MR0228171 (37 #3755) 65.55
Gautschi, Walter
Construction of Gauss-Christoffel quadrature formulas.
Math. Comp. **22** 1968 251–270

This important paper begins with a quantitative discussion of the ill-conditioned nature of the problem of the construction of Gaussian quadratures. It is found that this problem is about as ill-conditioned as the inversion of Hilbert matrices.

This defect is resolved by a discretization of the usual inner product and a set of discrete orthonormal polynomials are obtained. It is proved (via the convergence of the terms in the respective three-term recurrence relationships) that the roots of the discrete polynomials converge to the roots of the usual orthonormal polynomials.

The respective quadrature weights converge similarly and both the weights and the nodes can be made to converge more rapidly by the use of Newton's method. The latter again involves some ill-conditioned matrices, however. Numerical examples of the condition numbers and of the deviations in the discrete weights and nodes from the classical ones are given.

Reviewed by *R. E. Barnhill*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR0285117 (44 #2341a) 65.55

Gautschi, Walter

On the construction of Gaussian quadrature rules from modified moments.

Math. Comp. **24** 1970 245–260

MR0285118 (44 #2341b) 65.55

Gautschi, Walter

Tables of Gaussian quadrature rules for the calculation of Fourier coefficients.

Math. Comp. **24** 1970 *no. 110, loose microfiche suppl. A-D*

Author's summary: "Given a weight function $\omega(x)$ on (α,β), and a system of polynomials $\{p_k(x)\}_{k=0}^{\infty}$, with degree $p_k(x)=k$, we consider the problem of constructing Gaussian quadrature rules $\int_\alpha^\beta f(x)\omega(x)\,dx \doteq \sum_{r=1}^{n}\lambda_r{}^{(n)}f(\xi_r{}^{(n)})$ from "modified moments" $\nu_k=\int_\alpha^\beta p_k(x)\omega(x)\,dx$. Classical procedures take $p_k(x)=x^k$, but suffer from progressive ill-conditioning as n increases. A more recent procedure, due to R. A. Sack and A. F. Donovan ("An algorithm for Gaussian quadrature given modified moments", Numer. Math., to appear) takes for $\{p_k(x)\}$ a system of (classical) orthogonal polynomials. The problem is then remarkably well-conditioned, at least for finite intervals $[\alpha,\beta]$. In support of this observation, we obtain upper bounds for the respective asymptotic condition number. In special cases, these bounds grow like a fixed power of n. We also derive an algorithm for solving the problem considered, which generalizes one due to G. H. Golub and J. H. Welsch [Math. Comp. **23** (1969), 221–230; addendum; ibid. **23** (1969), no. 106, loose microfiche suppl. A1-A10; MR0245201 (39 #6513)]. Finally, some numerical examples are presented."

MR697783 (84e:65024) 65D30

Gautschi, Walter

How and how not to check Gaussian quadrature formulae.

BIT **23** (1983), *no. 2,* 209–216.

This is an important paper which practically demonstrates the dangers in ill-conditioned processes and the false sense of security that is obtained by the success of many validating tests. The particular problem addressed is the effective testing of Gaussian quadrature formulae. Two moment-related tests prove totally inadequate, yielding, for example, moments correct to 15 digits with the original formulae wrong in the second figure. More effective tests are generated based on computing the coefficients of either the three-term recurrence relations or of the orthogonal polynomials themselves.

Reviewed by *G. A. Evans*

MR723834 (85j:65010) 65D32
Gautschi, Walter (1-PURD-C); **Varga, Richard S.** (1-KNTS)
Error bounds for Gaussian quadrature of analytic functions.
SIAM J. Numer. Anal. **20** (1983), *no. 6,* 1170–1186.

