routines of the package ORTHPOL in [3].

4.2 Gauss quadrature with Airy weight function.

We define a weight function proportional to the one in Eq. (1.4) of [9],

$$w(x) = \frac{2^{\frac{2}{3}}\pi}{3^{\frac{5}{6}}\Gamma(\frac{2}{3})}x^{-\frac{2}{3}}e^{-x}\mathrm{Ai}((\tfrac{3}{2}x)^{\frac{2}{3}}), \quad 0 < x < \infty. \tag{4.8}$$

By (3.1), one has

$$w(x) = \frac{2^{\frac{1}{3}}}{3\Gamma(\frac{2}{3})}\, x^{-\frac{1}{3}} e^{-x} K_{\frac{1}{3}}(x). \tag{4.9}$$

As in [9], we use [7, 6.621.3] to obtain for the moments (correctly)

$$\mu_k = \int_0^\infty x^k w(x)dx = \frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma(\frac{2}{3})} \frac{k!\Gamma(k+\frac{1}{3})}{(6k+1)2^k\Gamma(k+\frac{1}{6})}. \tag{4.10}$$

In particular,

$$\mu_0 = \frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma(\frac{2}{3})} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})} = 1 \tag{4.11}$$

by virtue of the duplication formula [1, 6.1.19] for $\Gamma(2z)$, $z = \frac{1}{6}$. Thus, the weight function (4.8) is normalized.

The three-term recurrence relation (4.4) for this weight function can again be obtained by symbolic computation using the Maple script `cheb.mws` (cf. §4.1). Numerically, on the other hand, we may use, similarly as in Section 4.1, a three-pronged discretization method. First, integration against the weight function is "regularized" by means of the change of variable $x \mapsto x^3/3$; thus,

$$\int_0^\infty p(x)w(x)dx = \frac{2^{\frac{2}{3}}\pi}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})} \int_0^\infty p(\tfrac{1}{3}x^3)\mathrm{Ai}(2^{-\frac{2}{3}}x^2)\cdot e^{-x^3/3}dx. \tag{4.12}$$

The discretization is effected by using the decomposition $\mathbb{R}_+ = (0,2]\cup[2,6]\cup[6,\infty)$ and N-point Gauss–Legendre quadrature on the first two intervals, and N-point Gauss quadrature relative to the weight function $e^{-x^3/3}$, $0 < x < \infty$, on the last interval (after transforming the integral over $[6,\infty)$ to one over $[0,\infty)$). The latter quadrature rules have been generated by a "general-purpose" discretization method [4, p. 95]; see also [2]. In this way, when $n = 10$, then $N = 51$ (in double precision) yields about 12-digit accuracy, $N = 91$ (in quadruple precision) an accuracy of about 29 digits. For $n = 40$, the respective numbers are $N = 81$ and $N = 161$.

Values of the recursion coefficients α_k, β_k to 28 decimal digits are given in the appendix, Table A2, for $k = 0, 1, \ldots, 39$. These again permit the generation of Gauss and Gauss–Radau quadrature rules with up to 40 points. Double- and quadruple-precision fortran programs producing these coefficients are accessible at

`http://www.cs.purdue.edu/archives/2001/wxg/codes/` in the files`dOPairy.f` and `qOPairy.f`. The coefficients themselves to 28 decimals can be found in the file `coeffairy` at the same URL.

Acknowledgement.

The author gratefully acknowledges helpful comments from two anonymous referees.

A Appendix.

Recursion coefficients α_k and β_k for the orthogonal polynomials relative to the weight function $\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}(x)$, $0 < x < \infty$, are given in Table A.1.

Table A.1: Recursion coefficients for orthogonal polynomials with Bessel weight function.

