Graph Induced Complex: A Data Sparsifier for Homology Inference

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The Ohio State University

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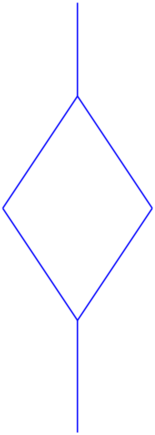
Space, Sample and Complex



Space

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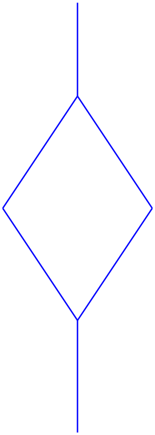

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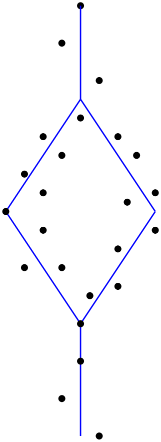
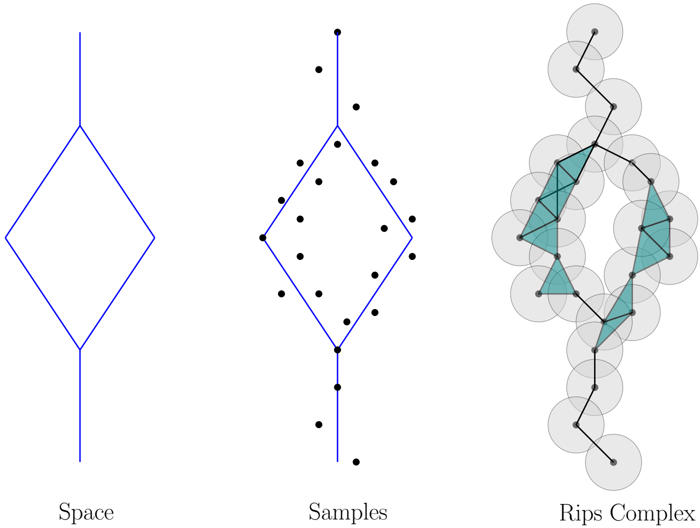


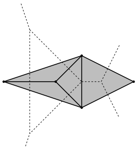
Samples

Space, Sample and Complex



Complexes

- Delaunay complex
  - difficult to compute in high dimensional spaces;



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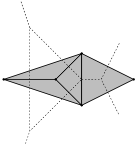
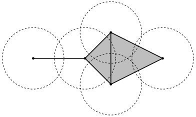


- Vietoris-Rips complex
  - too large (5000 points in $\mathbb{R}^3$ $\Rightarrow$ millions of simplicies);



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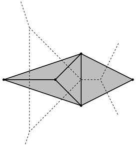

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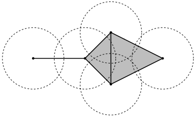

- Čech complex
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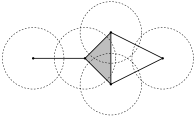
  - also large;



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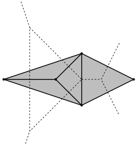

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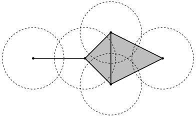

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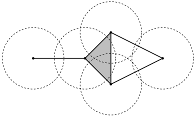
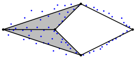


- Witness complex
  - manageable size;
  - lack topological inference;



Subsamples

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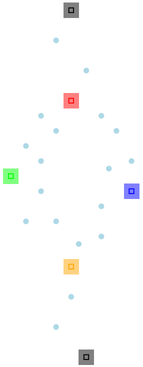
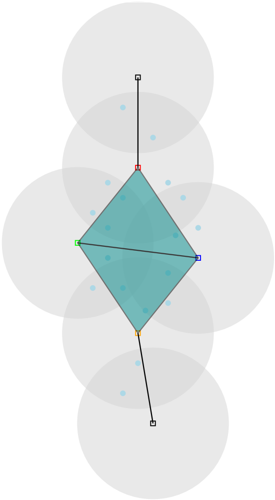


