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Luigi Gatteschi's work on asymptotics of special functions and their zeros

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Abstract A good portion of Gatteschi's research publications—about 65%—is devoted to asymptotics of special functions and their zeros. Most prominently among the special functions studied figure classical orthogonal polynomials, notably Jacobi polynomials and their special cases, Laguerre polynomials, and Hermite polynomials by implication. Other important classes of special functions dealt with are Bessel functions of the first and second kind, Airy functions, and confluent hypergeometric functions, both in Tricomi's and Whittaker's form. This work is reviewed here, and organized along methodological lines.

Keywords Luigi Gatteschi's work · Asymptotics · Special functions · Zeros

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1 Introduction

In asymptotics there are two kinds of theories: a qualitative theory, and a quantitative theory. They differ in the way the error of an asymptotic approximation is characterized. In the former, the error is estimated by an order-of-magnitude term $O(\omega(x))$, i.e., by a statement that there exists a positive,

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unspecified constant C such that the error is bounded in absolute value by $C\omega(x)$ as the variable (or parameter) x is in the neighborhood of a limit value x_0. Here, ω is a known, computable, positive function of x, for example a reciprocal power of x if $x_0 = +\infty$. A quantitative theory, in contrast, provides a numerical upper bound for the constant C, or better still, concrete numerical lower and upper bounds for the error, $\omega_-(x)$ and $\omega_+(x)$, along with a precise description of the domain of validity (in x). The approximation, in effect, then takes on the form of a two-sided inequality. Much of the older, classical theory of asymptotics is of a qualitative nature, while modern exigencies of computing require a quantitative theory. In the realm of special functions and their zeros, Luigi Gatteschi is without doubt one of the major exponents of, and contributor to, the quantitative theory of asymptotics. His results are not only of a concrete numerical nature, but often attain a degree of sharpness rarely found elsewhere in the literature.

In the following we briefly summarize Gatteschi's relevant work as it pertains to orthogonal polynomials, Bessel and Airy functions, and confluent hypergeometric functions. We arrange the presentation according to the type of methods used, and in each case proceed in more or less chronological order. Even though we can give only a quick and superficial account of finished results, it must be emphasized that, underneath it all, there is a great deal of hard analysis, imaginatively and skillfully executed.

2 The early influence of Szegö, Van der Corput, and Tricomi

Among important individuals who had an influence in shaping Luigi's formation as a research mathematician, one must mention Giovanni Sansone, who guided Luigi's first research efforts, Gabor Szegö and Johannes Van der Corput, with whom Luigi interacted during a visit in 1951 to Stanford University, and above all, from the start of Luigi's career at the University of Turin, Francesco Tricomi, who became his mentor.

2.1 A general method of Tricomi

Already in the very first papers of Luigi, dealing with zeros of Legendre and ultraspherical polynomials of large degrees, and high-order zeros of Bessel functions, an important ingredient is a method of Tricomi for deriving the asymptotics of zeros of functions from the asymptotics of the functions themselves (see [57], or [59, p. 151]). While Tricomi formulated his method in qualitative terms, Luigi in the special cases studied supplies concrete error bounds by tracing and estimating remainder terms in all Taylor expansions employed.

2.1.1 Zeros of ultrashperical polynomials

In the case of Legendre and ultraspherical polynomials, the results obtained in [10–12] are somewhat preliminary inasmuch as they cover only limited ranges of zeros. This deficiency is overcome later in [33], though at the expense of sharpness, where Tricomi's method is again applied to ultraspherical polynomials $P_n^{(\lambda)} = P_n^{(\lambda-1/2,\lambda-1/2)}$, $0 < \lambda < 1$. For the rth zero $\theta_{n,r}^{(\lambda)}$ of $P_n^{(\lambda)}(\cos\theta)$ it is found that for each $r = 1, 2, \ldots, \lfloor n/2 \rfloor$ (which, by symmetry, is all we need),

$$\theta_{n,r}^{(\lambda)} = \vartheta_{n,r}^{(\lambda)} + \frac{\lambda(1-\lambda)}{2(n+\lambda)(n+\lambda+1)} \cot \vartheta_{n,r}^{(\lambda)} + \rho, \tag{1}$$

where $\vartheta_{n,r}^{(\lambda)} = (r - (1-\lambda)/2)\pi/(n+\lambda)$, and[1]

$$|\rho| < \frac{\lambda(1-\lambda)}{(n+\lambda+1)(2r+\lambda-1)^2}, \quad 0 < \lambda < 1. \tag{2}$$

It can be seen that when r is fixed and $n \to \infty$, the first two terms on the right of (1), as well as the bound in (2), are all $\sim cn^{-1}$, with the respective constants c decreasing [actually, when $r = 1$, and in part also when $r = 2$, the constant c for the bound in (2) is a bit larger than the one for the second term]. On the other hand, when $r = \lfloor \delta n/2 \rfloor$, with $0 < \delta \le 1$ fixed, the two terms and bound are respectively $O(1)$, $O(n^{-2})$, and $O(n^{-3})$. For the zeros $x_{n,r}^{(\lambda)} = \cos\, \theta_{n,r}^{(\lambda)}$ themselves, one finds

$$x_{n,r}^{(\lambda)} = \xi_{n,r}^{(\lambda)} \left[1 - \frac{\lambda(1-\lambda)}{2(n+\lambda)(n+\lambda+1)} \right] + \varepsilon, \tag{3}$$

where $\xi_{n,r}^{(\lambda)} = \cos((r - (1-\lambda)/2)\pi/(n+\lambda))$, and

$$|\varepsilon| < \frac{1.55\, \lambda(1-\lambda)}{(n+\lambda)(n+\lambda+1)(2r+\lambda-1)}, \quad 0 < \lambda < 1. \tag{4}$$

2.1.2 Zeros of Bessel functions

Similarly complete are the results in [13] for the Bessel function J_ν, $0 \le \nu \le 1$. Thus, for the rth positive zero $j_{\nu,r}$ of J_ν, Luigi shows that

$$j_{\nu,r} = x_r - \frac{4\nu^2 - 1}{8x_r} + \varepsilon(\nu, r), \quad r = 1, 2, 3, \ldots, \tag{5}$$

where $x_r = (r + \nu/2 - 1/4)\pi$, and

$$|\varepsilon(\nu, r)| < \frac{(7.4A^2 + 1.1A)r}{64(6r-5)} (2r + \nu - 1)^{-3}, \quad A = |4\nu^2 - 1|, \tag{6}$$

valid for each $r = 1, 2, 3, \ldots$. The formula (6), in fact, quantifies the $O(r^{-3})$ term in a classical asymptotic formula of McMahon [54]. In another formula of McMahon for the rth zero of $J_0(kx)Y_0(x) - J_0(x)Y_0(kx)$, where J_0 and Y_0 are the zeroth-order Bessel functions of first and second kind, an $O(r^{-7})$ term is similarly quantified in [14] for $r \ge 2$ and values of the parameter k satisfying $1 < k < 3 + 2\sqrt{2}$. The calculations, however, are rather more formidable in this case. For the rth positive zero $j'_{\nu,r}$, $0 \le \nu \le 1$, of the derivative J'_ν of the

[1] The square in the second factor of the denominator is missing in Eq. (2.13_1) of [33].

Bessel function, a formula analogous to (5), (6), also due (without error bound) to McMahon, is derived in [20].

