MR0103289 (21 #2067) 33.00

Gautschi, Walter

Some elementary inequalities relating to the gamma and incomplete gamma function.

J. Math. and Phys. **38** 1959/60 77–81

The author gives lower and upper bounds of the form $c((x^p+c^{-1})^{1/p}-x)$ for $\exp(x^p)\int_x^\infty \exp(-t^p)dt$ in the range $p>1$, $0\le x<\infty$; the respective values of c are 2 and $\{\Gamma(1+p^{-1})\}^{p/(p-1)}$. As it stands, the proof is only valid if p is an integer, but, in a correction, the author has indicated a modification which validates it for all $p>1$.

Reviewed by *H. O. Pollak*

MR0104347 (21 #3102) 65.00 (33.00)

Gautschi, Walter

Exponential integral $\int_1^\infty e^{-xt}t^{-n}dt$ for large values of n.

J. Res. Nat. Bur. Standards **62** 1959 123–125

An asymptotic expansion for the integral in the title is derived, and numerical bounds for the error are given. The terms of the series are rational functions of n and x, and the remainder after the kth term is $O(n^{-k})$.

Reviewed by *W. Wasow*

MR0136074 (24 #B2113) 65.25
Gautschi, Walter
Recursive computation of the repeated integrals of the error function.
Math. Comp. **15** 1961 227–232

This paper is concerned with numerical computation of a function for satisfying a second-order difference equation

$$y_{n+1} + a_n y_n + b_n y_{n-1} = 0.$$

There are two independent solutions u_n, v_n, say, which may either oscillate with n in similar fashions, or which may behave exponentially. In the latter case, if u_n, v_n are suitably chosen, then, in general, $u_n/v_n \to 0$ as $n \to \infty$, while $v_n/u_n \to 0$ as $u \to -\infty$ (with similar, but modified, statements if the range of n is bounded at one or both ends by regions exhibiting oscillatory behaviour for u_n, v_n).

When evaluating u_n by recurrence for increasing n, rounding errors cause a multiple of v_n to appear, and eventually to dominate. This difficulty in evaluating u_n can be overcome by using the recurrence relation with n decreasing. The author discusses the error-behaviour for this process, with an application to the function called i^{n-1} erfc x by Hartree, which satisfies $ny_{n+1} + xy_n - y_{n-1} = 0$.

References to earlier use of this process are given.

Reviewed by *J. C. P. Miller*

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MR0115264 (22 #6066)
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR0293813 (45 #2889) 65D20

Gautschi, Walter

Efficient computation of the complex error function.

SIAM J. Numer. Anal. **7** (1970), 187–198.

The author is concerned with the computation of $w(z) = \exp(-z^2)erfc(-iz)$, $\operatorname{Im} z > 0$, the basic tool being $w(z) = i\pi^{-1}\int_{+\infty}^{-\infty}\exp(-t^2)(z-t)^{-1}\,dt$. The integral is approximated by Gauss-Hermite quadrature, the remainder being estimated by asymptotics.

The author remarks that, as it should be, quadrature is much more effective than straightforward asymptotic expansion. This is actually true for many functions defined by integrals and the deep root of this may be easily found in the theory of the asymptotic expansion of remainders as exposed by P. D. Tuan [Math. Comp. **25** (1971), 819–825].

The reviewer remarks that the part of the quadrature sum corresponding to the "large" zeros of Hermite polynomials is quite little, as compared with the remainder.

Reviewed by *P. Barrucand*

MR0350077 (50 #2570) 33A15

Gautschi, Walter

A harmonic mean inequality for the gamma function.

SIAM J. Math. Anal. **5** (1974), 278–281.

The author proves that the harmonic mean of $\Gamma(x)$ and $1/\Gamma(1/x)$ is greater than or equal to $\Gamma(1) = 1$ for arbitrary $x > 0$. He proves it first by using certain isolated numerical values of the psi function $\psi(x) = \Gamma'(x)/\Gamma(x)$ and its derivatives and then removes this deficiency by taking numerical values of standard constants such as π, $\ln 2$ and Euler's constant γ.

Reviewed by *S. Saran*

MR0350078 (50 #2571) 33A15
Gautschi, Walter
Some mean value inequalities for the gamma function.
SIAM J. Math. Anal. **5** (1974), 282–292.

