Computing Homology Cycles with Certified Geometry

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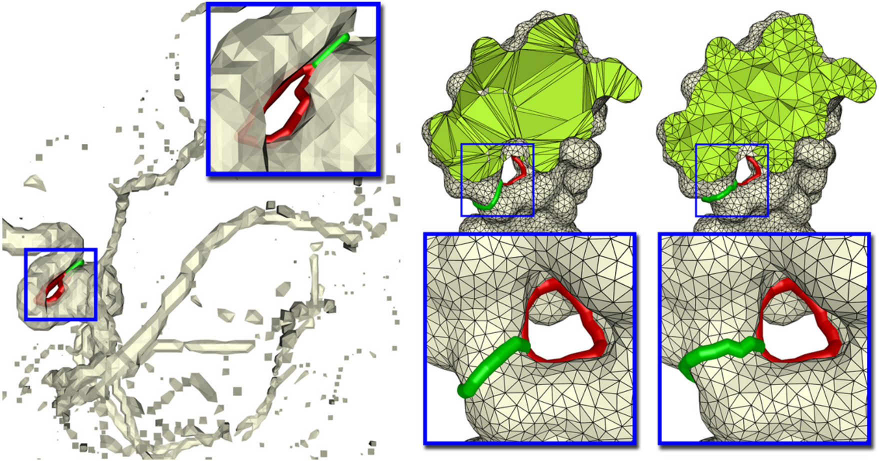
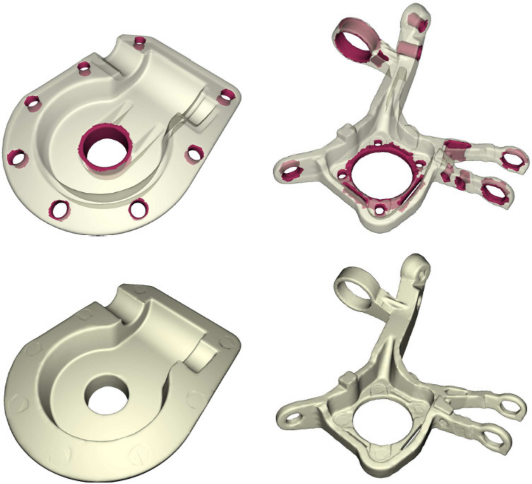
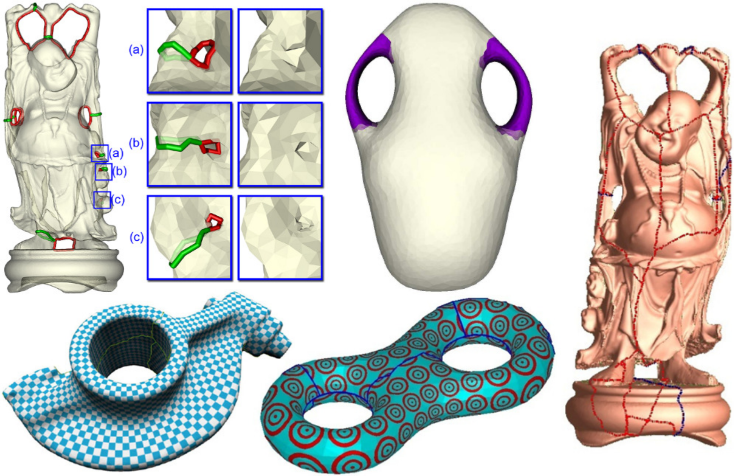
Cycles: Medical Imaging & Molecular Biology



Cycles: Computer-Aided Design



Cycles: Computer Graphics



Topological cycles: Homology

- Rank: Smith-Normal-Form; Special cases [DE95]
- Representative cycles:
  - Surfaces [VY90,DS95]
  - Volumes: [DG96]
  - General case: Persistence algorithm [ELZ00]
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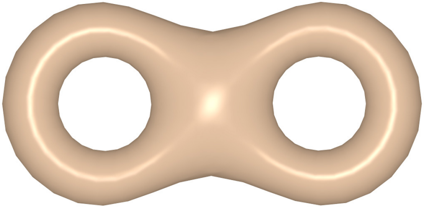