2.1.3 Zeros of Jacobi polynomials

The application of Tricomi's method to more general Jacobi polynomials $P_n^{(\alpha,\beta)}$ had to wait until 1980, when a suitable asymptotic expansion for $P_n^{(\alpha,\beta)}$ became available through the work of Hahn [53]. Using the first three terms of this expansion in conjunction with Tricomi's method (in fact, a slight extension thereof), and assuming $|\alpha| \le 1/2$, $|\beta| \le 1/2$, Luigi jointly with Pittaluga [50] proves that for the zeros $\theta_{n,r}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(\cos\theta)$ contained in any compact subinterval of $(-1, 1)$, there holds

$$\theta_{n,r}^{(\alpha,\beta)} = \vartheta_{n,r}^{(\alpha,\beta)} + \frac{1}{(2n+\alpha+\beta+1)^2}\left[\left(\frac{1}{4}-\alpha^2\right)\cot\left(\frac{1}{2}\vartheta_{n,r}^{(\alpha,\beta)}\right) - \left(\frac{1}{4}-\beta^2\right)\tan\left(\frac{1}{2}\vartheta_{n,r}^{(\alpha,\beta)}\right)\right] + O(n^{-4}), \tag{7}$$

where $\vartheta_{n,r}^{(\alpha,\beta)} = (2r+\alpha-1/2)\pi/(2n+\alpha+\beta+1)$. If $\alpha^2 = \beta^2 = 1/4$, not only the expression in brackets, but also the error term in (7) vanish. In the ultraspherical case $\alpha = \beta$, the result (7) is asymptotically in agreement with earlier ones in [11]. There is of course a result analogous to (7) for the zeros $x_{n,r}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(x)$ themselves. Numerical tests revealed that already for $n = 16$, these asymptotic approximations (with the error term removed) typically yield $4\frac{1}{2}$ – 6 correct significant digits for *all* zeros $x_{n,r}^{(\alpha,\beta)}$. Interestingly, if one of the parameters α, β has the value $\pm 1/2$, the accuracy is several orders higher near the appropriate boundary of $[-1, 1]$, a phenomenon duly explained by Luigi.

2.2 A general method of Gatteschi and Van der Corput's theory of enveloping series

2.2.1 Zeros of Bessel functions by Gatteschi's method

In [15], with the assistance of Van der Corput, Luigi develops a general procedure of his own for generating inequalities for the zeros of a function

$$f(x) = (1+\delta)\sin x + \varepsilon\cos x - \rho, \quad \delta > -1, \tag{8}$$

where δ, ε, and ρ may depend on x but are small in magnitude. This kind of functions is often encountered in asymptotic expansions (for large x) of certain Bessel-type functions. Luigi in [15] applies his new procedure to Bessel functions $J_\nu(x)$, where ν can now be arbitrary nonnegative, and supplements the results in [13] by estimating the zeros $j_{\nu,r}$ that are larger than $(2\nu+1)(2\nu+3)/\pi$. The same procedure is applied in [19] to Airy functions $\mathrm{Ai}(-x)$, $\mathrm{Bi}(-x)$ and their positive zeros.

In two of his late papers, [47] and [48], Luigi, jointly with Giordano, returns to his procedure and makes further applications to Bessel functions. In [47], McMahon's formula for $j_{\nu,r}$ is taken up again and in the case $|\nu| \le 1/2$ supplied with lower and upper bounds for the $O(r^{-5})$ term, and in the case $\nu > 1/2$ with similar bounds for the $O(r^{-3})$ and $O(r^{-5})$ terms. A two-term asymptotic approximation with explicit error bounds is obtained in [48] for the positive zeros $i_{\nu,r} > (r + \nu/2 - 3/4)\pi$, $r \ge 10$, of $(d/dx)[\sqrt{x}J_\nu(x)]$ in the case that $|\nu| \le 1/2$.

2.2.2 Bessel functions at and near the transition point

A new methodological element—Van der Corput's theory of "enveloping series"—appears in [21]. Given a series $\sum_{n=0}^{\infty} a_n$ (not necessarily convergent) and a majorizing series $\sum_{n=0}^{\infty} A_n$ thereof, i.e., $|a_n| \le A_n$ for all n, the series $\sum_{n=0}^{\infty} a_n$ is said to envelope a number (or function) s relative to the majorant $\sum_{n=0}^{\infty} A_n$, if for each $n = 0, 1, 2, \ldots$

$$s = \sum_{k=0}^{n-1} a_k + \vartheta_n A_n, \quad |\vartheta_n| \le 1.$$

Using two key theorems in Van der Corput's theory of enveloping series, one relating to the formal substitution of a series into another series, and another relating to integration of (functional) enveloping series, both applied to contour integral representations of Hankel functions, Luigi in [21] derives very impressive asymptotic expansions for $J_\nu(\nu)$ and $Y_\nu(\nu)$ as $\nu \to \infty$, both supplied with error estimates. They are not simple, involving as they do incomplete gamma functions and coefficients $A_k^{(m)}$ in the Taylor expansion of $\left(\frac{1}{5!} + \frac{1}{7!}z + \frac{1}{9!}z^2 + \cdots\right)^m$, $m = 2, 3, \ldots$ (which today, however, are easily obtainable by symbolic computation systems such as Maple). As an application, Luigi takes the first two terms of his expansion for $J_\nu(\nu)$ (the second term happening to be zero) and obtains

$$J_\nu(\nu) = \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi}\,\nu^{-1/3} - \theta\eta, \quad 0 < \theta \le 1,\ \nu \ge 6, \tag{9}$$

where[2]

$$\eta = \frac{1}{\pi\nu}\left(e^{-\nu\pi/\sqrt{3}} + .521e^{-(2\pi/\sqrt{3})^3\nu/6}\right) + \frac{1.4}{\pi}\left(\frac{6}{\nu}\right)^{5/3}.$$

This recovers an asymptotic formula of Cauchy, but endows it with an explicit error bound. The simpler bound $\eta < \nu^{-5/3}$ is given in the lecture [25].

As observed in [22], there is a slight inaccuracy (on p. 275) in [21], but the results obtained there are shown to continue to hold. Also, from the first term

[2]The first term in parentheses is misprinted in [21, Eq. (20′)] as $e^{-2\pi/\sqrt{3}}$.

of the asymptotic expansion for $Y_\nu(\nu)$ in [21], the following companion result to (9) is obtained,

$$Y_\nu(\nu) = -\frac{\Gamma(1/3)}{(4/3)^{1/3}\pi}\,\nu^{-1/3} + \rho, \quad |\rho| \le \frac{.252}{\pi\nu},\ \nu \ge 1. \tag{10}$$

An interesting consequence of this is

$$\left\{\begin{array}{l} |J_\nu(\nu x)| \\ |Y_\nu(\nu x)| \end{array}\right\} < \frac{1}{\sqrt{x}}\left[\frac{3.841}{\pi\nu^{1/3}} + \frac{.252}{\pi\nu}\right], \tag{11}$$

valid for $x > 1$, $\nu \ge 1$.

