On Inverses of Vandermonde and Confluent Vandermonde Matrices III*

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Summary. We derive lower bounds for the norm of the inverse Vandermonde matrix and the norm of certain inverse confluent Vandermonde matrices. They supplement upper bounds which were obtained in previous papers.

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1. Introduction

Norm estimates for the inverse of a Vandermonde matrix, or the inverse of confluent Vandermonde matrices, have been the subject of several previous papers [1, 2, 4]. The emphasis there was on upper bounds in the case of general complex nodes, or identities when the nodes are positive [1, 2] or real and symmetric with respect to the origin [4]. We now wish to supplement these results by providing lower bounds in the case of arbitrary complex nodes. We obtain these bounds by applying to appropriate polynomials Jensen’s formula in the theory of analytic functions.

2. Jensen’s Formula for Polynomials

Given a polynomial
\[ p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \] (2.1)
with complex coefficients \( a_\mu \), let \( \zeta_1, \zeta_2, \ldots, \zeta_n \) denote its zeros ordered such that
\[ |\zeta_1| \leq |\zeta_2| \leq \cdots \leq |\zeta_r| \leq 1 < |\zeta_{r+1}| \leq |\zeta_{r+2}| \leq \cdots \leq |\zeta_n|. \]
Jensen’s formula, applied to (2.1) on the unit circle, then gives [6]

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\[ |a_n \zeta_{r+1} \zeta_{r+2} \cdots \zeta_n| = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})| \, d\theta \right), \]

hence, letting \( M = \max_{0 \leq \theta \leq 2\pi} |p(e^{i\theta})|, \)

\[ |a_n \zeta_{r+1} \zeta_{r+2} \cdots \zeta_n| \leq M \leq \sum_{\mu=0}^{n} |a_\mu|. \] (2.2)

Thus,

\[ \sum_{\mu=0}^{n} |a_\mu| \geq |a_n| \prod_{v=1}^{n} \max (1, |\zeta_v|). \] (2.3)

Equality in (2.3) holds if and only if \( a_0 = a_1 = \cdots = a_{n-1} = 0, \) i.e., \( p(z) = a_n z^n. \) Indeed, if \( p(z) = a_n z^n, \) then (2.3) (with equality) is trivial. Conversely, if we have equality in (2.3), we must have equality in (2.2), hence, by Jensen’s formula, \( |p(e^{i\theta})| = M \) for \( 0 \leq \theta \leq 2\pi. \) Since

\[ |p(e^{i\theta})|^2 = \sum_{k,l=0}^{n} a_k \overline{a_l} e^{i(k-l)\theta} = \sum_{\lambda=-n}^{n} c_\lambda e^{i\lambda\theta} \]

is a trigonometric polynomial, with coefficients

\[ c_\lambda = \sum_{k=-\infty}^{\infty} a_k \overline{a}_{k-\lambda}, \quad c_{-\lambda} = \overline{c}_\lambda \]

(the convention \( a_\mu = 0, \) if \( \mu < 0 \) or \( \mu > n \) is used here), it can be constant equal to \( M^2 \) only if \( c_n = c_{n-1} = \cdots = c_1 = 0 \) and \( c_0 = M^2. \) The first condition, \( c_n = 0, \) implies \( a_n \overline{a}_0 = 0, \) hence \( a_0 = 0 \) (since \( a_n \neq 0 \)). The second condition, \( a_n \overline{a}_1 + a_{n-1} \overline{a}_0 = 0, \) then gives \( a_1 = 0, \) and continuing in this manner, we find recursively \( a_0 = a_1 = \cdots = a_{n-1} = 0. \)

3. Inverse Vandermonde Matrix

We denote the Vandermonde matrix of order \( n \) by

\[ V_n(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{-n} & z_2^{-n} & \cdots & z_n^{-n} \end{pmatrix} \] (3.1)

where \( z^T = [z_1, z_2, \ldots, z_n] \) is a vector of \( n \) complex numbers, called “nodes”. If the nodes are mutually distinct, then \( V_n(z) \) has an inverse, which we denote by

\[ V_n^{-1}(z) = [u_{\lambda\mu}]_{\lambda, \mu = 1}^{n}. \] (3.2)