The authors investigate the Gaussian quadrature formula

$$\int_{-1}^{1} f(t)w(t)\,dt = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} f(\tau_\nu^{(n)}) + R_n(f), \tag{1}$$

where the weight function $w(t)$ is nonnegative and integrable on $[-1,1]$, with $\int_{-1}^{1} w(t)\,dt > 0$. Let $f(z)$ be a function of a complex variable, regular in a domain $\mathcal{D}$ containing the interval $[-1,1]$, and let Γ be a contour in $\mathcal{D}$ which surrounds $[-1,1]$. Then, as is well known, the error of formula (1) can be represented as (2) $R_n(f) = (1/2\pi i)\int_\Gamma K_n(z)f(z)\,dz$, where (3) $K_n(z) = \rho_n(z)/\pi_n(z)$; here $\pi_n(z)$ is a polynomial of degree n orthogonal with respect to $w(t)$, and $\rho_n(z) = \int_{-1}^{1}[\pi_n(t)/(z-t)]w(t)\,dt$. The authors consider two types of contour Γ: the circle $C_r = \{z\colon |z| = r\}$, $r > 1$, and the ellipse $\mathcal{E}_\rho = \{z\colon z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}),\ 0 \le \theta \le 2\pi\},\ \rho > 1$.

The ordinary estimate for $|R_n(f)|$ derived from (2) contains as a factor $\max_{z\in\Gamma}|K_n(z)|$. The authors' primary result is the determination of a point on Γ at which $|K_n(z)|$ is a maximum. They obtain this result in the case $\Gamma = C_r$ for a broad class of weight functions, which includes the Jacobi weight $(1-t)^\alpha(1+t)^\beta$ for any $\alpha > -1$ and $\beta > -1$, and in the case $\Gamma = \mathcal{E}_\rho$ for the Jacobi weight when $\alpha = \beta = \pm\frac{1}{2}$ and $\beta = -\alpha = \frac{1}{2}$. To calculate the value of $K_n(z)$ at the point of maximum value, they use the representation (3) and the fact that $\pi(z)$ and $\rho_n(z)$ satisfy the same three-term recurrence relation. They calculate $\rho_n(z)$ from the recursion formula applied inversely. The authors give two numerical examples which illustrate the estimates obtained.

Reviewed by *I. P. Mysovskikh*

MR771039 (86j:65028) 65D32 (33A65 65A05 65B10 81-08 82-08)
Gautschi, Walter (1-PURD); **Milovanović, Gradimir V.**
Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series.
Math. Comp. **44** (1985), *no. 169,* 177–190.

The authors investigate several aspects concerning the derivation of effective quadrature rules for Stieltjes integrals of the type (1) $\int_0^\infty f(t)\,d\lambda(t)$ with the measure $d\lambda(t)$ defined either by (2) $d\lambda(t)=\varepsilon_r(t)dt$, $\varepsilon_r(t)=(t/(e^t-1))^r$, $r=1,2$, or by (3) $d\lambda(t)=\varphi_r(t)dt$, $\varphi_r(t)=(1/(e^t+1))^r$, $r=1,2$.

The interest in such an investigation is at least twofold. First, integrals (1) frequently occur in connection with the evaluation, in the independent particle approximation, of thermodynamic variables for solid state physics problems both for boson systems (which associate measures $d\lambda(t)$ given by (2)) and for fermion systems (which associate measures $d\lambda(t)$ given by (3)). Second, the integrals (1) are found to provide very effective tools for the summation of slowly convergent series whose general term is expressible in terms of a Laplace transform or its derivative.