An asymptotic estimate of $J_\nu(x)$ around the transition point $x = \nu$ is developed in [26], by using a Liouville–Steklov-type approach (cf. Section 3.1). It is shown that

$$J_\nu\big(\nu\exp(6^{-1/3}\nu^{-2/3}t)\big) = 3^{2/3}\Gamma(2/3)J_\nu(\nu)\mathrm{Ai}\big(-3^{-1/3}t\big) + \rho, \tag{12}$$

where for $\nu \ge 6$

$$|\rho| < \begin{cases} \dfrac{t^4 + 5.6\,t}{\pi\nu} & \text{if } 0 < t < 6^{1/3}\nu^{2/3}, \\ \dfrac{1}{\nu}\,[.005\,t^4\exp(4(|t|/3)^{3/2}) + 1.77\,|t|\exp(2(|t|/3)^{3/2})] & \text{if } t < 0. \end{cases} \tag{13}$$

Sharper estimates are obtained by a reapplication of the Liouville–Steklov method.

Luigi also gives an asymptotic estimate of the derivative $J'_\nu(x)$ at $x = \nu$,

$$J'_\nu(\nu) = \frac{1}{2\sqrt{3}\pi}\left[\Gamma(2/3)\left(\frac{6}{\nu}\right)^{2/3} - \frac{\Gamma(1/3)}{30}\left(\frac{6}{\nu}\right)^{4/3}\right] + \vartheta\,\frac{2}{\nu^2}, \quad |\vartheta| < 1,\ \nu \ge 6, \tag{14}$$

an interesting subsidiary result.

3 Methods based on differential equations

Linear second-order differential equations, which are at the heart of much of special function theory, can be used in many ways to obtain asymptotic approximations and inequalities. There are two techniques, in particular, that Luigi frequently, and early on, availed himself of: One is the method of Liouville–Steklov (sometimes also attributed to Fubini), which is based on transforming the differential equation into a Volterra integral equation; the other is the use of Sturm-type comparison theorems.

3.1 The method of Liouville–Steklov

3.1.1 Hilb's formula and zeros of Legendre polynomials

Already in one of his early papers, [16], Luigi applies the method of Liouville–Steklov, following Szegö's treatment in [55, Section 8.62], to the differential equation satisfied by $(\sin\theta)^{1/2}P_n(\cos\theta)$. (By symmetry, it suffices to consider the interval $0 \le \theta \le \pi/2$.) This yields immediately Hilb's formula,

$$P_n(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0((n+1/2)\theta) + \sigma, \tag{15}$$

where for large n, when θ is away from the origin (i.e., $\theta \ge cn^{-1}$ for some positive constant c), the error is $\sigma = \theta^{1/2}O(n^{-3/2})$, otherwise $\sigma = O(n^{-2})$. In his quest for quantification, Luigi derives explicit inequalities for the error σ: In the first case,

$$|\sigma| < .358\,\theta^{-1/2}n^{-5/2} + .394\,\theta^{1/2}n^{-3/2} \quad \text{if } \pi/2n < \theta \le \pi/2 \tag{16}$$

(which may also be written as $|\sigma| < .622\,\theta^{1/2}n^{-3/2}$; cf. [23, Eq. (2)]), and in the second case,

$$|\sigma| < .09\,\theta^2 \quad \text{if } 0 < \theta \le \pi/2n. \tag{17}$$

This is then applied to obtain two-sided inequalities for the zeros $\theta_{n,r}$ (in ascending order) of $P_n(\cos\theta)$, namely[3]

$$0 < \frac{j_{0,r}}{n+1/2} - \theta_{n,r} < (1.6 + 3.7r)n^{-4}, \quad n = 1, 2, \ldots, \lfloor n/2 \rfloor, \tag{18}$$

where $j_{0,r}$ is the rth positive zero of the Bessel function J_0.

A reapplication of the Liouville–Steklov method to the same differential equation, but now with (15) inserted in the integral of the Volterra integral equation, in [23] yields an improved two-term asymptotic approximation for $P_n(\cos\theta)$, and in consequence also two-term approximations for the zeros $\theta_{n,r}$ of $P_n(\cos\theta)$, and likewise for the zeros $x_{n,r}$ of $P_n(x)$. Thus, for example,

$$x_{n,r} = 1 - \frac{j_{0,r}^2}{2(n+1/2)^2} + \frac{j_{0,r}^2 + j_{0,r}^4}{24(n+1/2)^4} + O(n^{-6}), \tag{19}$$

which for $n = 16$, $r = 1$ and $r = 2$ (neglecting the error term), yields approximations for the respective zeros having errors 2.28×10^{-8} resp. 2.15×10^{-6}.

3.1.2 Hilb's formula for ultraspherical polynomials

Hilb's formula for ultraspherical polynomials is supplied with error bounds in [18] and applied to the zeros of $P_n^{(\lambda)}$. A slightly different application of the method of Liouville–Steklov, especially if applied successively as suggested by Szegö [55, Section 8.61(2)] in the case of Legendre polynomials, yields more accurate approximations of ultraspherical polynomials $P_n^{(\lambda)}$, valid in any compact subinterval of $(-1, 1)$, and of their zeros contained therein [34].

[3]There is a misprint in Eq. (17) of [16], where the number 16 in the denominator should be 10. The upper bound given there (and in our Eq. (18)) has been checked by us on the computer and was found to be too small, at least for larger values of n. The reason for this may be inaccuracies in the numerical constants supplied.

3.1.3 Hilb's formula for Jacobi polynomials

There is a Hilb's formula also for Jacobi polynomials $P_n^{(\alpha,\beta)}$, $\alpha > -1$ and β arbitrary real [55, Section 8.63], which in the classical form reads as follows,

$$\theta^{-1/2}\left(\sin\frac{1}{2}\theta\right)^{\alpha+1/2}\left(\cos\frac{1}{2}\theta\right)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos\theta)$$
$$= 2^{-1/2}N^{-\alpha}\,\frac{\Gamma(n+\alpha+1)}{n!}\,J_\alpha(N\theta)+\sigma_\alpha(n,\theta), \tag{20}$$

where $N = n + (\alpha+\beta+1)/2$ and $\sigma_\alpha = \theta^{1/2}O(n^{-3/2})$ away from the origin, and $\sigma_\alpha = \theta^{\alpha+2}O(n^\alpha)$ otherwise. In [31], this is improved in two ways: First, the method of Liouville–Steklov is refined, with the result that in (20) the number N can be replaced by

$$\nu = \left[\left(n+\frac{\alpha+\beta+1}{2}\right)^2+\frac{1-\alpha^2-3\beta^2}{12}\right]^{1/2} \tag{21}$$

and the error term improved to $\sigma_\alpha = \theta^{5/2}O(n^{-3/2})$ and $\sigma_\alpha = \theta^{\alpha+4}O(n^\alpha)$ away from, and near the origin, respectively.[4] Secondly, the method of Liouville–Steklov is iterated once more, similarly as in [23], producing a two-term approximation,

$$\theta^{-1/2}\left(\sin\frac{1}{2}\theta\right)^{\alpha+1/2}\left(\cos\frac{1}{2}\theta\right)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos\theta)$$
$$= 2^{-1/2}\nu^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left[\left(1-\frac{4-\alpha^2-15\beta^2}{1440\,\nu^2}\,\theta^2\right)J_\alpha(\nu\theta)\right.$$
$$\left.+\frac{4-\alpha^2-15\beta^2}{720\,\nu^3}\,\theta\left(\frac{1}{2}\nu^2\theta^2+\alpha^2-1\right)J_\alpha'(\nu\theta)\right]+\rho_\alpha(n,\theta), \tag{22}$$

with the remainder term further improved to respectively $\rho_\alpha = \theta^{9/2}O(n^{-3/2})$ and $\rho_\alpha = \theta^{\alpha+6}O(n^\alpha)$. The result (22) can easily be specialized to ultraspherical polynomials (i.e., to $\alpha=\beta=\lambda-1/2$) and to Legendre polynomials ($\lambda = 1/2$). In the latter case, by expressing J_0' in terms of J_0 and J_2, one obtains the rather simple formula

$$\left(\frac{\sin\theta}{\theta}\right)^{1/2}P_n(\cos\theta) = J_0(\nu\theta)-\frac{\theta^3}{360\,\nu}+\frac{\theta^2}{360\,\nu^2}J_2(\nu\theta)+\rho(n,\theta), \tag{23}$$

with $\nu = [(n+1/2)^2+1/12]^{1/2}$ and $\rho = \theta^{9/2}O(n^{-3/2})$ resp. $\rho = \theta^6\,O(1)$.

