Reprinted from JOURNAL OF MATHEMATICS AND PHYSICS
Vol. 38, No. 1, April, 1959
*Printed in U.S.A.*

# SOME ELEMENTARY INEQUALITIES RELATING TO THE GAMMA AND INCOMPLETE GAMMA FUNCTION*

BY WALTER GAUTSCHI

**1.** In a recent note Y. Komatu [3] has proved the inequality

$$\frac{1}{x+\sqrt{x^2+2}} < e^{x^2}\int_x^\infty e^{-t^2}\,dt < \frac{1}{x+\sqrt{x^2+1}} \quad (0 \leqq x < \infty). \tag{1}$$

The deviation of the bounds from the estimated function decreases monotonically to zero as $x$ varies from zero to infinity. H. O. Pollak [5] has improved the upper bound by showing that

$$e^{x^2}\int_x^\infty e^{-t^2}\,dt \leqq \frac{1}{x+\sqrt{x^2+4/\pi}} \tag{2}$$

with a deviation increasing from zero to a maximum value and decreasing, from there on, monotonically to zero.

We shall prove in section 3 the more general inequality

$$\begin{aligned} &\tfrac{1}{2}((x^p+2)^{1/p}-x) < e^{x^p}\int_x^\infty e^{-t^p}\,dt \\ &\leqq c_p\left(\left(x^p+\frac{1}{c_p}\right)^{1/p}-x\right),\ c_p = \left\{\Gamma\left(1+\frac{1}{p}\right)\right\}^{p/(p-1)} \quad (0 \leqq x < \infty) \end{aligned} \tag{3}$$

where $p$ is any real number $> 1$.[1] For $p = 2$ the right-hand inequality in (3) reduces to (2) while the left-hand inequality reduces to the corresponding inequality in (1). The deviations of the bounds in the general $p$-case behave the same way as in the special case $p = 2$. Also, (3) remains valid if we replace $c_p$ by 1. The quality of the bounds is indicated in Fig. 1.

By an easy transformation we can write (3) in terms of the complementary gamma function $\Gamma(a, x) = \int_x^\infty e^{-t}t^{a-1}\,dt$ as follows:

$$\begin{aligned} &\tfrac{1}{2}p((x+2)^{1/p}-x^{1/p}) < e^x\Gamma(p^{-1}, x) \\ &\leqq pc_p((x+c_p^{-1})^{1/p}-x^{1/p}) \quad (0 \leqq x < \infty). \end{aligned} \tag{4}$$

In particular, if $p \to \infty$, we obtain an inequality for the exponential integral $E_1(x) = \Gamma(0, x)$:

$$\tfrac{1}{2}\ln(1+2x^{-1}) \leqq e^x E_1(x) \leqq \ln(1+x^{-1}) \quad (0 < x < \infty). \tag{5}$$

This improves an inequality due to E. Hopf [1]; the bounds in (5) exhibit the logarithmic singularity of $E_1(x)$ at $x = 0$.

* This paper was prepared under a National Bureau of Standards contract with American University.

[1] The integral in (3) for $p = 3$ occurs in heat transfer problems [2], for $p = 4$ in the study of electrical discharge through gases [6]. An application of (3) for general $p$ is given in [4].

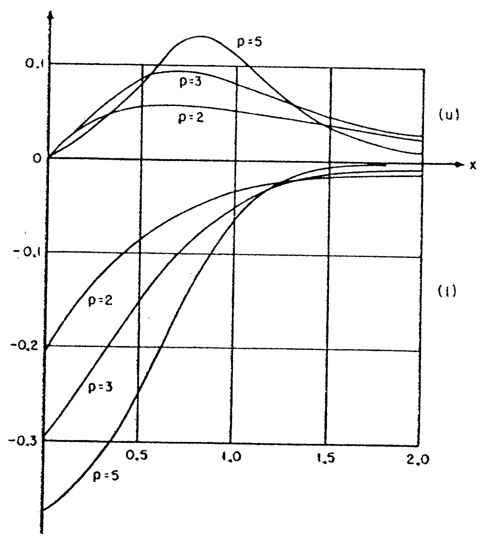


FIG. 1. Relative error of upper ($u$) and lower ($l$) bounds in (3)