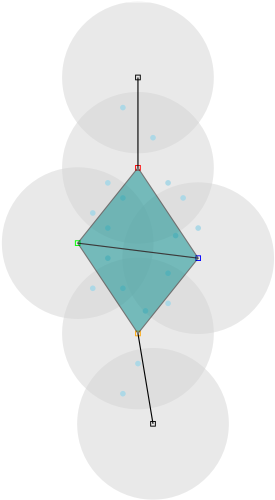
Q

Subsamples



$\mathcal{R}^{r'}(Q)$

Subsamples



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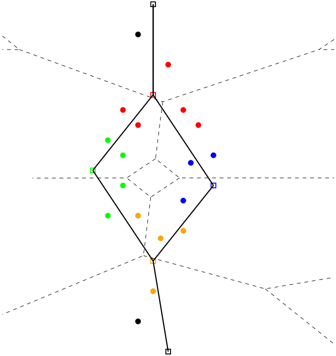

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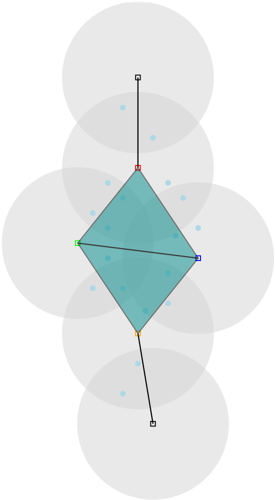

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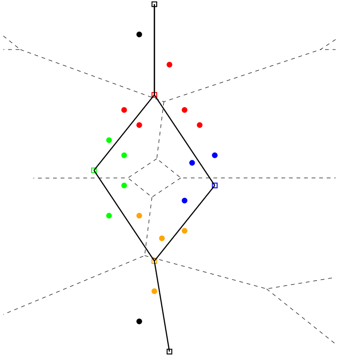

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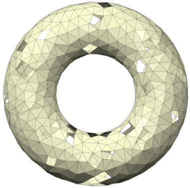
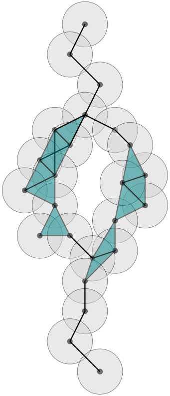


Our solution



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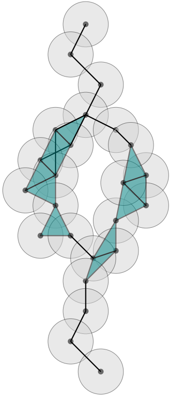

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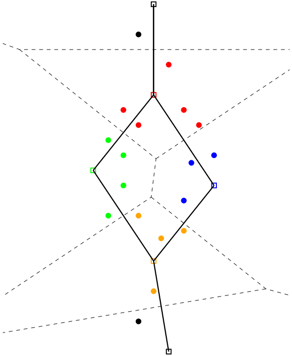
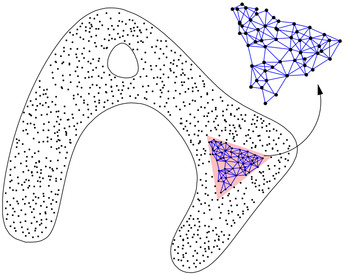


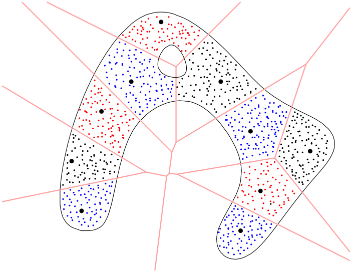
$\mathcal{G}^r(P, Q)$

Input Assumptions



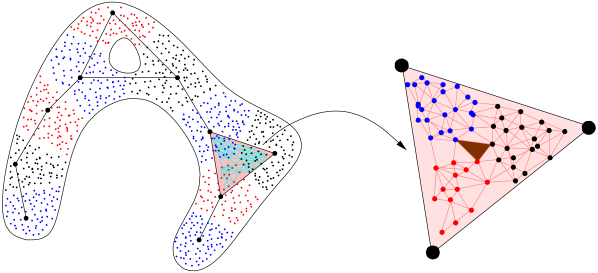
- P finite point set;
- (P, d) metric space;
- $G(P)$ be a graph;

Subsampling



- $Q \subset P$ a subset;
- $\nu(p)$: the closest point of $p \in P$ in Q;