[4] In the second of these formulae, the factor $\theta^{\alpha+4}$ is misprinted as $\theta^{\alpha+1}$ in the original Eq. (19) of [31].

Luigi now once again applies Tricomi's theorem (cf. Section 2.1) to derive from the asymptotic approximations in [31] asymptotic results for zeros of Jacobi polynomials in terms of zeros of Bessel functions, and vice versa. In the ultraspherical case, for example, he finds for the rth positive zero $j_{s,r}$ of J_s, $-1/2 < s < 1/2$, when r is fixed, the following asymptotic approximation,

$$j_{s,r} = \nu\theta_{n,r} + \frac{1-4s^2}{360\,\nu}\,\theta_{n,r}^3 - (1-s^2)\,\frac{1-4s^2}{180\,\nu^3}\,\theta_{n,r} + O(n^{-6}), \tag{24}$$

where $\nu = \left[(n+s+1/2)^2 + (1-4s^2)/12\right]^{1/2}$ and $\theta_{n,r} = \theta_{n,r}^{(s+1/2)}$ is the rth zero of $P_n^{(s+1/2)}(\cos\theta)$.

In [2], the method of Liouville-Steklov is used to derive a new asymptotic approximation of Hilb's type for Jacobi polynomials $P_n^{(\alpha,\beta)}$, $|\alpha| \le 1/2$, $|\beta| \le 1/2$, with realistic and explicit error bounds, and from it an asymptotic estimate of the zeros $\theta_{n,r}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(\cos\theta)$ obtained previously in a different manner by Frenzen and Wong [7]. Continuation of this work in [41, 42] led to a number of significant improvements.

The classical Hilb's formula (20) for Jacobi polynomials is applied in [17] to study the relative extrema of $P_n^{(\alpha,\beta)}$. If $y_{n,r}$ are their abscissae, and $y_{n,r} = \cos\varphi_{n,r}$, a short and elegant proof is given of the limit relation

$$\lim_{n\to\infty}\left(\sin\frac{1}{2}\,\varphi_{n,r}\right)^{\alpha}\left(\cos\frac{1}{2}\,\varphi_{n,r}\right)^{\beta} P_n^{(\alpha,\beta)}(y_{n,r}) = J_\alpha(j_{\alpha+1,r}). \tag{25}$$

3.2 Methods based on Sturm comparison theorems

Sturm-type comparison theorems, for example in the form stated by Szegö in [55, Section 1.82], are a natural tool for comparing zeros of one type of special functions with zeros of another type, the types of special functions depending on the choice of differential equations that are being compared. This is a recurring theme in Luigi's work and gives rise to many interesting inequalities.

3.2.1 Zeros of Jacobi polynomials and Bessel functions

In [32], the comparison is between zeros $\theta_{n,r}^{(\alpha,\beta)}$ of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$ and zeros $j_{\alpha,r}$ of Bessel functions J_α, which, under the assumption $|\alpha| \le 1/2$, $\beta \le 1/2$, finds expression in the inequalities

$$j_{\alpha,r}\left[N^2 + \frac{1}{4} - \frac{\alpha^2+\beta^2}{2} - \frac{1-4\alpha^2}{\pi^2}\right]^{-1/2} < \theta_{n,r}^{(\alpha,\beta)} < j_{\alpha,r}\left[N^2 + \frac{1-\alpha^2-3\beta^2}{12}\right]^{-1/2}, \tag{26}$$

valid for $r = 1, 2, \ldots, \lfloor n/2 \rfloor$, where $N = n + (\alpha+\beta+1)/2$.

The first zero, $j_\nu = j_{\nu,1}$ of the Bessel function J_ν, $\nu > 0$, and also the abscissa j'_ν of its first maximum, are studied in [49], where Sturm's theorem is used in a form given by Watson in [61, Section 15.83] and is slightly extended and combined, in part, with Tricomi's theorem (cf. Section 2.1). The result can be written in the form

$$j_\nu = \nu \exp\left(2^{-1/3}\nu^{-2/3}a_1 - 1.623\,\vartheta\,\nu^{-4/3}\right),$$
$$j'_\nu = \nu \exp\left(2^{-1/3}\nu^{-2/3}a'_1 - 1.06\,\vartheta'\,\nu^{-4/3}\right), \tag{27}$$

where $0 < \vartheta, \vartheta' < 1$, and $a_1 = 2.33810741$, $a'_1 = 1.01879297$ are the first zero, resp. maximum, of the Airy function $\mathrm{Ai}(-x)$. The bounds implied by (27) compare favorably with earlier estimates by Schafheitlin and Tricomi.

Restricting ν to the "principal" interval $|\nu| < 1/2$, Luigi, together with Giordano, in [46] obtains a very sharp upper bound for j_ν, namely

$$j_\nu < \Theta(\nu)K(\nu), \quad -1/2 < \nu < 1/2, \tag{28}$$

where

$$\Theta(\nu) = \arccos\sqrt{\frac{10\nu + 35 + 2\sqrt{10\nu^2 + 55\nu + 70}}{4\nu^2 + 32\nu + 63}}$$

is the first zero $\theta_{5,1}^{(\nu)}$ of $P_5^{(\nu,\nu)}(\cos\theta)$,

$$K(\nu) = \left[(\nu + 11/2)^2 + \left(\frac{1}{4} - \nu^2\right)\left(\frac{1}{\sin^2\phi(\nu)} - \frac{1}{\phi^2(\nu)}\right)\right]^{1/2},$$

and

$$\phi(\nu) = \frac{\sqrt{\nu+1}\left(\sqrt{\nu+2}+1\right)}{\nu + 11/2}.$$

Outside the principal interval, there holds

$$j_\nu < \frac{1}{3}\,\Theta(\nu)\sqrt{6\nu^2 + 99\nu + 273}, \quad \nu \notin (-1/2, 1/2). \tag{29}$$

These inequalities are generally sharper (often considerably so) than the best inequalities (valid for $\nu > -1$) known in the literature.