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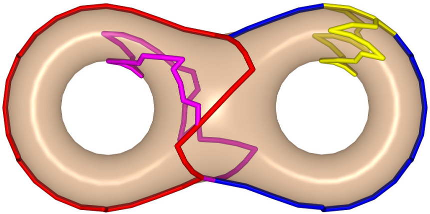

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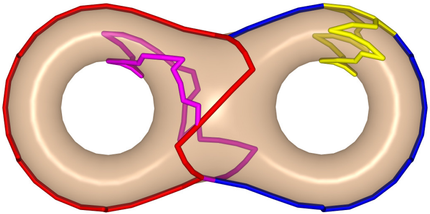

- Goal: ‘Geometry-oblivious’ to ‘Geometry-aware’

OHBP: Optimal Homology Basis Problem

- Compute an optimal set of cycles forming a basis

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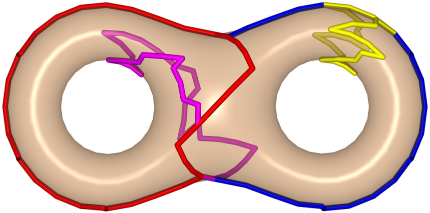

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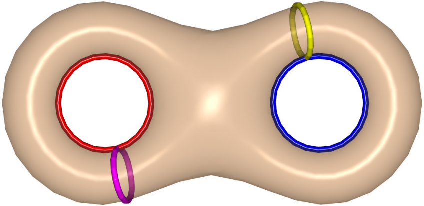

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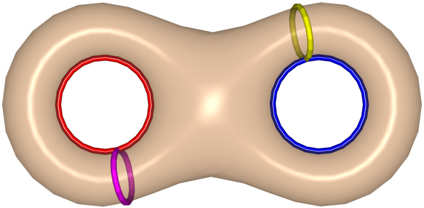

- First solution for surfaces: Erickson-Whittlesey [SODA05]
- General problem NP-hard: Chen-Freedman [SODA10]
- H_1 basis for simplicial complexes: D.-Sun-Wang [SoCG10]

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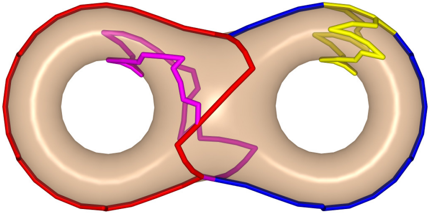

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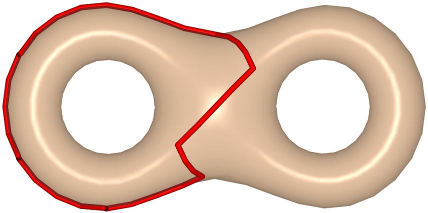

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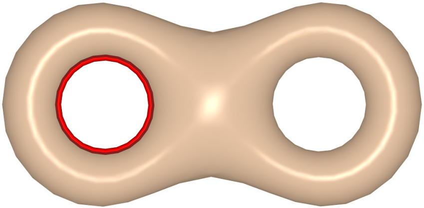

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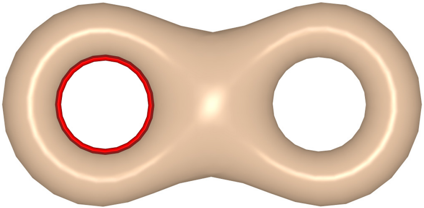

- Surfaces: Colin de Verdière-Lazarus [DCG05], Colin de Verdière-Erickson [SODA06], Chambers-Erickson-Nayyeri [SoCG09]
- General problem NP-hard: Chen-Freedman [SODA10]
- Special cases: D.-Hirani-Krishnamoorthy [STOC10]

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Definition

A smallest basis of $\mathrm{H}_p(\mathcal{T})$ is a set of p-cycles with minimal weight that generates $\mathrm{H}_p(\mathcal{T})$

Three results

- A polynomial time algorithm for computing a smallest basis of $H_1(\mathcal{K})$ for any weighted finite simplicial complex $\mathcal{K}$