As usual, the derivation of an n-point Gaussian quadrature rule (consisting of the quadrature sum and an estimate of the associated error) for the integration of (1) rests on the knowledge of the first n terms of a suitably defined sequence of (monic) polynomials $\pi_k(\cdot)=\pi_k(\cdot,d\lambda)$, orthogonal with respect to the measure $d\lambda$. More precisely, the coefficients α_k,β_k have to be known in the recursion formula (4) $\pi_{k+1}(t)=(t-\alpha_k)\pi_k(t)-\beta_k\pi_{k-1}(t)$, $k=0,1,2,\cdots$, $\pi_0(t)=1$, $\pi_{-1}(t)=0$. Evaluation of the condition numbers associated with several methods for the generation of the coefficients α_k,β_k leads the authors to the conclusion that Gauss-Legendre discretization is effective for the evaluation of (1), (2), while a composite Fejér rule can be successfully used for the evaluation of (1), (3). Several numerical examples support this conclusion for integrands $f(t)$ showing bounded variation on $[0,\infty)$. For singular integrands, however (the case of a square root singularity at $t=0$ is considered), the well-known convergence difficulties of the quadrature sums are overcome by isolation of a neighbourhood of the singularity and appropriate modification of the measure, $f(t)d\lambda(t)=g(t)d\mu(t)$, such that the new integrand $g(t)$ is a function with bounded variation.

For the four measures defined by equations (2) and (3), tables of coefficients α_k,β_k, computed with 25-figure accuracy, are reported for $0\le k\le 39$. Values of the associated error constants are reported in the tables as well. From these data, Gaussian quadrature formulae can be constructed, with up to 40 points. Examples of n-point formulae, $n=5(5)40$, are reported by the authors in the supplements section at the end of the same issue of the journal.

To the reviewer's knowledge, the present paper provides the first systematic investigation on the derivation of quadrature rules able to provide high-precision accuracy to the above integrals. The evaluation of the remainder of the quadrature sum using a higher order derivative of the integrand, as done in the paper, is, however, impractical for the implementation of automatic quadrature routines. As a means for reaching such a goal, it would probably be worthwhile to investigate the possibility of supplementing the obtained Gaussian quadrature sums with Kronrod-type extensions, similar in spirit to those currently in use for Gauss-Legendre quadrature sums on finite intervals [see, e.g., R. Piessens et al., *QUADPACK*, Springer, Berlin, 1983; MR0712135 (85b:65022)].

Reviewed by *Gh. Adam*

MR1034931 (91d:65044) 65D32
Gautschi, Walter (1-PURD-C); **Tychopoulos, E.** (GR-ATHN); **Varga, R. S.** (1-KNTS-CM)
A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rule.
SIAM J. Numer. Anal. **27** (1990), *no. 1,* 219–224.

The authors consider the remainder term $R_n(f)$ for the Gauss quadrature rule with a Chebyshev weight function of the second kind, namely, $\int_{-1}^{1} f(t)(1-t^2)^{1/2}dt = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} f(\tau_\nu) + R_n(f)$. The remainder $R_n(t)$ is given by $R_n(t) = (2\pi i)^{-1} \int_{\mathcal{E}_p} K_n(z) f(z)\, dz$, with $\mathcal{E}_p = \{z\colon z = \frac{1}{2}(u+u^{-1}),$ $u = pe^{i\theta}, 0 \le \theta \le 2\pi\}$. This note establishes more precisely than a previous paper [Gautschi and Varga, same journal **20** (1983), no. 6, 1170–1186; MR0723834 (85j:65010)] where the maximum of $K_n(z)$ is attained.

Reviewed by *G. A. Evans*

MR1092121 (92a:65078) 65D32

Gautschi, Walter (1-PURD-C); **Li, Shikang [Li, Ken S.]** (1-PURD)

The remainder term for analytic functions of Gauss-Radau and Gauss-Lobatto quadrature rules and with multiple end points.