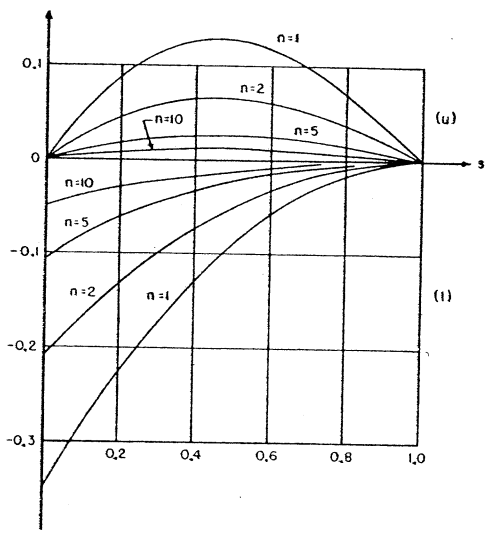


FIG. 2. Relative error of upper ($u$) and lower ($l$) bounds in (6)

**2.** From (4) we can deduce a simple inequality for the gamma function if we set $p = 1/s$, $x = 0$ and replace $c_p$ by 1:

$$2^{s-1} \leqq \Gamma(1 + s) \leqq 1 \qquad (0 \leqq s \leqq 1).$$

We shall prove in section 4 the sharper and more general inequality

$$(6) \qquad e^{(s-1)\psi(n+1)} \leqq \frac{\Gamma(n + s)}{\Gamma(n + 1)} \leqq n^{s-1} \qquad (0 \leqq s \leqq 1, n = 1, 2, 3, \cdots)$$

where $\psi(n+1) = \sum_{k=1}^{n} 1/k - \gamma$ and $\gamma = 0.57721 \ldots$ is the Euler-Mascheroni constant. We have equality in (6) only for $s = 1$ on the left-hand side and for $s = 0$ and $s = 1$ on the right-hand side. Fig. 2 illustrates the quality of the bounds. Since $\psi(n) < \ln n$ we have also

$$(7) \qquad \left(\frac{1}{n+1}\right)^{1-s} \leqq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leqq \left(\frac{1}{n}\right)^{1-s} \qquad (0 \leqq s \leqq 1).$$

It may be of interest to note that by letting $n \to \infty$ in (7) we obtain a simple proof of Euler's product formula in the segment $0 \leqq s \leqq 1$. In fact, (7) is equivalent to

$$(8) \qquad \frac{n!(n+1)^{s-1}}{(s+1)(s+2)\cdots(s+n-1)} \leqq \Gamma(1+s) \leqq \frac{(n-1)!n^s}{(s+1)(s+2)\cdots(s+n-1)}.$$

Setting

$$\gamma_n(s) = \frac{(n-1)!n^s}{(s+1)(s+2)\cdots(s+n-1)}$$

we can write (8) in the form

$$\left(\frac{1}{1+1/n}\right)^{1-s} \gamma_n(s) \leqq \Gamma(1+s) \leqq \gamma_n(s).$$

Therefore

$$0 \leqq \gamma_n(s) - \Gamma(1+s) \leqq \Gamma(1+s)\{(1+1/n)^{1-s} - 1\} = 0(1/n) \quad (n \to \infty).$$

**3.** *Proof of* (3). Let

$$(9) \qquad \Delta_p(a, x) = ae^{-x^p}((x^p + a^{-1})^{1/p} - x) - \int_x^\infty e^{-t^p}\, dt \ (a > 0).$$

We have to prove that

$$(10) \qquad \Delta_p(c_p, x) \geqq 0, \qquad \Delta_p(\tfrac{1}{2}, x) < 0 \qquad (0 \leqq x < \infty).$$