We are interested in the \( l_\infty \)-norm of (3.2),

\[ \|V_n^{-1}(z)\|_\infty = \max_{1 \leq \lambda \leq n} \sum_{\mu=1}^{n} |u_{\lambda\mu}|. \]
Theorem 3.1. If $z_1, z_2, \ldots, z_n$ are mutually distinct complex numbers, and $n>1$, then
\[
\|V_n^{-1}(z)\|_\infty > \max_{1 \leq \lambda \leq n} \prod_{v=1}^{n} \max \left( 1, \frac{|z_v|}{|z_\lambda - z_v|} \right). \tag{3.3}
\]

Proof. We recall [4] that the elements $u_{\lambda, \mu}$ in (3.2) are the coefficients of the fundamental Lagrange interpolation polynomials associated with the nodes $z_v,$
\[
\prod_{v=1}^{n} \frac{z - z_v}{z_\lambda - z_v} = u_{\lambda, 1} + u_{\lambda, 2} z + \cdots + u_{\lambda, n} z^{n-1}. \tag{3.4}
\]
Applying (2.3) and the remark following (2.3) to the polynomial of degree $n-1$ in (3.4), we find
\[
\sum_{\mu=1}^{n} |u_{\lambda, \mu}| > \left( \prod_{v=1, v \neq \lambda} \frac{1}{|z_\lambda - z_v|} \right) \prod_{v=1}^{n} \max \left( 1, |z_v| \right). \tag{3.5}
\]
If $\lambda_0$ is the index $\lambda$ for which the right-hand expression in (3.5) attains its maximum, then that maximum is less than $\sum_{\mu=1}^{n} |u_{\lambda_0, \mu}|$, hence less or equal than
\[
\max_{1 \leq \lambda \leq n} \sum_{\mu=1}^{n} |u_{\lambda, \mu}|. \tag{3.6}
\]
This establishes (3.3) and proves Theorem 3.1.

The lower bound in (3.3) supplements the upper (attainable) bound in [1], which is of the same form as (3.3) except that the $l_\infty$-norm of the 2-vectors $[1, z_v]$ in the numerator factors is replaced by the $l_1$-norm.

4. Inverse Confluent Vandermonde Matrices

The technique used in the proof of Theorem 3.1 can be adapted to confluent Vandermonde matrices. We illustrate this with the particular matrix
\[
U_{2n}(z) = \\
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
z_1 & z_2 & \cdots & z_n & 1 & 1 & \cdots & 1 \\
z_1^2 & z_2^2 & \cdots & z_n^2 & 2z_1 & 2z_2 & \cdots & 2z_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_1^{2n-1} & z_2^{2n-1} & \cdots & z_n^{2n-1} & (2n-1)z_1^{2n-2} & (2n-1)z_2^{2n-2} & \cdots & (2n-1)z_n^{2n-2}
\end{pmatrix} \tag{4.1}
\]
considered previously in [1], [2].

Theorem 4.1. If $z_1, z_2, \ldots, z_n$ are mutually distinct complex numbers, and $n>1$, then
\[
\|U_{2n}^{-1}(z)\|_\infty > \max_{1 \leq \lambda \leq n} b_\lambda \prod_{v=1}^{n} \left( \frac{\max \left( 1, |z_v| \right)}{|z_\lambda - z_v|} \right)^2, \tag{4.2}
\]
where \( b_\lambda \) is the larger of the two quantities

\[
\begin{align*}
   b^{(1)}_\lambda &= \max \{ 1, |z_\lambda| \}, \\
   b^{(2)}_\lambda &= \max \left( 2 \left| \sum_{v \neq \lambda} 1/(z_\lambda - z_v) \right|, \left| 1 + 2z_\lambda \sum_{v \neq \lambda} 1/(z_\lambda - z_v) \right| \right). 
\end{align*}
\]  