Graph Induced Complex



- Graph induced complex $\mathcal{G}(P, Q, d)$: $\{q_1, \ldots, q_{k+1}\} \subseteq Q$;
  - a $(k+1)$-clique in $G(P)$ with vertices $p_1, \ldots, p_{k+1}$;
  - $\nu(p_i) = q_i$;

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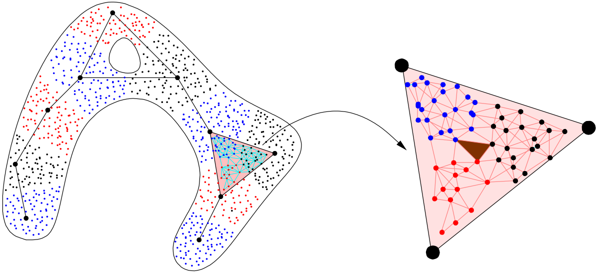

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Remark: $\mathcal{G}(P, Q, d)$ depends on the metric d;

- Euclidean distance d_E;
- Graph based distance d_G;

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  - $\forall p \in P$, $\exists q \in Q$ such that $d(p, q) \leq \delta$;

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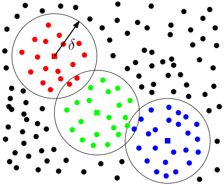

Q_i

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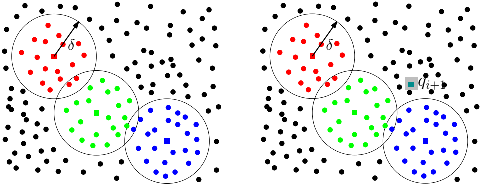

Q_i Q_{i+1}

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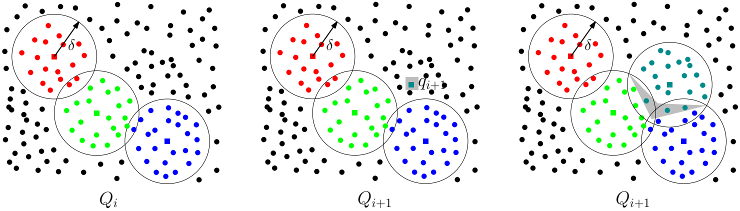

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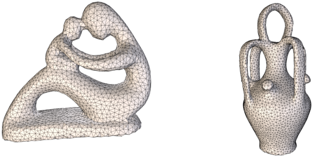

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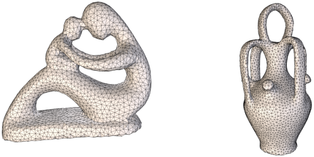
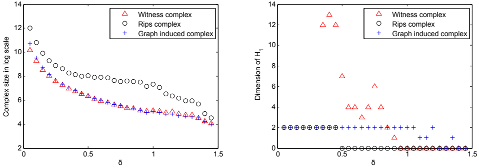


- Improved H_1 inference in $\mathbb{R}^n$ by d_G from a lean subsample ;

Klein bottle in $\mathbb{R}^4$



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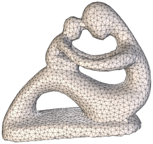

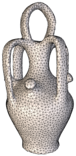

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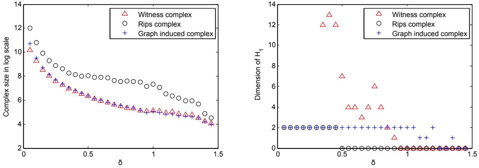

- Topological inference for compact sets in $\mathbb{R}^n$;

$$\mathcal{G}^{\alpha}(P, Q, d) \rightarrow \mathcal{G}^{4(\alpha+2\delta)}(P, Q', d)$$

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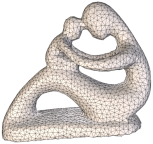

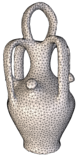

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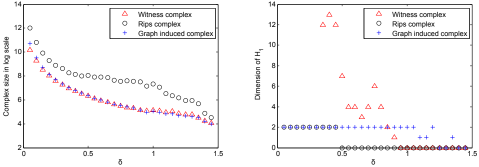

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Simplicial map

- $f : \mathcal{K} \to \mathcal{L}$ *simplicial map*
  - for every simplex $\sigma = \{v_1, v_2, \ldots, v_k\} \in \mathcal{K}$, $f(\sigma) = \{f(v_1), f(v_2), \ldots, f(v_k)\}$ is a simplex in $\mathcal{L}$