J. Comput. Appl. Math. **33** (1990), *no. 3,* 315–329.

This work is a continuation of earlier papers [Gautschi, E. Tychopoulos and R. S. Varga, SIAM J. Numer. Anal. **27** (1990), no. 1, 219–224; MR1034931 (91d:65044); Gautschi and Varga, ibid. **20** (1983), no. 6, 1170–1186; MR0723834 (85j:65010)] in which the remainder term for Gaussian quadrature rules was considered. The extension encompasses rules of the Lobatto and Radau type. The remainder is expressed using a contour integral representation, and interest focuses on precisely where the kernel in the representation attains its maximum modulus on the elliptic contours considered. Attention is confined to the four Chebyshev weight functions $(1-t^2)^{-1/2}$, $(1-t^2)^{1/2}$, $(1-t)^{-1/2}(1+t)^{1/2}$ and $(1-t)^{1/2}(1+t)^{-1/2}$. Some of the more difficult results are obtained by numerical and asymptotic means rather than proof.

Reviewed by *G. A. Evans*

MR1113924 (93a:41071a) 41A80 (41A55 65D30)
Gautschi, Walter (1-PURD-C)
On the remainder term for analytic functions of Gauss-Lobatto and Gauss-Radau quadratures.
Proceedings of the U.S.-Western Europe Regional Conference on Padé Approximants and Related Topics (Boulder, CO, 1988).
Rocky Mountain J. Math. **21** (1991), *no. 1,* 209–226.

MR1138156 (93a:41071b) 41A80 (41A55 65D30)
Gautschi, Walter
Corrections to: "On the remainder term for analytic functions of Gauss-Lobatto and Gauss-Radau quadratures".
Rocky Mountain J. Math. **21** (1991), *no. 3,* 1143.

The author studies the kernels $K_N(\cdot\,;w)$ in the contour integral representation of the remainder term
$$R_N(f)=(2\pi i)^{-1}\int_\Gamma K_N(z;w)f(z)\,dz$$
of Gauss-Lobatto and Gauss-Radau quadratures with N nodes and an integrable weight function w. He determines the location of their maxima on circular contours for general weight functions and derives explicit formulae for the Lobatto and Radau kernels $K_N(\cdot\,;w)$ for any of the four Chebyshev weight functions $w_1(t)=(1-t^2)^{-1/2}$, $w_2(t)=(1-t^2)^{1/2}$, $w_3(t)=(1-t)^{-1/2}(1+t)^{1/2}$, and $w_4(t)=(1-t)^{1/2}(1+t)^{-1/2}$. In the last section, the author presents results concerning the location of the maximum of $|K_N(z;w)|$ as z varies on the circle $C_r=\{z\in\mathbf{C}\colon\ |z|=r\}$ or the ellipse $\mathcal{E}_\rho=\{z\in\mathbf{C}\colon\ z=(\rho e^{i\theta}+\rho^{-1}e^{-i\theta})/2,\,0\le\theta\le 2\pi\}$, both for Lobatto and Radau type formulae, and for the four Chebyshev weights. For example, he obtains that the kernel of the $(n+2)$-point Lobatto formula for the weight w_1 attains its maximum on the ellipse $\mathcal{E}_\rho$ on the real axis, and that maximum is given by
$$4\pi(\rho-\rho^{-1})^{-1}(\rho^{2n+2}-1)^{-1}.$$
The topics of earlier papers with R. S. Varga [SIAM J. Numer. Anal. **20** (1983), no. 6, 1170–1186; MR0723834 (85j:65010)] and with E. Tychopoulos and Varga [ibid. **27** (1990), no. 1, 219–224; MR1034931 (91d:65044)] are related to this paper.

{For the entire collection containing the first paper see MR 92b: 41001.}

Reviewed by *Gradimir V. Milovanović*

MR1102590 (92c:65032) 65D32 (41A55)

Gautschi, Walter (1-PURD-C); **Li, Shikang [Li, Ken S.]** (1-PURD)

Gauss-Radau and Gauss-Lobatto quadratures with double end points.

J. Comput. Appl. Math. **34** (1991), *no. 3,* 343–360.

The authors consider generalized Gauss-Radau and Gauss-Lobatto quadrature formulae having endpoints of multiplicity 2. Explicit formulae for these quadrature rules for all four Chebyshev weight functions are developed. Some properties of the corresponding orthogonal polynomials are studied as well.