The author determines the infimum of the harmonic mean of $\Gamma(x_1)\cdots\Gamma(x_n)$ under the constraints $\prod_{k=1}^n x_k = 1$ for all $x_k > 0$. He verifies that this infimum is equal to $\Gamma(1) = 1$ if $n \le 8$ and is less than 1 when $n > 8$. He also proves that the geometric mean of these gammas is not less than one under the same constraints, and that the geometric mean is the power mean with the smallest exponent for which this is true.

Reviewed by *S. Saran*

MR0391476 (52 #12297) 65D15
Gautschi, Walter
Computational methods in special functions—a survey.
Theory and application of special functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975), pp. 1–98. *Math. Res. Center, Univ. Wisconsin Publ., No.* 35, *Academic Press, New York,* 1975.

As the title indicates, this is a survey of the state of the art in the computation of the special functions of mathematical physics, particularly as applied to methods suitable for automatic computers.

The survey is subdivided into four general categories. First discussed are methods in which the function to be computed is approximated by elementary functions. Included in this group are truncated Taylor and Čebyšev series, truncated asymptotic series, and rational approximations. The latter type may be derived in a number of ways, notably by the application of the principle of "best uniform (Čebyšev) approximation" first treated by N. I. Ahiezer [English translation, *Theory of approximation*, Ungar, New York, 1956; MR0095369 (20 #1872); German translation of the second Russian edition, Akademie Verlag, Berlin, 1967; MR0222516 (36 #5567)], and by numerous others in more recent publications. Such approximations are characterized by the equi-oscillation property, wherein the error curve assumes extreme values with alternating signs in the interval of approximation. A second type of rational approximation consists of those obtained from continued fractions, the Padé table and the q-d and ε-algorithms.

Section 2 deals with computation by means of recursive relations. The question of stability of both first- and higher-order recursion schemes is discussed. In the case of second-order relations, the concept of minimal solutions is of importance. Such solutions may be obtained either by backward recursion or from the associated continued fraction. Numerous algorithms, notably those of Olver and Miller, for implementing the above are mentioned.

The third section discusses the use of nonlinear recurrence algorithms. While the processes described in this section appear to be confined almost exclusively to elliptic integrals and functions through quadratic transformation of the modulus, the ideas find limited use in the computation of other hypergeometric functions. Ascending and descending Gauss and Landen transformations (or the equivalent arithmetic-geometric-mean principle) are treated in detail, with a discussion of relative speed and accuracy.

The final section describes some of the existing and readily accessible software in current use. Published algorithms in various journals and users' publications furnish a good source of such material. Other sources are special interest groups such as NATS in this country and NAG in the U.K., as well as the libraries of industrial and government installations.

An extensive bibliography follows the text.

{For the entire collection see MR0379915 (52 #819).}

Reviewed by *H. E. Fettis*

MR0442204 (56 #590) 30A22 (65B99)

Gautschi, Walter

Anomalous convergence of a continued fraction for ratios of Kummer functions.

Math. Comp. **31** (1977), *no. 140*, 994–999.

For the ratios of Kummer functions, the author shows a continued fraction that apparently converges to the wrong limit for certain values of z. The point is illustrated in the special cases of Bessel functions and incomplete gamma functions.

Reviewed by *E. Frank*

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1. W. GAUTSCHI, ”An evaluation procedure for incomplete gamma functions,” *ACM Trans. Mathematical Software*. (To appear.)
2. W. GAUTSCHI & J. SLAVIK, ”On the computation of modified Bessel function ratios,” *Math. Comp.* (To appear.) cf. MR 57 #10025
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6. H. S. WALL, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948; reprint, Chelsea, New York, 1967. MR **10**, 32. MR0025596 (10,32d)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR0470267 (57 #10025) 33A40

Gautschi, Walter; **Slavik, Josef**

On the computation of modified Bessel function ratios.

Math. Comp. **32** (1978), *no. 143*, 865–875.

In this paper, the authors give a detailed comparison between the use of a continued fraction of Gauss, and one of Perron, for the evaluation of ratios of modified Bessel functions $I_\nu(x)/I_{\nu-1}(x)$, $x > 0$, $\nu > 0$. After detailed comparisons, they show that Perron's continued fraction is substantially superior when $x \gg \nu$, and only moderately inferior otherwise. This advantage is limited to modified Bessel functions of a real argument.

Reviewed by *James M. Horner*

There is no review available for [68].

MR547763 (81f:65015) 65D20
Gautschi, Walter
A computational procedure for incomplete gamma functions. (Italian)
Rend. Sem. Mat. Univ. Politec. Torino **37** (1979), *no. 1*, 1–9.