k	dalpha(k)	dbeta(k)
0	0.57735026918962576450914878 05D+00	0.10000000000000000000000000 00D+01
1	0.25403411844343533638402546 34D+01	0.55555555555555555555555555 56D+00
2	0.45303250992775569729406193 92D+01	0.30800000000000000000000000 00D+01
3	0.65252632053670247545844211 46D+01	0.75973088960101947114934127 92D+01
4	0.85220916815723164954105718 18D+01	0.14111288950643590484767646 85D+02
5	0.10519870690325796817093273 97D+02	0.22623283756519972141184890 43D+02
6	0.12518205342004118426986613 10D+02	0.33133934495118279090152051 75D+02
7	0.14516897450708475899825403 47D+02	0.45643601984576447974202422 03D+02
8	0.16515835338909712945017052 03D+02	0.60152511693871654445741830 86D+02
9	0.18514950703700216501467869 56D+02	0.76660815122790272370295821 99D+02
10	0.20514199134596376599961869 34D+02	0.95168619641531512764202519 70D+02
11	0.22513550350966397331553300 55D+02	0.11567600451295364022414377 43D+03
12	0.24512982906925447589739452 21D+02	0.13818303016614297368305199 92D+03
13	0.26512481134031380745844017 42D+02	0.16268974388883212711028968 29D+03
14	0.28512033284217889886196819 52D+02	0.18919618349092681270044297 81D+03
15	0.30511630353379984494986073 62D+02	0.21770237975590841525221055 42D+03
16	0.32511265309289715164513256 88D+02	0.24820835813554385367738681 70D+03
17	0.34510932569315403938872506 2D+02	0.28071413995440991178675067 45D+03
18	0.36510627637760244734914891 60D+02	0.31521974328668330503422661 85D+03
19	0.38510346848218176853857670 70D+02	0.35172518360774645238197113 69D+03
20	0.40510087176814417926038440 69D+02	0.39023047428733980675972347 09D+03
21	0.42509846104386257094909113 93D+02	0.43073562696885280867041753 67D+03
22	0.44509621513140441276089535 58D+02	0.47324065186525903076257111 71D+03
23	0.46509411608040199994936432 28D+02	0.51774555799300470520410315 95D+03
24	0.48509214856221347986475885 26D+02	0.56425035335901673936349506 13D+03
25	0.50509029939747541853289263 80D+02	0.61275504511180750405912503 02D+03
26	0.52508855718368037952718831 28D+02	0.66325963966474388749188552 82D+03
27	0.54508691199868481800078655 62D+02	0.71576414279749246320929380 47D+03
28	0.56508535516250920027796913 10D+02	0.77026855974017792568324480 17D+03
29	0.58508387904435593090423523 07D+02	0.82677289524371904890660617 45D+03
30	0.60508247690504090767465010 85D+02	0.88527715363901570667787427 62D+03
31	0.62508114276740777255813280 07D+02	0.94578133888707076553849392 25D+03
32	0.64507987130903653941908285 50D+02	0.10082854546216859666327285 10D+04
33	0.66507865772851844459428789 7D+02	0.10727895041860321418401763 89D+04
34	0.68507749789220609720584340 23D+02	0.11392934906641335695677578 33D+04
35	0.70507638782774714709547449 45D+02	0.12077974169081041187988146 82D+04
36	0.72507532411394094660919210 10D+02	0.12783012855618146698424410 05D+04
37	0.74507430361355164228014596 83D+02	0.13508050990815465982217713 77D+04
38	0.76507332347871680219924933 44D+02	0.14253088597540870680261682 99D+04
39	0.78507238111751765159290348 05D+02	0.15018125697126426815746413 94D+04

Recursion coefficients α_k and β_k for the orthogonal polynomials relative to the weight function $\frac{2^{2/3}\pi}{3^{5/6}\Gamma(2/3)}\, x^{-\frac{2}{3}} e^{-x} \mathrm{Ai}\big((\frac{3}{2}x)^{\frac{2}{3}}\big)$, $0 < x < \infty$, are given in Table A.2.

Table A.2: Recursion coefficients for orthogonal polynomials with Airy weight function.