Differentiating (9) with respect to $x$ we find

$$(11) \qquad u^{p-1}(u^p - 1)e^{x^p} \frac{\partial}{\partial x} \Delta_p(a, x) = (1-a)u^{2p-1} - (p-a)u^p + (p+a-1)u^{p-1} - a$$

where

$$(12) \qquad u = [1 + (1/ax^p)]^{1/p}, \qquad u \geqq 1.$$

Denoting the polynomial on the right-hand side of (11) by $g_p(u)$ we have

$$(13) \qquad g_p(1) = g_p'(1) = 0, \qquad g_p''(1) = p(p-1)(1-2a).$$

Consider now the case $a = c_p$. We first note that

$$(14) \qquad \Delta_p(a, \infty) = 0, \qquad \Delta_p(c_p, 0) = 0.$$

Next we notice that the coefficients of $g_p(u)$ alternate in sign. Since there are three sign changes we conclude from Descartes' rule that $g_p(u)$ has either three positive zeros or one. (13) shows that two zeros are located at $u = 1$; thus $g_p(u)$ has exactly three positive zeros. Furthermore, since $g_p''(1) < 0$ and $g_p(\infty) = \infty$, the third zero must be larger than 1. Therefore, as $u$ increases from 1 to $\infty$ the polynomial $g_p(u)$ decreases from zero to a minimum value and from there on increases monotonically to $\infty$. On account of (11), (12) and (14) this means that $\Delta_p(c_p, x)$ increases from zero to a maximum value and from there on decreases monotonically to zero as $x$ varies from zero to $\infty$. This proves the first relation in (10).

Consider next the case $a = \frac{1}{2}$. Again, Descartes' rule applies and from (13) it follows that all three positive zeros of $g_p(u)$ coincide at $u = 1$. Therefore $g_p(u) > 0$ for $u > 1$, from which follows $(\partial/\partial x)\Delta_p(\frac{1}{2}, x) > 0$ for $x > 0$. This proves the second relation in (10) since $\Delta_p(\frac{1}{2}, \infty) = 0$.

**4.** *Proof of* (6). Consider

$$f(s) = \frac{1}{1-s} \ln \left\{\frac{\Gamma(n+s)}{\Gamma(n+1)}\right\} \qquad (0 \leqq s < 1).$$

We have $f(0) = \ln(1/n)$ and by using the rule of Bernoulli-L'Hospital

$$\lim_{s\to 1} f(s) = -\lim_{s\to 1} \psi(n+s) = -\psi(n+1)$$

where $\psi(x) = (d/dx)[\ln \Gamma(x)]$. We show that $f(s)$ is monotonically decreasing. We have

$$(1-s)f'(s) = f(s) + \psi(n+s).$$

Letting

$$\varphi(s) = (1-s)[f(s) + \psi(n+s)]$$

we have

$$\varphi(0) = \psi(n) - \ln n < 0, \qquad \varphi(1) = 0, \qquad \varphi'(s) = (1-s)\psi'(n+s).$$

Since $\psi'(n+s) > 0$ it follows that $\varphi(s) < 0$ and therefore $f'(s) < 0$ for $0 < s < 1$. Thus

$$-\psi(n+1) \leqq f(s) \leqq \ln(1/n)$$

which is equivalent to (6).

REFERENCES

[1] E. Hopf, Mathematical Problems of Radiative Equilibrium, Cambridge Tracts in Mathematics and Mathematical Physics, No. 31, Cambridge University Press 1934, p. 26.

[2] K. Yamagata, A Contribution to the Theory of Non-isothermal Laminar Flow of Fluids inside a Straight Tube of Circular Cross Section, Mem. Fac. Engr., Kyushu Imp. Univ. **8** (1940), pp. 365–449.

[3] Y. KOMATU, Elementary inequalities for Mills' ratio, Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1955), pp. 69–70.
[4] G. FRANKLIN MONTGOMERY, On the Transmission Error Function for Meteor-Burst Communication, Proc. IRE **46** (1958), pp. 1423–1424.
[5] H. O. POLLAK, A Remark on "Elementary Inequalities for Mills' Ratio" by Yûsaku Komatu, Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1956), p. 40.
[6] W. O. SCHUMANN, Elektrische Durchbruchfeldstärke von Gasen, Springer Berlin 1923.

AMERICAN UNIVERSITY
WASHINGTON, D. C.

(Received March 27, 1958)