This study (the subject of a lecture given by the author for the Mathematics Seminar of Turin) deals with the calculation of the incomplete gamma functions.

The author notes that the numerical calculation of special functions, when one deals with functions of one real variable, is a rather simple problem to which one can apply known methods. On the other hand, in the case of functions of one complex variable, and especially in the case of a function of several real variables, general algorithms of rational approximation either do not exist or have not been sufficiently developed in practice. For this reason the present work turns out to be particularly interesting. It studies the incomplete gamma functions $\gamma(a,x)=\int_0^x e^{-t}t^{a-1}\,dt$, $\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt$. These functions are functions of two real variables and are of particular interest for real values of the variable a and nonnegative values of the variable x. The results of the author's research on this topic are accurately presented in this work.

Reviewed by *C. Dagnino*

MR1737497 (2001g:33003) 33B20 (01A60 33-03 33F05)
Gautschi, Walter (1-PURD)
The incomplete gamma functions since Tricomi.
Tricomi's ideas and contemporary applied mathematics (Rome/Turin, 1997), 203–237, *Atti Convegni Lincei*, 147, *Accad. Naz. Lincei*, *Rome*, 1998.

This interesting paper charts the history and developments in the understanding of the properties of the incomplete gamma functions $\gamma(a, z)$ and $\Gamma(a, z)$ from Tricomi's work in the 1950s to the present day. These functions were described affectionately by Tricomi as the "Cinderella" of special functions.

The author begins by describing Tricomi's main results on the asymptotic expansions and his investigations into the zeros of these functions, and briefly describes two applications he made of this theory to number theory and the problem of random walks. In the remainder of this review paper the author then describes more recent work on these functions. This includes a discussion of uniform asymptotic expansions together with an exponentially improved expansion for the (closely related) exponential integral $E_p(z) = z^{p-1}\Gamma(1 - p, z)$. This latter expansion plays an important role in the modern discussion of the Stokes phenomenon and hyperasymptotic expansions. There then follows a section on inversion of the incomplete gamma functions and a discussion of their zeros, both real and complex. Some inequalities and monotonicity properties are mentioned and there is a section on numerical methods of computation of the incomplete gamma functions.

The paper contains a list of 160 references, which is a very valuable source of information for the reader desirous of additional, more detailed information on these functions.

{For the entire collection see MR1752930 (2000k:00051)}

Reviewed by *Richard B. Paris*

MR1697463 65D20
Gautschi, Walter (1-PURD-C)
A note on the recursive calculation of incomplete gamma functions. (English summary)
ACM Trans. Math. Software **25** (1999), *no. 1,* 101–107.

{There will be no review of this item.}

MR1876879 (2002m:33029) 33F05 (33C05 33C15 65D20)
Gautschi, Walter (1-PURD-C)

Gauss quadrature approximations to hypergeometric and confluent hypergeometric functions. (English summary)

J. Comput. Appl. Math. **139** (2002), *no. 1,* 173–187.

We often encounter the evaluation of hypergeometric functions in many branches of applied sciences. This paper presents approximations for the ordinary and confluent hypergeometric functions ${}_2F_1$ and ${}_1F_1$. More specifically, the well-known integral representations of these functions are used to obtain practical formulas by means of Gaussian quadrature, which are valid in large domains of parameters and argument values.

Reviewed by *Hasan Taşeli*

MR1896388 (2003d:33010) 33C10 (33C45 33F05 65D20 65D32)
Gautschi, Walter (1-PURD-C)
Computation of Bessel and Airy functions and of related Gaussian quadrature formulae. (English summary)
BIT **42** (2002), *no. 1,* 110–118.

In this paper the author describes procedures for the high-precision calculation of the modified Bessel function $K_\nu(x)$, $0 < \nu < 1$, and the Airy function $\mathrm{Ai}(x)$, $x > 0$, as prerequisites for generating Gaussian quadrature rules having these functions as weight function.

Reviewed by *Khélifa Trimèche*

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pp. 351–360. MR0678740 (84a:65102)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2189767 (2006m:33022) 33F05 (33C10 65D20 65D30)
Gautschi, Walter (1-PURD-C)
Numerical quadrature computation of the Macdonald function for complex orders. (English summary)
BIT **45** (2005), *no. 3,* 593–603.

The MacDonald function $K_\nu(z)$, also known as the modified Bessel function, is one of the most important of the standard special functions. Thus, efficient and accurate methods for its computation are of importance. In this paper, the author considers the computation when $\nu = \alpha + i\beta$, $\beta \neq 0$ and $z = x$ with x real and positive.