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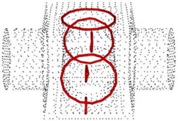
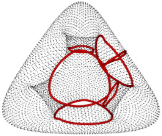
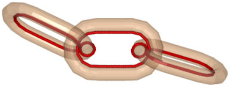
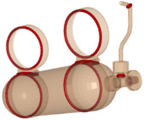

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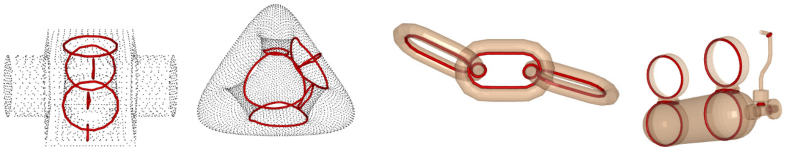

- Given a p-cycle in a simplicial complex, compute a smallest cycle in its class.

Theorem 1 [D.-Sun-Wang 10]

Theorem

Let $\mathcal{K}$ be a finite simplicial complex with non-negative weights on edges. A smallest basis for $\mathrm{H}_1(\mathcal{K})$ can be computed in $O(n^4)$ time where $n = |\mathcal{K}|$

Greedy Set

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The greedy set G is an *ordered* set of cycles $\{c_1, \ldots, c_k\}$ satisfying the following conditions:

(a) c_1 is the smallest cycle in $\mathcal{L}$ nontrivial in $\mathrm{H}_1(\mathcal{K})$

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- If a set of cycles $\mathcal{L}$ in $\mathcal{K}$ contains a smallest basis, then the *greedy* set G chosen from $\mathcal{L}$ is a smallest basis by matroid theory

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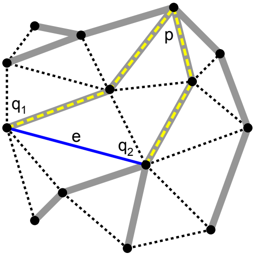

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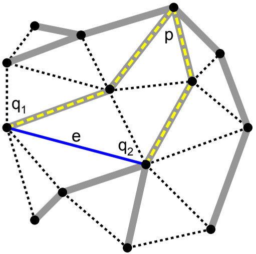

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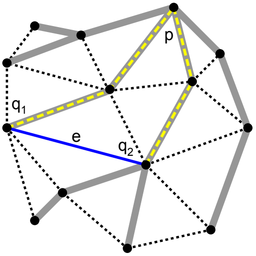

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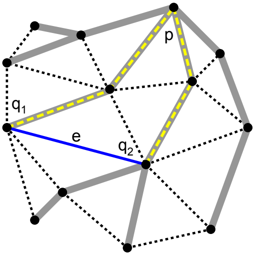

Definition

The canonical cycle for a non-tree edge e is defined as

$$T(e) = \mathrm{sp}_T(q_1, q_2) \circ e$$

Candidate Cycles

Proposition

Let C_p be the set of all canonical cycles with respect to p:

$$C_p = \{T(e) : e \in E \backslash E_T\}$$

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$\cup_{p \in P} G_p$ *from* $\cup_{p \in P} C_p$ *and a smallest basis from* $\cup_{p \in P} G_p$ *can be computed by persistence algorithm in time* $O(n^4)$.

Improved algorithm by simplex annotations

Definition

An annotation for p-simplices is a function $a : \mathcal{K}_p \rightarrow (\mathbb{Z}_2)^g$ where any two p-cycles are homologous iff $\Sigma_{\sigma \in c} a(\sigma) = \Sigma_{\sigma \in c'} a(\sigma)$.

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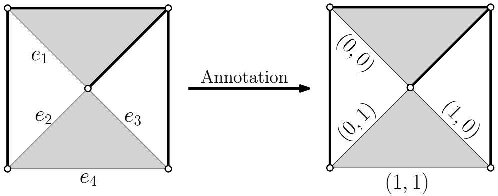

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Theorem (Busaryev-Cabello-Chen-D.-Wang 11)

For a simplicial complex with n simplices, annotations can be computed in $O(n^{\omega})$ time.