3.2.2 Zeros of Laguerre polynomials

The application of Sturm's theorem (again in Szegö's form) to zeros $0 < \lambda_{n,1}^{(\alpha)} < \lambda_{n,2}^{(\alpha)} < \cdots < \lambda_{n,n}^{(\alpha)}$ of Laguerre polynomials $L_n^{(\alpha)}$ is carried out in [37]. Two types of comparison differential equations are used, one giving rise to Bessel functions, the other to Airy functions. In the former case, under the assumption $-1 < \alpha \le 1$, Luigi finds that

$$\lambda_{n,r}^{(\alpha)} < \nu\cos^2\left(\frac{1}{2}\,x_{n,r}^{(\alpha)}\right), \quad r = 1, 2, \ldots, n, \tag{30}$$

where $x_{n,r}^{(\alpha)}$ is the root of the equation

$$x - \sin x = \pi - \frac{4\,j_{\alpha,r}}{\nu}, \tag{31}$$

and $\nu = 4n + 2\alpha + 2$. In the latter case, he shows that

$$\lambda_{n,r}^{(\alpha)} > \nu \cos^2\left(\frac{1}{2}\, x_{n,r}^{*(\alpha)}\right) \quad \text{if } -1/2 \le \alpha \le 1/2, \tag{32}$$

and

$$\lambda_{n,r}^{(\alpha)} < \nu \cos^2\left(\frac{1}{2}\, x_{n,r}^{*(\alpha)}\right) \quad \text{if } -1 < \alpha \le -2/3 \text{ or } \alpha \ge 2/3, \tag{33}$$

where $x_{n,r}^{*(\alpha)}$ is the root of the equation

$$x - \sin x = \frac{8}{3\nu}\, a_{n+1-r}^{3/2} \tag{34}$$

and a_k the kth zero in ascending order of $\mathrm{Ai}(-x)$.

Since Hermite polynomials are related to Laguerre polynomials with parameters $\alpha = \pm 1/2$, and $j_{1/2,r} = r\pi$, $j_{-1/2,r} = (r - 1/2)\pi$, the inequalities (30) and (32) yield upper[5] and lower bounds for the positive zeros $0 < h_{n,\lfloor (n+1)/2 \rfloor + r}$, $r = 1, 2, \ldots, \lfloor n/2 \rfloor$, of the Hermite polynomial H_n:

$$h_{n,\lfloor (n+1)/2\rfloor + r} < \sqrt{2n+1} \times \begin{cases} \cos\left[\dfrac{1}{2}\, x\left(\dfrac{2n-4r+3}{2n+1}\,\pi\right)\right], & n \text{ even}, \\ \cos\left[\dfrac{1}{2}\, x\left(\dfrac{2n-4r+1}{2n+1}\,\pi\right)\right], & n \text{ odd}, \end{cases} \tag{35}$$

where $x = x(y)$ is the inverse function of $y = \sin x - x$ and

$$h_{n,\lfloor (n+1)/2\rfloor + r} > \sqrt{2n+1}\, \cos\left[\frac{1}{2}\, x\left(\frac{8}{3(2n+1)}\, a_{\lfloor n/2\rfloor + 1 - r}^{3/2}\right)\right], \; r = 1, 2, \ldots, \lfloor n/2 \rfloor. \tag{36}$$

All these inequalities are remarkably sharp.

3.2.3 Zeros of confluent hypergeometric functions

Since Laguerre polynomials $L_n^{(\alpha)}$ are special cases of confluent hypergeometric functions $\Phi(a, c; x)$ and $\Psi(a, c; x)$ (in Tricomi's notation), namely $a = -n$, $c = 1 + \alpha$, it is natural to try extending the inequalities obtained in [37] for Laguerre polynomials to confluent hypergeometric functions. This is done in [39], where Sturm-type comparison theorems are used in both Szegö's and Watson's form. With regard to the first ("regular") confluent hypergeometric function $\Phi(a, c; x)$, it is known that, if $c > 0$, there are no positive zeros of $\Phi(a, c; x)$ if $a \ge 0$, and precisely $-\lfloor a \rfloor$ positive zeros if $a < 0$. Under the assumption $a < 0$, $0 < c \le 2$, Luigi then proves that for the rth positive zero ϕ_r there holds

$$\phi_r < 4k \cos^2\left(\frac{1}{2}\, x_r\right), \quad r = 1, 2, \ldots, s, \tag{37}$$

[5] In the upper bound of (35) for n odd, the numerator $2n - 4r + 1$ in [37] is misprinted as $2n - 4r + 3$.

where $k = \frac{1}{2}c - a$, $s = \lfloor \frac{1}{4} - a \rfloor$, and x_r is the root of the equation

$$x - \sin x = \pi - \frac{j_{c-1,r}}{k}. \tag{38}$$

Note that (38) is identical with (31) in the case $a = -n$, $c = 1 + \alpha$ of Laguerre polynomials, since $k = \frac{1}{2}(1 + \alpha) + n = \nu/4$. Also, $s = \lfloor \frac{1}{4} - a \rfloor$ is either the total number of positive zeros, or one less, depending on whether $a - \lfloor a \rfloor$ is less than, or greater or equal to, 1/4.

As for the ("irregular") confluent hypergeometric function $\Psi(a, c; x)$, it is known that, if $c \geq 1$, it has no positive zeros if $a \geq 0$, and precisely $-\lfloor a \rfloor$ positive zeros ψ_r if $a < 0$. Here, assuming $a < 0$, $1 \leq c \leq 2$, Luigi proves the inequality

$$\psi_r < 4k\cos^2\left(\frac{1}{2}x_r^0\right), \quad r = 1, 2, \ldots, -\lfloor a \rfloor, \tag{39}$$

where $k = \frac{1}{2}c - a$ and x_r^0 is the root of

$$x - \sin x = \pi - \frac{j_{c-1,r}^0}{k}, \quad k = \frac{1}{2}c - a, \tag{40}$$

with $j_{c-1,r}^0$ the rth positive zero of $\cos((a - \lfloor a \rfloor)\pi)J_{c-1}(x) - \sin((a - \lfloor a \rfloor)\pi)$ $Y_{c-1}(x)$. The case $0 < c < 1$ can be reduced to $1 < c < 2$ by applying the identity $\Psi(a - c + 1, 2 - c; x) = x^{c-1}\Psi(a, c; x)$.

Using a different differential equation for comparison in Sturm's theorem, Luigi derives additional inequalities for ϕ_r and ψ_r, where the former reduce to the inequalities (32), (33) in the case of Laguerre polynomials. Another interesting special case is $a = (1 - \nu)/2$, $\nu > 1$ and $c = 3/2$, which leads to parabolic cylinder functions D_ν and upper and lower bounds for their positive zeros $\delta_{\nu,r}$, $r = 1, 2, \ldots, -\lfloor (1 - \nu)/2 \rfloor$.

3.2.4 Inequalities from asymptotic estimates

Applications of Sturm's theorem of a somewhat different character are made in [36] and [43], where known asymptotic estimates containing order-of-magnitude terms are shown to actually become inequalities if the O-term is omitted. Such is the case, e.g., in a result of Frenzen and Wong [7, Corollary 2] concerning the zeros $\theta_{n,r}^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}(\cos\theta)$, which in the hands of Luigi becomes the inequality

$$\theta_{n,r}^{(\alpha,\beta)} \geq \frac{1}{N}\, j_{\alpha,r} - \frac{1}{4N^2}\left[\left(\frac{1}{4} - \alpha^2\right)\left(\frac{2}{t} - \cot\frac{1}{2}t\right) + \left(\frac{1}{4} - \beta^2\right)\tan\frac{1}{2}t\right],$$

$$N = n + \frac{\alpha + \beta + 1}{2}, \quad t = \frac{1}{N}\, j_{\alpha,r}, \tag{41}$$

valid for $|\alpha| \leq 1/2$, $|\beta| \leq 1/2$ and $r = 1, 2, \ldots, n$, with equality holding if $\alpha^2 = \beta^2 = 1/4$. In fact, (41) can be improved by replacing N in the definition of t (but not elsewhere) by $\nu = [N^2 + (1 - \alpha^2 - 3\beta^2)/12]^{1/2}$. A similar *upper* bound can be obtained by switching the parameters α and β and using a well-known identity relating $P_n^{(\alpha,\beta)}$ with $P_n^{(\beta,\alpha)}$. These inequalities are quite sharp, especially near the respective end points π and 0. Sometimes, the upper bound in (26) may be better for the first few values of r than the upper bound obtainable from (41) by switching α and β, and likewise the lower bound obtainable similarly from the upper bound of (26) may be better than (41) for the last few values of r. Thus, in applications, (41) and (26) should be considered conjointly. All these inequalities are easily specialized to the ultraspherical case $\alpha = \beta$.