k	dalpha(k)	dbeta(k)
0	0.14285714285714285714285714D+00	0.10000000000000000000000000D+01
1	0.11105088302150725651337646 44D+01	0.67503924646781789638932496 08D-01
2	0.21039232623033664697190466 50D+01	0.61764355206314549464773663 22D+00
3	0.31006129236194680602040417 49D+01	0.16652223695797287639935981 04D+01
4	0.40985229900533503002596773 56D+01	0.32116654379338133014426731 08D+01
5	0.50970480179122379522111913 66D+01	0.52574397914282669985698940 03D+01
6	0.60959347721427352894050676 51D+01	0.78027628718161412450789162 30D+01
7	0.70950557426151749696957611 15D+01	0.10847755827663314637317852 13D+02
8	0.80943387049797903045725228 53D+01	0.14392494051279503008919829 57D+02
9	0.90937392474629931366999661 23D+01	0.18437028129221289287730985 12D+02
10	0.10093228346099565290956749 25D+02	0.22981393910509113133201030 04D+02
11	0.11092786118994277963895130 21D+02	0.28025617876564395301895131 39D+02
12	0.12092398425735241831496679 65D+02	0.33569720237015983129850790 15D+02
13	0.13092054896137796054305526 64D+02	0.39613716825662470924767630 52D+02
14	0.14091747728196667776798864 68D+02	0.46157620319174718546589178 80D+02
15	0.15091470923783667057958572 63D+02	0.53201441052135684780032004 69D+02
16	0.16091219785281719839537880 68D+02	0.60745187580452629573767385 63D+02
17	0.17090990573969440137935584 92D+02	0.68788867081900320364309351 16D+02
18	0.18090780272089838774453518 34D+02	0.77332485647813606942498593 41D+02
19	0.19090586413340360720405440 31D+02	0.86376048499994747266102355 45D+02
20	0.20090406959676781196253599 98D+02	0.95919560154988984853375918 88D+02
21	0.21090240210178251886147338 37D+02	0.10596302455052778144792737 89D+03
22	0.22090084732554680300801113 79D+02	0.11650644514426334671872770 61D+03
23	0.23089939310933302448587962 06D+02	0.12754982499186694087724614 53D+03
24	0.24089802905539742800085714 54D+02	0.13909316809525760664644148 52D+03
25	0.25089674621197700228155754 63D+02	0.15113647302448387041451470 93D+03
26	0.26089553682454295865206351 28D+02	0.16367974581630744705402971 92D+03
27	0.27089439413744318231582362 92D+02	0.17672298715087261457351113 46D+03
28	0.28089331223429595213827617 89D+02	0.19026619880858475806527647 75D+03
29	0.29089228590849281318522073 32D+02	0.20430938240798204060362816 48D+03
30	0.30089131055731901297556390 56D+02	0.21885253942561325212625418 85D+03
31	0.31089038209476330674573859 95D+02	0.23389567121288418866806529 98D+03
32	0.32088949687923881056051201 54D+02	0.24943877901041894670919058 99D+03
33	0.33088865165329150952378707 07D+02	0.26548186396036985718116802 51D+03
34	0.34088784349301510252239270 20D+02	0.28202492711702309565791942 38D+03
35	0.35088706976537763785392615 56D+02	0.29906796945597970768703581 57D+03
36	0.36088632809203769272943704 07D+02	0.31661099188213910906157393 19D+03
37	0.37088561631851495611747249 58D+02	0.33465399523667056339923916 24D+03
38	0.38088493248780320814079768 38D+02	0.35319698030312511600256047 49D+03
39	0.39088427481768837660306608 97D+02	0.37223994781281404077676699 42D+03

REFERENCES

1. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, NBS Applied Mathematics Series, Vol. 55, U.S. Government Printing Office, Washington, DC, 1964.
2. W. Gautschi, *How and how not to check Gaussian quadrature formulae*, BIT, 23 (1983), pp. 209–216.
3. W. Gautschi, *Algorithm 726: ORTHPOL—A package of routines for generating orthogonal polynomials and Gauss-type quadrature rules*, ACM Trans. Math. Software, 20 (1994), pp. 21–62.
4. W. Gautschi, *Orthogonal polynomials: applications and computation*, Acta Numerica, 5 (1996), pp. 45–119.
5. R. G. Gordon, *New method for constructing wavefunctions for bound states and scattering*, J. Chem. Phys., 51 (1969), pp. 14–25.
6. R. G. Gordon, *Constructing wavefunctions for nonlocal potentials*, J. Chem. Phys., 52 (1970), pp. 6211–6217.
7. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, San Diego, CA, 2000.
8. D. W. Lozier and F. W. J. Olver, *Numerical evaluation of special functions*, in Mathematics of Computation 1943–1993: A half-century of computational mathematics, Vancouver, BC, 1993, W. Gautschi, ed., Proc. Sympos. Appl. Math., Vol. 48, Amer. Math. Soc., Providence, RI, 1994, pp. 79–125.
9. Z. Schulten, D. G. M. Anderson, and R. G. Gordon, *An algorithm for the evaluation of the complex Airy functions*, J. Comput. Phys., 31 (1979), pp. 60–75.
10. N. M. Temme, *On the numerical evaluation of the modified Bessel function of the third kind*, J. Comput. Phys., 19 (1975), pp. 324–337.
11. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1958.
12. R. Wong, *Quadrature formulas for oscillatory integral transforms*, Numer. Math., 39 (1982), pp. 351–360.