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Theorem (BCCDW 11)

For a simplicial complex $\mathcal{K}$ with n simplices, a smallest basis for $\mathrm{H}_1(\mathcal{K})$ can be computed in time $O(n^\omega + n^2 g^{\omega-1})$ using simplex annotations where g is the dimension of $\mathrm{H}_1(\mathcal{K})$.

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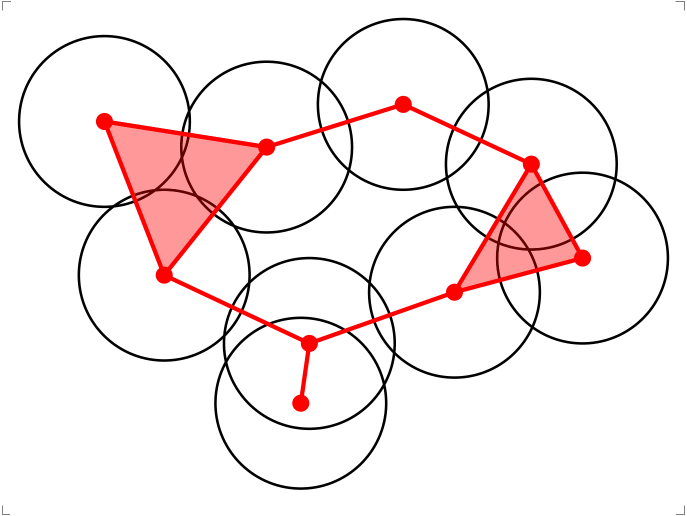
- Compute a *complex* $\mathcal{K}$ from P
- Compute a smallest basis of $\mathsf{H}_1(\mathcal{K})$
- Argue that if P is *dense*, a subset of computed 1-cycles approximate a smallest basis of $\mathsf{H}_1(\mathcal{M})$ within constant factors

Rips complex $\mathcal{R}^r(P)$



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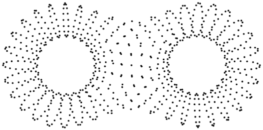

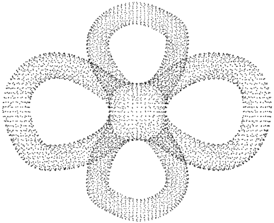

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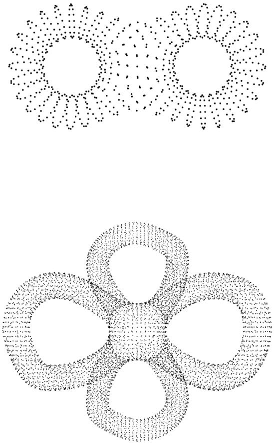

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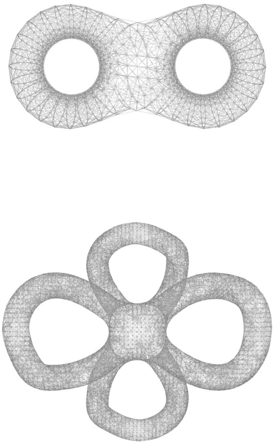

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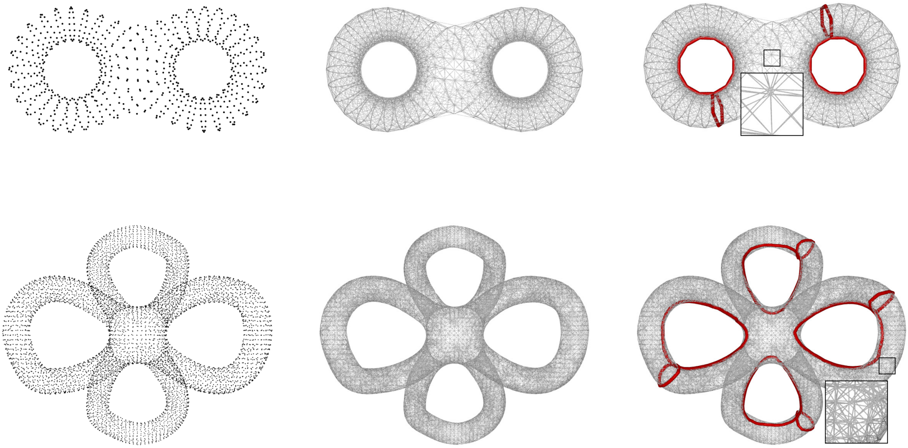