Similarly, by omitting the O-term in (7), the right-hand side becomes an upper bound in the ultraspherical case $\alpha = \beta$.

For the zeros $j_{\nu,r}$ of Bessel functions J_ν, the removal of the O-terms in some asymptotic (for large ν) estimates of Olver is conjectured in [43] to lead to upper and lower bounds, specifically to

$$\nu x_{\nu,r} < j_{\nu,r} < \nu x_{\nu,r} + g_\nu(x_{\nu,r}), \quad r = 1, 2, 3, \dots, \tag{42}$$

where $x_{\nu,r}$ is the root of the equation $\sqrt{x^2-1} - \arctan\sqrt{x^2-1} = (2/3\nu)a_r^{3/2}$,

$$g_\nu(x) = \frac{x}{\nu}\frac{1}{(x^2-1)^{1/2}}\left[\frac{-5\nu}{48\,a_r^{3/2}} + \frac{5}{24(x^2-1)^{3/2}} + \frac{1}{8(x^2-1)^{1/2}}\right], \tag{43}$$

and a_r is th rth zero of $\mathrm{Ai}(-x)$. The lower bound is actually proved to hold for $\nu > 0$ and all r, and the upper bound for $\nu > 0$ and all r sufficiently large. In fact, if in (43) the right-hand side is multiplied by the factor $1 + 2^{1/3}/(280\,a_r\nu^{4/3})$, then the conjecture is proved to hold for $\nu \geq 1/2$ and all r. Heavy use is made in this work of symbolic computation with Maple V.

4 Uniform expansions

4.1 Zeros of Laguerre polynomials

Asymptotic estimates of the zeros of Laguerre polynomials $L_n^{(\alpha)}$ that resemble the inequalities in (30)–(34) are obtined in [38] from the initial terms of uniform asymptotic expansions for Laguerre polynomials due to Frenzen and Wong [8]. With $x_{n,r}^{(\alpha)}$ again denoting the root of (31), and setting $\tau_{n,r}^{(\alpha)} = \cos^2\left(\frac{1}{2}\,x_{n,r}^{(\alpha)}\right)$, from the expansion [8, Eq. (4.7)] Luigi finds the asymptotic estimate

$$\lambda_{n,r}^{(\alpha)} = \nu\tau_{n,r}^{(\alpha)} - \frac{1}{2\nu}\left[\frac{(1-4\alpha^2)\nu}{2\,j_{\alpha,r}}\left(\frac{\tau_{n,r}^{(\alpha)}}{1-\tau_{n,r}^{(\alpha)}}\right)^{1/2} + \frac{4\alpha^2-1}{2} + \frac{\tau_{n,r}^{(\alpha)}}{1-\tau_{n,r}^{(\alpha)}} + \frac{5}{6}\left(\frac{\tau_{n,r}^{(\alpha)}}{1-\tau_{n,r}^{(\alpha)}}\right)^2\right] + O\big(\nu^{-3}\big), \tag{44}$$

where $\nu = 4n + 2\alpha + 2$, and the O-term is uniformly bounded for all $r = 1, 2, \dots, \lfloor qn \rfloor$, with $0 < q < 1$ fixed.

A companion estimate, valid in the range $r = \lfloor pn \rfloor, \lfloor pn \rfloor + 1, \dots, n$, $0 < p < 1$, which overlaps with the range for (44) when $p \leq q$, is similarly obtained from the expansion [8, Eq. (5.13)]. With $x_{n,r}^{*(\alpha)}$ again denoting the root of (34), and setting $\tau_{n,r}^{*(\alpha)} = \cos^2\left(\frac{1}{2}\, x_{n,r}^{*(\alpha)}\right)$, the asymptotic estimate now reads

$$\lambda_{n,r}^{(\alpha)} = \nu\tau_{n,r}^{*(\alpha)} + \frac{1}{\nu}\left[\frac{5\nu}{24\, a_{n+1-r}^{3/2}}\left(\frac{\tau_{n,r}^{*(\alpha)}}{1-\tau_{n,r}^{*(\alpha)}}\right)^{1/2} + \frac{1}{4} - \alpha^2 \right.$$
$$\left. -\frac{1}{2}\,\frac{\tau_{n,r}^{*(\alpha)}}{1-\tau_{n,r}^{*(\alpha)}} - \frac{5}{12}\left(\frac{\tau_{n,r}^{*(\alpha)}}{1-\tau_{n,r}^{*(\alpha)}}\right)^2\right] + O\big(\nu^{-3}\big), \tag{45}$$

where ν is as above in (44).

In the case where r is fixed and $\nu \to \infty$, the estimate (44) can be sharpened to

$$\lambda_{n,r}^{(\alpha)} = \frac{j_{\alpha,r}^2}{\nu}\left[1 + \frac{j_{\alpha,r}^2 + 2(\alpha^2-1)}{3\,\nu^2}\right] + O\big(\nu^{-5}\big), \quad r \text{ fixed}, \tag{46}$$

which is an old estimate of Tricomi from the 1940s. Likewise, the estimate (45) for $r = n+1-s$ and s fixed can be sharpened to

$$\lambda_{n,n+1-s}^{(\alpha)} = \nu - 2^{2/3} a_s \nu^{1/3} + \frac{1}{5}\, 2^{4/3} a_s^2 \nu^{-1/3} + O\big(\nu^{-1}\big), \quad s \text{ fixed}, \tag{47}$$

which is another of Tricomi's earlier estimate.

4.2 Zeros of confluent hypergeometric functions

Two types of uniform asymptotic expansions for Whittaker's confluent hypergeometric functions $M_{\kappa,\mu}$, $W_{\kappa,\mu}$, given by Dunster [5], are used in [9] to develop asymptotic estimates (for large κ) of the positive zeros $m_r^{(\kappa,\mu)}$, $w_r^{(\kappa,\mu)}$ of $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$, respectively. If specialized to Laguerre polynomials, $\kappa = n + (\alpha+1)/2$, $\mu = \alpha/2$, they yield approximations for the zeros $\lambda_{n,r}^{(\alpha)}$ of $L_n^{\alpha)}$ that are now applicable for unrestrictedly large values of both n and α. Uniformity of the results, of course, comes at a price of increased complexity of the formulae.