(4.3)

**Proof.** We have [2]

\[
U_{2n}^{-1} = \begin{bmatrix} V \\ W \end{bmatrix}, \quad V = [v_{\lambda \mu}], \quad W = [w_{\lambda \mu}],
\]

where

\[
\begin{align*}
   l^{(2)}_\lambda (z) [1 - 2l'_{\lambda}(z)(z - z_\lambda)] &= \sum_{\mu = 1}^{2n} v_{\lambda \mu} z^{\mu - 1} \\
   l^{(2)}_\lambda (z)(z - z_\lambda) &= \sum_{\mu = 1}^{2n} w_{\lambda \mu} z^{\mu - 1}
\end{align*}
\]

(4.4)

and \( l_{\lambda}(z) \) denotes the fundamental Lagrange interpolation polynomial in (3.4). Applying (2.3) to the polynomials in (4.4), and taking note of the remark following (2.3), one finds

\[
\begin{align*}
   \sum_{\mu = 1}^{2n} |v_{\lambda \mu}| &> b^{(2)}_\lambda \prod_{\nu \neq \lambda} \left( \frac{\max \{ 1, |z_\nu| \}^2}{|z_\lambda - z_\nu|} \right), \\
   \sum_{\mu = 1}^{2n} |w_{\lambda \mu}| &> b^{(1)}_\lambda \prod_{\nu \neq \lambda} \left( \frac{\max \{ 1, |z_\nu| \}^2}{|z_\lambda - z_\nu|} \right)
\end{align*}
\]

where \( b^{(1)}_\lambda, b^{(2)}_\lambda \) are as defined in (4.3). Denoting the products \( \prod \) on the right by \( \pi_\lambda \), and observing that \( \| U_{2n}^{-1} \|_\infty = \max \left( \max_{\lambda, \mu = 1}^{2n} |v_{\lambda \mu}|, \max_{\lambda, \mu = 1}^{2n} |w_{\lambda \mu}| \right) \), an argument similar to the one after (3.5) will show that

\[
b^{(1)}_\lambda \pi_\lambda < \| U_{2n}^{-1} \|_\infty, \quad b^{(2)}_\lambda \pi_\lambda < \| U_{2n}^{-1} \|_\infty
\]

for all \( \lambda = 1, 2, \ldots, n \), hence \( \max (b^{(1)}_\lambda, b^{(2)}_\lambda) \pi_\lambda < \| U_{2n}^{-1} \|_\infty \) for all \( \lambda = 1, 2, \ldots, n \). This proves Theorem 4.1.

The lower bound in (4.2) supplements the (attainable) upper bound in [2], which is of the same form as (4.2) except that the \( l_\infty \)-norm of the 2-vectors \([1, z_v]\) in the numerator factors, and the \( l_\infty \)-norms defining \( b^{(1)}_\lambda \) and \( b^{(2)}_\lambda \), are all replaced by the respective \( l_1 \)-norms. In the case of positive nodes \( z_v \) another (usually sharper) lower bound can be found in [3, Theorem 2.1].

5. Examples

**Example 5.1** (roots of unity). \( z_v = e^{2\pi i(v-1)/n} \), \( v = 1, 2, \ldots, n \). In view of

\[
l_{\lambda}(z) = \frac{1}{n} \sum_{\mu = 1}^{n} \left( \frac{z}{z_\mu} \right)^{\mu - 1}, \quad \lambda = 1, 2, \ldots, n,
\]

where

\[
b^{(1)}_\lambda = \max \{ 1, |z_\lambda| \}, \quad b^{(2)}_\lambda = \max \left( 2 \left| \sum_{v \neq \lambda} 1/(z_\lambda - z_v) \right|, \left| 1 + 2z_\lambda \sum_{v \neq \lambda} 1/(z_\lambda - z_v) \right| \right).
\]