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Rips complex

1-cycles

Approximation Theorem [DSW10]

Theorem

Let $\mathcal{M} \subset \mathbb{R}^d$ be a smooth, closed manifold with ℓ as the length of a smallest basis of $\mathsf{H}_1(\mathcal{M})$ and $k = \operatorname{rank} \mathsf{H}_1(\mathcal{M})$. Given a set $P \subset \mathcal{M}$ of n points which is an ε-sample of $\mathcal{M}$ and $4\varepsilon \leq r \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(\mathcal{M}), \rho_c(\mathcal{M})\}$, one can compute a set of k 1-cycles G within $r/2$ Hausdorff distance of a basis of $\mathsf{H}_1(\mathcal{M})$ in $O(nn_e^2 n_t)$ time where

$$\frac{1}{1 + \frac{4r^2}{3\rho^2(\mathcal{M})}}\ell \leq \mathrm{Len}(\mathrm{G}) \leq (1 + \frac{4\varepsilon}{\mathrm{r}})\ell.$$

Here n_e, n_t are the number of edges and triangles in $\mathcal{R}^{2r}(P)$

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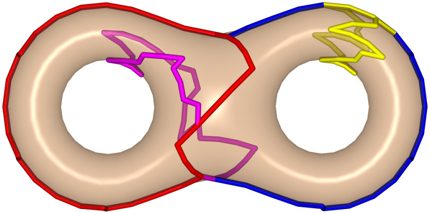

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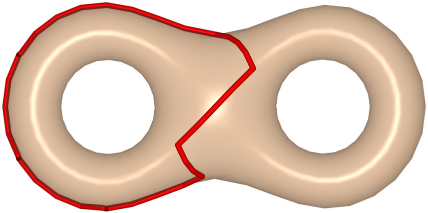

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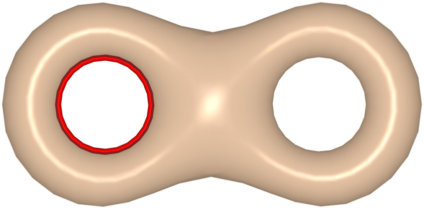

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- We characterize the complexes for which this is true
- For such complexes, the optimal cycle can be computed in polynomial time ☺

Total Unimodularity

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Theorem

Let A be an $m \times n$ totally unimodular matrix and $\mathbf{b}$ an integral vector, i.e. $\mathbf{b} \in \mathbb{Z}^m$. Then the polyhedron $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ is integral meaning that $\mathcal{P}$ is the convex hull of the integral vectors contained in $\mathcal{P}$. In particular, the extreme points (vertices) of $\mathcal{P}$ are integral. Similarly the polyhedron $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \geq \mathbf{b}\}$ is integral.

Optimization

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- Consider the *integer linear program*

Program

$$\begin{aligned} & \min \mathbf{f}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \\ \text{and} \quad & \mathbf{x} \in \mathbb{Z}^n. \end{aligned}$$

Optimization

- Consider an integral vector $\mathbf{b} \in \mathbb{Z}^m$ and a real vector $\mathbf{f} \in \mathbb{R}^n$.
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Corollary

Let A be a totally unimodular matrix. Then the integer linear program above can be solved in time polynomial in the dimensions of A.

Central Idea

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- Find the conditions under which the constraint matrix of the program is totally unimodular.
- For this class of problems, relax the integer linear program to a linear program by dropping the constraint that the variables be integral.

Optimization Program

- Assume that $\mathcal{K}$ contains m p-simplices and n $(p+1)$-simplices.