The author uses the fact that the real and imaginary values of the function can each be given by an integral representation using a mixture of $\sin$, $\cos$, $\sinh$, $\cosh$ and $\exp(-x\cosh t)$. This last term decays extremely rapidly, and to deal with this the author transforms the integrals to ones with this decay absorbed into the function $w(t) = \exp(-\exp(t))$.

He then develops Gaussian quadrature weights and points for $w(t)$ over the range $[0, \infty)$, which are used as the basis for Matlab codes for K_ν.

The paper is extremely clearly written and should be read by anyone wishing to use the MacDonald function. The author is to be commended for providing web access to the software, a facility which ought to be standard for all papers on computing special functions.

Reviewed by *Allan J. Mac Leod*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2355984 (2009c:33022) 33C45 (33B10 33C50 33C52)
Gautschi, Walter (1-PURD-C); **Leopardi, Paul** (5-NSW-SMS)
Conjectured inequalities for Jacobi polynomials and their largest zeros. (English summary)
Numer. Algorithms **45** (2007), *no. 1-4,* 217–230.

Let $x_n^{(\alpha,\beta)} = \cos\Theta_n^{(\alpha,\beta)}$, $0 < \Theta_n^{(\alpha,\beta)} < \pi$, be the largest zero of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$. In this paper the authors give some conjectures related to the inequality

$$n\Theta_n^{(\alpha,\beta)} < (n+1)\Theta_{n+1}^{(\alpha,\beta)}.$$

All inequalities are only conjectured to hold, but compelling evidence is provided, both numerical and analytic, in support of their validity.

Reviewed by *Ridha Sfaxi*

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1. Gatteschi, L.: New inequalities for the zeros of Jacobi polynomials. SIAM J. Math. Anal. **18,** 1549–1562 (1987) MR0911648 (88m:33021)
2. Leopardi, P.: Positive weight quadrature on the sphere and monotonicities of Jacobi polynomials. Numer. Algor. doi:10.1007/s11075–007-9073–7 (2007) MR2355973 (2008i:65040)
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2457099 (2009m:33018) 33C45
Gautschi, Walter (1-PURD-C)

On a conjectured inequality for the largest zero of Jacobi polynomials. (English summary)

Numer. Algorithms **49** (2008), *no. 1-4,* 195–198.

Let $x_n^{(\alpha,\beta)} = \cos\theta_n^{(\alpha,\beta)}$ be the largest zero of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, where $\alpha,\beta > -1$. It was conjectured by the author and P. C. Leopardi [Numer. Algorithms **45** (2007), no. 1-4, 217–230; MR2355984 (2009c:33022)] that if the inequality

$$n\theta_n^{(\alpha,\beta)} < (n+1)\theta_{n+1}^{(\alpha,\beta)} \tag{1}$$

holds for $n = 1$, then it holds for all $n \geq 1$. The region where (1) holds for $n = 1$ is described by a curve $\beta = \beta(\alpha)$ in the sense that (1) holds for $n = 1$ if (α,β) is above the curve, and fails if (α,β) is on or below the curve. This curve, which is the solution of an explicit equation in α and β, is monotonically descending from the point $(-1, 0)$, through $(-\frac{1}{2}, -\frac{1}{2})$, to the point $(1, -1)$.

In this paper, the author proves that (1) fails for n large enough if $\alpha+\beta+1 < 0$ and in the half-open segment line from $(-\frac{1}{2}, -\frac{1}{2})$ (inclusive) to $(0, -1)$, and holds for n large enough, otherwise. Then, he observes that the curve $\beta = \beta(\alpha)$ is in the region $\alpha+\beta+1 < 0$ if $-1 < \alpha < -\frac{1}{2}$, thus disproving the conjecture. Finally, he reformulates the conjecture excluding this region.

Reviewed by *Mario Pérez Riera*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2487229 (2010e:33011) 33C45 (33F99)
Gautschi, Walter (1-PURD-C)
On conjectured inequalities for zeros of Jacobi polynomials. (English summary)
Numer. Algorithms **50** (2009), *no. 1,* 93–96.

In this paper, based on extensive computations, the author formulates some conjectures concerning the domain of validity in the (α,β)-plane of the inequalities $n\,\theta_{n,r}^{(\alpha,\beta)} < (n+1)\,\theta_{n+1,r}^{(\alpha,\beta)}$, $r = 1, 2, \ldots, n$, where $x_{n,r}^{(\alpha,\beta)} = \cos\theta_{n,r}^{(\alpha,\beta)}$ are the zeros in descending order of the Jacobi polynomials $P_n^{(\alpha,\beta)}$.