In [45], Luigi develops two new uniform asymptotic expansions for Whittaker functions, one involving Bessel functions, the other Airy functions. Using three terms of the former, he then derives asymptotic estimates of the respective zeros, which are simpler than those obtained previously in [9] and valid as $\kappa \to \infty$ for fixed μ. Thus, for the rth positive zero of $M_{\kappa,\mu}(x)$ he finds

$$m_r^{(\kappa,\mu)} = 4\kappa\xi_r + \frac{1}{2\kappa}\left(\frac{\xi_r}{1-\xi_r}\right)^{1/2}\left[\frac{\kappa}{2}\,\frac{16\mu^2-1}{j_{2\mu,r}} - 2\mu^2\left(\frac{1-\xi_r}{\xi_r}\right)^{1/2} \right.$$
$$\left. + \frac{1}{24}\,\frac{4\xi_r^2 - 12\xi_r + 3}{(1-\xi_r)^{3/2}\xi_r^{1/2}}\right] + O(\kappa^{-3}), \tag{48}$$

where $\xi_r = \xi_r^{(\kappa,\mu)}$ is the root of the equation

$$\arcsin\sqrt{\xi} + \sqrt{\xi - \xi^2} = \frac{j_{2\mu,r}}{2\kappa}\,. \tag{49}$$

The O-term is uniformly bounded for all $r = 1, 2, 3, \ldots$ such that $m_r^{(\kappa,\mu)} \le 4q\kappa$ with q fixed, $0 < q < 1$. Similarly, for the rth positive zero of $W_{\kappa,\mu}(x)$,

$$w_r^{(\kappa,\mu)} = 4\kappa\tau_r + \frac{1}{2\kappa}\left(\frac{\tau_r}{1-\tau_r}\right)^{1/2}\left[\frac{\kappa}{2}\,\frac{16\mu^2-1}{j^0_{2\mu,r}} - 2\mu^2\left(\frac{1-\tau_r}{\tau_r}\right)^{1/2} + \frac{1}{24}\,\frac{4\tau_r^2-12\tau_r+3}{(1-\tau_r)^{3/2}\tau_r^{1/2}}\right] + O\big(\kappa^{-3}\big), \tag{50}$$

where $\tau_r = \tau_r^{(\kappa,\mu)}$ is the root of

$$\arcsin\sqrt{\tau} + \sqrt{\tau-\tau^2} = \frac{j^0_{2\mu,r}}{2\kappa}, \tag{51}$$

and $j^0_{2\mu,r}$ the rth positive zero of $\sin((\kappa-\mu)\pi)J_{2\mu}(x) - \cos((\kappa-\mu)\pi)Y_{2\mu}(x)$.

Three terms of Luigi's Airy-type asymptotic expansion yield estimates valid for all zeros $m_r^{(\kappa,\mu)}$, $w_r^{(\kappa,\mu)}$ larger than $4p\kappa$, with p fixed, $0 < p < 1$. Specifically, with $n = \lceil \kappa - \mu - 1/2 \rceil$ denoting the number of positive zeros,

$$m_r^{(\kappa,\mu)} = 4\kappa\xi_r^* + \frac{1}{24\kappa}\left(\frac{\xi_r^*}{1-\xi_r^*}\right)^{1/2} \times\left[\frac{5\kappa}{c_{n+1-r}^{3/2}} - \frac{1}{2}\,\frac{(48\mu^2-4)(\xi_r^*-1)^2+4\xi_r^*+1}{(1-\xi_r^*)^{3/2}\xi_r^{*1/2}}\right] + O\big(\kappa^{-3}\big), \tag{52}$$

where $\xi_r^* = \xi_r^{*(\kappa,\mu)}$ is the root of the equation

$$\arccos\sqrt{\xi} - \sqrt{\xi-\xi^2} = \frac{c_{n+1-r}^{2/3}}{3\kappa} \tag{53}$$

and c_k the kth positive zero in ascending order of $\sin((\kappa-\mu)\pi)\mathrm{Ai}(-x) + \cos((\kappa-\mu)\pi)\mathrm{Bi}(-x)$, and

$$w_r^{(\kappa,\mu)} = 4\kappa\tau_r^* + \frac{1}{24\kappa}\left(\frac{\tau_r^*}{1-\tau_r^*}\right)^{1/2} \times\left[\frac{5\kappa}{a_{n+1-r}^{3/2}} - \frac{1}{2}\,\frac{(48\mu^2-4)(\tau_r^*-1)^2+4\tau_r^*+1}{(1-\tau_r^*)^{3/2}\tau_r^{*1/2}}\right] + O\big(\kappa^{-3}\big), \tag{54}$$

where $\tau_r^* = \tau_{n,r}^*$ is the root of

$$\arccos\sqrt{\tau} - \sqrt{\tau-\tau^2} = \frac{a_{n+1-r}^{2/3}}{3\kappa}, \tag{55}$$

and a_k the kth positive zero in ascending order of $\mathrm{Ai}(-x)$.

5 Miscellanea

In this section, a few of Luigi's papers are collected, which do not fit into the classification scheme we have adopted.

5.1 Retouching asymptotic formulae

The idea of "retouching" asymptotic formulae, going back to Tricomi [56], consists in introducing into the asymptotic approximation small correction terms, which can be compactly tabulated or presented graphically so as to enable a quick and relatively accurate determination of the desired quantity. The idea is particularly useful if two or more variables are involved. In [27], Luigi experiments with this idea in connection with asymptotic formulae for Bessel functions $J_\nu(x)$, $Y_\nu(x)$ in the range $x \geq 10$ and arbitrary ν with $-1 < \nu < 1$. He is able, in this way, to produce approximations accurate to about six decimals. He does the same in [28] for Laguerre polynomials $L_n(x)$, $n \geq 7$, in the oscillatory region $0 \leq x \leq 4n+2$, where retouching is applied to two asymptotic formulae, one appropriate for the left tenth, the other for the remaining part, of the interval.

Retouching of sorts is taking place also in the paper [29], dedicated to the computation of all zeros of the generalized Laguerre polynomial $L_n^{(\alpha)}$, $\alpha > -1$. Classical results need to distinguish between zeros in three zones: a central zone and two lateral zones. Appropriate retouching of the asymptotic formula for the central zone gives rise to a unique procedure for computing *all* zeros. It involves the first $\lfloor n/2 \rfloor$ zeros of the Bessel function $J_\alpha(x)$ and of the Airy function $\mathrm{Ai}(-x)$.

5.2 Reversing asymptotic approximations

Hilb-type formulae such as (15) and their generalizations to ultraspherical and Jacobi polynomials are intended to approximate these polynomials in terms of Bessel functions, and likewise for the respective zeros. There is no intrinsic reason why this process cannot be turned around and thus be used to approximate Bessel functions in terms of, say, ultraspherical polynomials. This in fact is done in [30], where an improved Hilb formula for ultraspherical polynomials $P_n^{(\nu+1/2)}(\cos\theta)$, $\nu > -1/2$, is used to compute Bessel functions $J_\nu(x)$ in terms of them, the variable x being an appropriate multiple (depending on ν and n) of θ. Luigi's intention was to bridge in this way the gap of moderately large x, where neither the power series expansion of $J_\nu(x)$ (for small x) nor its asymptotic expansion (for large x) is numerically satisfactory. Strong competitors, however, are computational algorithms based on three-term recurrence relations satisfied by Bessel functions, which have been developed by one of us (W. G.) and others at just about the same time.