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Program

$$\begin{array}{rl} & \min ||W\mathbf{x}||_1 \\ \text{such that} & \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\mathbf{y} \\ \text{and} & \mathbf{x} \in \mathbb{Z}^m, \mathbf{y} \in \mathbb{Z}^n. \end{array}$$

Integer Linear Program

Program

$$\begin{aligned} & \min \sum_i |w_i|(x_i^+ + x_i^-) \\ \text{subject to} \quad & \mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\mathbf{y} \\ & \mathbf{x}^+, \mathbf{x}^- \geq 0 \\ & \mathbf{x}^+, \mathbf{x}^- \in \mathbb{Z}^m, \mathbf{y} \in \mathbb{Z}^n. \end{aligned}$$

Linear Program

Program

$$
\begin{aligned}
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Lemma

If $B = [\partial_{p+1}]$ is totally unimodular then so is $[I \ -I \ -B \ B]$.

Theorem

If the boundary matrix $[\partial_{p+1}]$ of a finite simplicial complex of dimension greater than p is totally unimodular, the optimal homologous chain problem for p-chain can be solved in polynomial time.

Orientable Manifolds

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For a finite simplicial complex triangulating a $(p+1)$-dimensional compact orientable manifold, $[\partial_{p+1}]$ is TU irrespective of the orientation.

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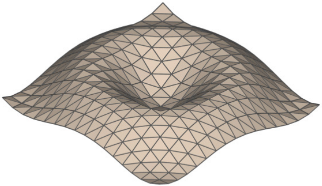

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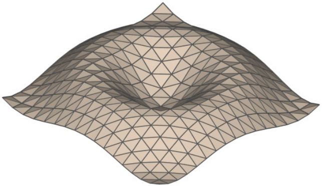

Corollary

For a finite simplicial complex triangulating a $(p+1)$-dimensional compact orientable manifold, OHCP can be solved for p-chains in polynomial time.

Total Unimodularity and Relative Torsion

Definitions

A pure simplicial complex of dimension p is a simplicial complex formed by a collection of p-simplices and their proper faces.
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Theorem (DHK10)

$[\partial_{p+1}]$ is totally unimodular if and only if $\mathrm{H}_p(\mathcal{L}, \mathcal{L}_0)$ is torsion-free, for all pure subcomplexes $\mathcal{L}_0, \mathcal{L}$ of $\mathcal{K}$ of dimensions p and $p+1$, respectively, where $\mathcal{L}_0 \subset \mathcal{L}$. Hence, OHCP for p-chains in such complexes are polynomial time solvable by linear programs.

A Special Case

Theorem

Let $\mathcal{K}$ be a finite simplicial complex embedded in $\mathbb{R}^{d+1}$. Then, $\mathrm{H}_d(\mathcal{L}, \mathcal{L}_0)$ is torsion-free for all pure subcomplexes $\mathcal{L}_0$ and $\mathcal{L}$ of dimensions d and $d+1$ respectively, such that $\mathcal{L}_0 \subset \mathcal{L}$.

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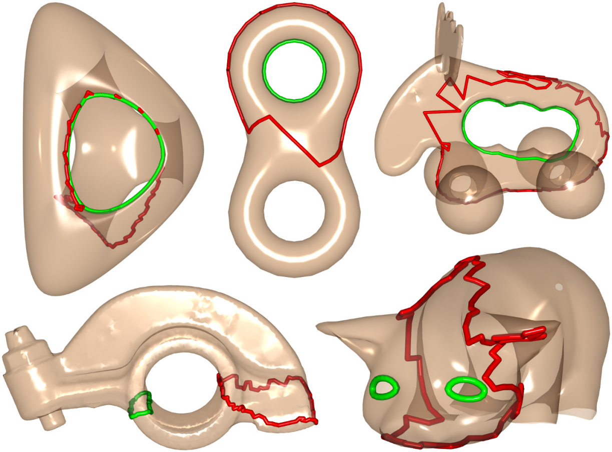
Corollary

Given a d-chain $\mathbf{c}$ in a weighted finite simplicial complex embedded in $\mathbb{R}^{d+1}$, an optimal chain homologous to $\mathbf{c}$ can be computed by a linear program.

Computed Optimal Cycles



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- What about efficient updates?

Thank You