{This review is also provided for [W. Gautschi, Numer. Algorithms **50** (2009), no. 3, 293–296; MR2487240].}

Reviewed by *Nicolae Cotfas*

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1. Gautschi, W.: On a conjectured inequality for the largest zero of Jacobi polynomials. Numer. Algorithms (2008). doi:10.1007/s11075–008-9207–6 MR2457099 (2009m:33018)
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2487240 (2010e:33012) 33C45 (33F99)
Gautschi, Walter (1-PURD-C)
New conjectured inequalities for zeros of Jacobi polynomials. (English summary)
Numer. Algorithms **50** (2009), *no. 3,* 293–296.

In this paper, based on extensive computations, the author formulates some conjectures concerning the domain of validity in the (α,β)-plane of the inequalities $(n+(\alpha+\beta+1)/2)\,\theta_{n,r}^{(\alpha,\beta)} > (n+(\alpha+\beta+3)/2)\,\theta_{n+1,r}^{(\alpha,\beta)}$, $r=1,2,\dots,n$, where $x_{n,r}^{(\alpha,\beta)}=\cos\theta_{n,r}^{(\alpha,\beta)}$ are the zeros in descending order of the Jacobi polynomials $P_n^{(\alpha,\beta)}$.

{This review is also provided for [W. Gautschi, Numer. Algorithms **50** (2009), no. 1, 93–96; MR2487229].}

Reviewed by *Nicolae Cotfas*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2779993 (2012a:33018) 33C45 (41A17)
Gautschi, Walter (1-PURD-C)
How sharp is Bernstein's inequality for Jacobi polynomials? (English summary)
Electron. Trans. Numer. Anal. **36** (2009/10), 1–8.

In order to present numerical evidence about the sharpness of the constant appearing in a Bernstein-type inequality for Jacobi polynomials $P_n^{(\alpha,\beta)}$, when larger domains of the parameters (α,β) are considered, the author computationally explores some analytic facts involving inequalites of Bernstein type for Jacobi polynomials $P_n^{(\alpha,\beta)}$, with domain $|\alpha|\le 1/2$, $|\beta|\le 1/2$ [see P. Baratella, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **120** (1986), no. 5-6, 207–223 (1987); MR0953391 (89f:33016); Y. S. Chow, L. Gatteschi and R. S. C. Wong, Proc. Amer. Math. Soc. **121** (1994), no. 3, 703–709; MR1209419 (94i:33008)] and the Darboux formula [G. Szegő, *Orthogonal polynomials*, fourth edition, Amer. Math. Soc., Providence, RI, 1975; MR0372517 (51 #8724) (Theorem 8.21.8)]. Also, for the Jacobi orthonormal polynomials $\widehat{P}_n^{(\alpha,\beta)}$, the author describes computational experiments which suggest that $C=.66198126\dots$ is the best constant implied in the Erdélyi–Magnus–Nevai conjecture

$$(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}[\widehat{P}_n^{(\alpha,\beta)}(x)=]^2=O(\max[1,(\alpha^2+\beta^2)^{1/4}]),$$

on the domain $|\alpha|\le 1/2$, $|\beta|\le 1/2$.

Reviewed by *Yamilet Quintana*

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MR2786727 (Review) 33F05 (33E20 65D20 65D30)
Gautschi, Walter (1-PURD-C)

The Lambert W-functions and some of their integrals: a case study of high-precision computation. (English summary)

Numer. Algorithms **57** (2011), *no. 1,* 27–34.

Summary: "The real-valued Lambert W-functions considered here are $w_0(y)$ and $w_{-1}(y)$, solutions of $we^w = y$, $-1/e < y < 0$, with values respectively in $(-1, 0)$ and $(-\infty, -1)$. A study is made of the numerical evaluation to high precision of these functions and of the integrals $\int_1^\infty [-w_0(-xe^{-x})]^\alpha x^{-\beta} dx$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $\int_0^1 [-w_{-1}(-xe^{-x})]^\alpha x^{-\beta} dx$, $\alpha > -1$, $\beta < 1$. For the latter we use known integral representations and their evaluation by nonstandard Gaussian quadrature, if $\alpha \neq \beta$, and explicit formulae involving the trigamma function, if $\alpha = \beta$."

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