The authors discuss the application of the developed quadrature formulae with several numerical examples.

Reviewed by *Georgi Grozev*

MR1248398 (94m:65044) 65D32

Gautschi, Walter (1-PURD-C)

Gauss-type quadrature rules for rational functions. (English summary)

Numerical integration, IV (Oberwolfach, 1992), 111–130, *Internat. Ser. Numer. Math.*, 112, *Birkhäuser, Basel*, 1993.

The quadrature rules investigated here are designed for integrands which have poles outside the interval of integration, but are otherwise regular. They are n-point rules which exactly integrate m chosen rational functions and polynomials of degree $2n-m-1$, where $0 \leq m \leq 2n$. The new quadrature rules are classified in terms of classical Gaussian formulae with modified weight functions, and are shown to exist for simple real or conjugate complex poles, for real poles of order 2, and for some combinations. Two methods for generating such quadrature formulae are described; both may be implemented by using routines in the author's ORTHPOL package [ACM Trans. Math. Software, Algorithm 762; per revr.]. Several numerical examples illustrate the behaviour of the quadrature rules and features of the methods used to generate them.

{For the entire collection see MR1248389 (94e:65010)}

Reviewed by *John P. Coleman*

MR1491435 (99a:65030) 65D30 (41A55)

Gautschi, Walter (1-PURD-C); **Milovanović, Gradimir V.** (YU-NISEE)

s-orthogonality and construction of Gauss-Turán-type quadrature formulae. (English summary)

Special issue dedicated to William B. Gragg (Monterey, CA, 1996).
J. Comput. Appl. Math. **86** (1997), *no. 1,* 205–218.

Summary: "Using the theory of s-orthogonality and reinterpreting it in terms of the standard orthogonal polynomials on the real line, we develop a method for constructing Gauss-Turán-type quadrature formulae. The determination of nodes and weights is very stable. For finding all weights, our method uses an upper triangular system of linear equations for the weights associated with each node. Numerical examples are included."

Reviewed by *I. P. Mysovskikh*

MR1754926 (2002a:65049) 65D30
Gautschi, Walter (1-PURD-C); **Gori, Laura [Gori Nicolò-Amati, Laura]** (I-ROME-M); **Pitolli, Francesca** (I-ROME-M)

Gauss quadrature for refinable weight functions. (English summary)

Appl. Comput. Harmon. Anal. **8** (2000), *no. 3,* 249–257.

Summary: "A two-parameter class of refinable functions is considered and Gaussian quadrature rules having these functions as weight functions are constructed. A discretization method is described for generating the recursion coefficients of the required orthogonal polynomials. Numerical results are also presented." 2000Academic Press

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1794345 (2002a:41030) 41A55 (65D30 65D32)
Gautschi, Walter (1-PURD-C); **Gori, Laura [Gori Nicolò-Amati, Laura]** (I-ROME-AS); **Lo Cascio, M. Laura** (I-ROME-AS)

Quadrature rules for rational functions. (English summary)

Numer. Math. **86** (2000), *no. 4,* 617–633.

The paper deals with quadrature rules of the following form: Let $d\lambda(t)$ be a positive measure on $\mathbf{R}$ and $\{\tau_i^G\}_{i=1}^n$ the nodes of the n-point quadrature rule for $d\lambda$. A Gauss-Kronrod rule for rational functions is a formula of the type

$$\int_{\mathbf{R}} g(t)d\lambda(t) = \sum_{i=1}^{n} \lambda_i^K g(\tau_i^G) + \sum_{j=1}^{n+1} \lambda_j^{*K} g(\tau_j^K) + R_n^K(g),$$

which is exact for all functions of the space

$$S_{3n+2} = Q_m \oplus P_{3n+1-m}, \quad 0 \le m \le 3n+2,$$

where $Q_m = \operatorname{span}\{g\colon\ g(t) = (1+\zeta_\mu)^{-r},\ r = 1,\dots,r_\mu,\ \mu = 1,\dots,M,\ \sum_{\mu=1}^{M} r_\mu = m\}$ and P_k is the space of polynomials of degree $\le k$.