(4.3)
Table 1. Norm estimates for Example 5.2

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>lower true</th>
<th>upper true</th>
<th>lower true</th>
<th>upper true</th>
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<td>5</td>
<td>3</td>
<td>7.24(-1) 1.89</td>
<td>2.89</td>
<td>1.57</td>
<td>1.79(1)</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>1.17</td>
<td>1.47(1)</td>
<td>3.75(1)</td>
<td>8.29</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>4.25</td>
<td>2.03(2)</td>
<td>5.45(2)</td>
<td>1.46(2)</td>
</tr>
<tr>
<td>20</td>
<td>11</td>
<td>1.17(1)</td>
<td>2.76(3)</td>
<td>1.20(4)</td>
<td>1.52(3)</td>
</tr>
</tbody>
</table>

Table 2. Norm estimates for Example 5.3

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<th>upper true</th>
<th>lower true</th>
<th>upper true</th>
</tr>
</thead>
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<td>3.93</td>
<td>4.76</td>
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<td>4.45(1)</td>
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<td>7.45</td>
<td>1.69(1)</td>
<td>5.71(1)</td>
<td>6.08(2)</td>
</tr>
<tr>
<td>20</td>
<td>2.27(1)</td>
<td>5.36(1)</td>
<td>2.02(2)</td>
<td>7.54(3)</td>
</tr>
</tbody>
</table>

Table 3. Norm estimates for Example 5.4

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<th>$n$</th>
<th>lower true</th>
<th>upper true</th>
<th>lower true</th>
<th>upper true</th>
</tr>
</thead>
<tbody>
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<td>1.62</td>
<td>2.74</td>
<td>3.13</td>
<td>6.69</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5.71</td>
<td>1.09(1)</td>
<td>1.27(1)</td>
<td>1.38(2)</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>3.07(1)</td>
<td>6.82(1)</td>
<td>8.63(1)</td>
<td>5.71(3)</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>1.60(2)</td>
<td>3.60(2)</td>
<td>4.49(2)</td>
<td>1.88(5)</td>
</tr>
</tbody>
</table>

we obtain from (3.4), and from (4.4) after a little computation,

$$\| V_n^{-1}(z) \|_\infty = 1, \quad \| U_{2n}^{-1}(z) \|_\infty = 2 - \frac{1}{n}. \quad (5.1)$$

The lower bounds in (3.3) and (4.2) both evaluate to $1/n$, while the upper bounds in [1], [2] are $2n^{-1}/n$ and $(2n-1)4n^{-1}/n^2$, respectively.

Example 5.2 (roots of unity on half-circle). $z_v = e^{2\pi i (v-1)/N}$, $v = 1, 2, \ldots, n$, where $n = \lceil N/2 \rceil + 1$.

The true norms of $V_n^{-1}$ and $U_{2n}^{-1}$, as well as the lower bounds of Theorems 3.1 and 4.1 and the upper bounds in [1], [2] are shown in Table 1 for $N = 5(5)20^1$. It is interesting to note how deletion of the roots of unity on a half-circle results in substantially larger values of $\| V_n^{-1} \|_\infty$ and $\| U_{2n}^{-1} \|_\infty$.

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1 The integers in parentheses indicate exponents of 10
Example 5.3. $e_n(z_v)=0$, $v=1, 2, \ldots, n$, where $e_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$. Using the zeros of $e_n$, tabulated in [5], we obtain the results in Table 2.

Example 5.4. $e_n(z_v)=0$, $\text{Im } z_v \geq 0$, $v=1, 2, \ldots, n$, where $n = \left\lceil \frac{N+1}{2} \right\rceil$.

Similarly as in Example 5.2, deletion of the zeros in the lower half-plane has the effect of increasing the norms of $V^{-1}_n$ and $U^{-1}_{2n}$ (see Table 3).

References


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