5.3 Bernstein-type inequalities

A well-known inequality for Legendre polynomials is Bernstein's inequality

$$(\sin\theta)^{1/2}|P_n(\cos\theta)| < (2/\pi)^{1/2}n^{-1/2}, \quad 0 \le \theta \le \pi, \tag{56}$$

where the constant $(2/\pi)^{1/2}$ is best possible. This result has been sharpened and generalized to ultraspherical polynomials by various authors. A generalization to Jacobi polynomials is due to Baratella [1]. By improving the constant in Baratella's result, Luigi jointly with Chow and Wong in [4] proves, for $|\alpha| \le 1/2$, $|\beta| \le 1/2$, that

$$\left(\sin\frac{1}{2}\theta\right)^{\alpha+1/2}\left(\cos\frac{1}{2}\theta\right)^{\beta+1/2}|P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{\Gamma(q+1)}{\Gamma(1/2)}\binom{n+q}{n}N^{-q-1/2}, \\ N = n+(\alpha+\beta+1)/2, \;\; 0 \le \theta \le \pi, \tag{57}$$

where $q = \max(\alpha, \beta)$. The numerical constant in (57) is best possible (cf. [52]).

5.4 Jacobi polynomials in the complex plane

In [6], Elliott obtained an asymptotic expansion for Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ which is valid uniformly for all z in the complex plane cut along the real axis from $-\infty$ to 1, with a neighborhood of $z = -1$ deleted, and with regard to the parameters α and β holds for arbitrary real β but only for $\alpha \ge 0$. From this expansion, Luigi in [3], together with Baratella, derives one- and two-term asymptotic approximations for $P_n^{(\alpha,\beta)}(z)$ with the same region of validity for z as stated above, and the same assumption on β, but with the restriction $\alpha \ge 0$ relaxed to $\alpha > -1$ by a judicious use of the differential equation and differentiation formulae satisfied by Jacobi polynomials. Analogous approximations that are valid in the z-plane cut along the real axis from -1 to $+\infty$, with a neighborhood of $z = 1$ deleted, can be obtained by switching α and β and using the reflection formula for Jacobi polynomials.

5.5 An expansion of Jacobi polynomials in Laguerre polynomials

In [40], for $\alpha > -1$, $\beta > -1$, the following curious expansion is derived,

$$P_n^{(\alpha,\beta)}(x) = \frac{(2k+t)^{n+\alpha+\beta+1}}{(2k)^{n+\alpha+1}(2k-t)^{\beta}}\, e^{-t}\sum_{m=0}^{\infty} A_m\left(k, \frac{\alpha+1}{2}\right)\left(\frac{t}{2k}\right)^m L_n^{(\alpha+m)}(t), \tag{58}$$

where

$$t = 2k\frac{1-x}{3+x}, \quad k = n+\beta+\frac{\alpha+1}{2}, \quad |t| < 2k,$$

and $A_m = A_m(k, \ell)$ satisfies the recurrence relation

$$(m+1)A_{m+1} = (m+2\ell-1)A_{m-1} - 2kA_{m-2}, \quad m = 2, 3, \dots,$$

with $A_0 = 1$, $A_1 = 0$, $A_2 = \ell$. The condition $|t| < 2k$ translates into $x > -1$. In the special case $\beta = 0$, the expansion is due to Tricomi; see [58, Eq. (26)] as corrected in [60, p. 98].

5.6 Surveys

On a number of occasions, Luigi has taken time out to survey recent progress he and others had made. In an early lecture, [24], beautifully written, he explains the nature of asymptotics, the need for error bounds and techniques to obtain them, for special functions as well as for their zeros, all carefully illustrated on the example of Legendre polynomials.[6]

In [35], work on asymptotic estimates for the zeros of Jacobi polynomials and Bessel functions is reviewed.[7] There are also many original results in this survey, for example a new application of (22) to obtain the following estimate for the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$, $|\alpha| \le 1/2$, $|\beta| \le 1/2$,

$$\theta_{n,r}^{(\alpha,\beta)} = \frac{j_{\alpha,r}}{\nu}\left[1 - \frac{4-\alpha^2-15\beta^2}{720\,\nu^4}\left(\frac{1}{2}\,j_{\alpha,r}^2 + \alpha^2 - 1\right)\right] + j_{\alpha,r}^5\;O\big(n^{-7}\big) \tag{59}$$

valid for $r = 1, 2, \dots, \lfloor \gamma n \rfloor$, with γ fixed in $0 < \gamma < 1$, and ν defined as in (21). Moreover, when r is fixed, (59) with the error term replaced by $O(n^{-7})$, is shown to hold for any $\alpha > -1$ and arbitrary real β. If solved for $j_{\alpha,r}$, it yields a good approximation for the first few zeros of the Bessel function J_α. The simplified $O(n^{-5})$ version of (59), with $r = 1$, has been found useful by one of us (W. G.) to discuss (in [51]) a conjectured inequality involving $\theta_{n,1}^{(\alpha,\beta)}$ and $\theta_{n+1,1}^{(\alpha,\beta)}$.

The final sections of [35] discuss inequalities holding between zeros of Jacobi polynomials and zeros of Bessel functions, some of which sharpening (26), and others extending (26), with the bounds switched, to[8] $|\alpha| > 1/2$, $|\beta| > 1/2$. In particular, many interesting and sharp upper and lower bounds are obtained for the first zero $j_{\alpha,1}$, and first few zeros $j_{\alpha,r}$, of the Bessel function J_α.

Asymptotic estimates and inequalities for the zeros $\lambda_{n,r}^{(\alpha)}$ of Laguerre polynomials $L_n^{(\alpha)}$ are reviewed in [44] and, here too, supplemented by new results.

[6]There are some misprints that may distract the reader: $\varepsilon_1(*)$ at the bottom of p. 88 should be $\varepsilon_1(x^*)$; on p. 89, second text line, x should read x^*; and in the displayed equation that follows, the first term on the left should be multiplied by δ.

[7]For unexplained reasons, the numbers in Table 1 differ somewhat from those in the corresponding table in [50, p. 85]. In the survey, plots for these numbers are also provided.

[8]In [35, Theorem 5.1 ii)], the inclusion sign $\in$ should be $\notin$.

Thus, e.g., the formula (45) is used to derive a very sharp and interesting estimate for the last few zeros of $L_n^{(\alpha)}$, namely, when s is fixed and $n \to \infty$,

$$\lambda_{n,n+1-s}^{(\alpha)} = \nu - 2^{2/3} a_s \nu^{1/3} + \frac{1}{5} 2^{4/3} a_s^2 \nu^{-1/3} + \left(\frac{11}{35} - \alpha^2 + \frac{12}{175} a_s^3\right) \nu^{-1} + \left(\frac{92}{7875} a_s^4 - \frac{16}{1575} a_s\right) 2^{2/3} \nu^{-5/3} + \left(\frac{15152}{3031875} a_s^5 - \frac{1088}{121275} a_s^2\right) 2^{1/3} \times \nu^{-7/3} + O\left(\nu^{-3}\right), \tag{60}$$

where $\nu = 4n + 2\alpha + 2$. This, in fact, improves an old $O(\nu^{-1})$ result of Tricomi. In obtaining (60), heavy use is made of Maple V. Luigi also conjectures that in the case $|\alpha| \leq 1/2$, when O-terms are omitted, the right-hand side of (44) becomes a lower bound for all $r = 1, 2, \ldots, n$, whereas the right-hand side of (45) becomes an upper bound for all, except the first few, zeros, and for all zeros if $-.4999 \leq \alpha \leq 1/2$.

The last three sections of [44] review results obtained by Luigi and others in the case where the parameter α is large compared to n, or both parameters α and n are large.

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