Rational quadrature formulae of Gaussian or Newton-Cotes type have proven to be quite effective if the poles are chosen so as to simulate the poles present in the integrand.

In this paper rational versions of other important quadrature rules are considered, e.g. rational Gauss-Kronrod formulae which have the above form. Moreover, Gauss-Turán and Cauchy principal value quadrature rules are investigated. Results concerning the construction of these formulae are shown. Many examples are given in order to show how effective the use of rational quadrature formulae is if the integrand contains poles.

Reviewed by *Hans Strauss*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1827155 (2002c:65043) 65D32 (41A55)
Gautschi, Walter (1-PURD)
High-order Gauss-Lobatto formulae. (English summary)
Mathematical journey through analysis, matrix theory and scientific computation (Kent, OH, 1999).
Numer. Algorithms **25** (2000), *no. 1-4,* 213–222.

Summary: "Currently, the method of choice for computing the $(n+2)$-point Gauss-Lobatto quadrature rule for any measure of integration is to first generate the Jacobi matrix of order $n+2$ for the measure at hand, then modify the three elements at the right lower corner of the matrix in a manner proposed in 1973 by Golub, and finally compute the eigenvalues and first components of the respective eigenvectors to produce the nodes and weights of the quadrature rule. In general, this works quite well, but when n becomes large, underflow problems cause the method to fail, at least in the software implementation provided by us in 1994. The reason is the singularity (caused by underflow) of the 2×2 system of linear equations that is used to compute the modified matrix elements. It is shown here that in the case of arbitrary Jacobi measures, these elements can be computed directly, without solving a linear system, thus allowing the method to function for essentially unrestricted values of n. In addition, it is shown that all weights of the quadrature rule can be computed explicitly, which not only obviates the need to compute eigenvectors, but also provides better accuracy. Numerical comparisons are made to illustrate the effectiveness of this new implementation of the method."

{For the entire collection see MR1827140 (2001j:00034)}

Reviewed by *I. P. Mysovskikh*

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3. W. Gautschi, Orthogonal polynomials: applications and computation, in: *Acta Numerica 1996*, ed. A. Iserles (Cambridge Univ. Press, Cambridge, 1996) pp. 45–119. MR1624591 (99j:65019)
4. G.H. Golub, Some modified matrix eigenvalue problems, SIAM Rev. 15 (1973) 318–334. MR0329227 (48 #7569)
5. G. Szegô, *Orthogonal Polynomials*, 4th ed., American Mathematical Society Colloquium Publications, Vol. 23 (Amer. Math. Soc., Providence, RI, 1978). MR0106295 (21 #5029)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1808697 (2001k:65049) 65D32
Gautschi, Walter (1-PURD-C)
Gauss-Radau formulae for Jacobi and Laguerre weight functions. (English summary)
1999 International Symposium on Computational Sciences, to honor John R. Rice (West Lafayette, IN).
Math. Comput. Simulation **54** (2000), *no. 4-5,* 403–412.

Two families of Gauss-Radau formulae are explicitly found for the Jacobi weight $(1-t)^\alpha(1+t)^\beta$ over the intervals $[-1,1]$ and $[0,\infty]$.

The method of generation is by finding the eigenvalues of the modified Jacobi matrix [G. H. Golub, SIAM Rev. **15** (1973), 318–334; MR0329227 (48 #7569)], which yield the required abscissae. The weights can be found explicitly, thus avoiding the computation of eigenvectors. The formula for the boundary weight is derived separately from the interior weights after some manipulation.

The final section deals with the equivalent problem for an infinite range Laguerre measure.

{For the entire collection see MR1807515 (2001h:65004)}

Reviewed by *G. A. Evans*

MR2211041 (2006k:65070) 65D32 (41A15)

Gautschi, Walter (1-PURD-C)

Generalized Gauss-Radau and Gauss-Lobatto formulae. (English summary)

BIT **44** (2004), *no. 4,* 711–720.

Summary: "Computational methods are developed for generating Gauss-type quadrature formulae having nodes of arbitrary multiplicity at one or both end points of the interval of integration. Positivity properties of the boundary weights are investigated numerically, and related conjectures are formulated. Applications are made to moment-preserving spline approximation."

Reviewed by *Valdir A. Menegatto*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2280368 (2007k:42067) 42C05 (65D30)
Gautschi, Walter (1-PURD-C)
The circle theorem and related theorems for Gauss-type quadrature rules. (English summary)
Electron. Trans. Numer. Anal. **25** (2006), 129–137 (*electronic*).

Summary: “In 1961, P. J. Davis and P. Rabinowitz [J. Math. Anal. Appl. **2** (1961), 428–437; MR0128613 (23 1652)] established a beautiful ‘circle theorem’ for Gauss and Gauss-Lobatto quadrature rules. They showed that, in the case of Jacobi weight functions, the Gaussian weights, suitably normalized and plotted against the Gaussian nodes, lie asymptotically for large orders on the upper half of the unit circle centered at the origin. Here analogous results are proved for rather more general weight functions—essentially those in the Szegö class—not only for Gauss and Gauss-Lobatto, but also for Gauss-Radau formulae. For much more restricted classes of weight functions, the circle theorem even holds for Gauss-Kronrod rules. In terms of potential theory, the semicircle of the circle theorem can be interpreted as the reciprocal density of the equilibrium measure of the interval $[-1, 1]$. Analogous theorems hold for weight functions supported on any compact subset Δ of $(-1, 1)$, in which case the (normalized) Gauss points approach the reciprocal density of the equilibrium measure of Δ. Many of the results are illustrated graphically.”

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2505838 (2010c:65033) 65D30 (41A10 65D32)
Gautschi, Walter (1-PURD)

High-order generalized Gauss-Radau and Gauss-Lobatto formulae for Jacobi and Laguerre weight functions. (English summary)

Numer. Algorithms **51** (2009), *no. 2,* 143–149.

Generalized Gauss-Radau and Gauss-Lobatto formulae are quadrature formulae of maximum polynomial degree of exactness that involve boundary points of arbitrary multiplicity $r \geq 2$. The computation of these formulae for arbitrary weight functions has been discussed in [W. Gautschi, BIT **44** (2004), no. 4, 711–720; MR2211041 (2006k:65070)], where reference is also made to the respective MATLAB routines `gradau.m` and `globatto.m`. It has been noted that these routines break down in the case of Jacobi and Laguerre measures when the order of the quadrature rules becomes very large because of underflow, resp. overflow, of the respective monic orthogonal polynomials. In this paper the author circumvents the problem by rescaling of the polynomials and other corrective measures. Numerical experiments show that formulae can be generated of orders as high as 1,000.

Reviewed by *Francesco Dell'Accio*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2720632 (2011j:65051) 65D30 (41A10 41A55)
Gautschi, Walter (1-PURD-C)

Gauss quadrature routines for two classes of logarithmic weight functions. (English summary)

Numer. Algorithms **55** (2010), *no. 2-3,* 265–277.

The implemented algorithms for Gauss quadrature formulas computing two types of integrals of logarithmic weight functions using Gauss-Laguerre and Gauss-Jacobi rules and a modified Chebyshev algorithm are presented. The procedures to compute the weight coefficients in terms of Euler's gamma-functions are implemented using Matlab with 130-digit precision. The relative errors are about 1.5×10^{-32} (that is, 32 digit accuracy) for the logarithmic Gauss-Jacobi quadrature.

Reviewed by *Alexandru M. Bica*

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