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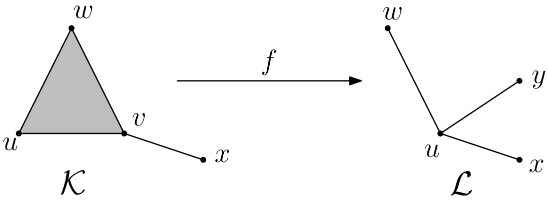

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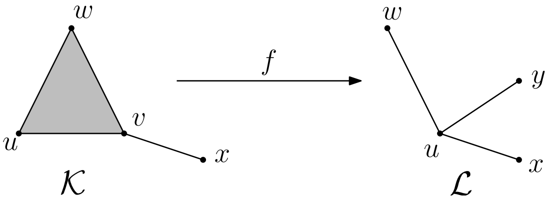

$[u]$	$\rightarrow$	$[u]$
$[v]$	$\rightarrow$	$[u]$
$[w]$	$\rightarrow$	$[w]$
$[x]$	$\rightarrow$	$[x]$

$[uv]$	$\rightarrow$	$[u]$
$[uw]$	$\rightarrow$	$[uw]$
$[vw]$	$\rightarrow$	$[uw]$
$[vx]$	$\rightarrow$	$[vx]$

$[uvw] \rightarrow [uw]$

Contiguous maps

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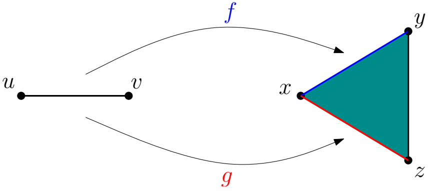

$f([v]) = [y]$
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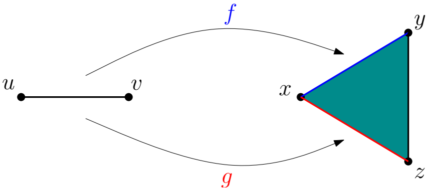

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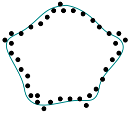
Fact

If $f : \mathcal{K}_1 \to \mathcal{K}_2$ and $g : \mathcal{K}_1 \to \mathcal{K}_2$ are contiguous, then the induced homomorphisms $f_* : \mathsf{H}_n(\mathcal{K}_1) \to \mathsf{H}_n(\mathcal{K}_2)$ and $g_* : \mathsf{H}_n(\mathcal{K}_1) \to \mathsf{H}_n(\mathcal{K}_2)$ are equal.

H_1 inference in $\mathbb{R}^n$

Sample P from a space $\mathcal{M}$;



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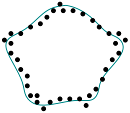
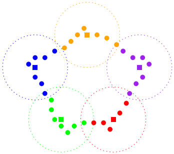


Subsample Q : δ-sample, δ-sparse;



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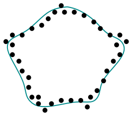

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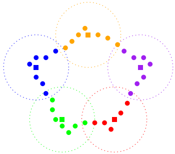
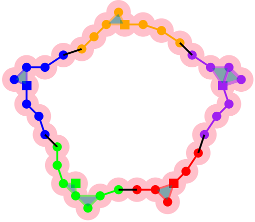


$G^\alpha(P)$ = 1-skeleton of $\mathcal{R}^\alpha(P)$



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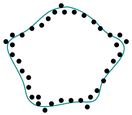

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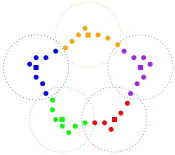

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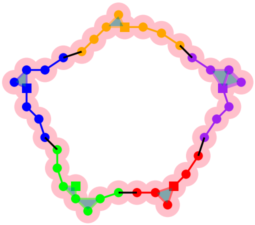

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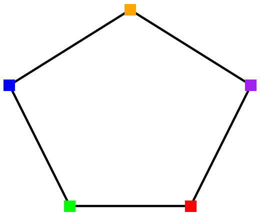

H_1 inference in $\mathbb{R}^n$

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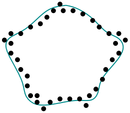

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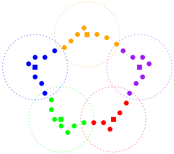

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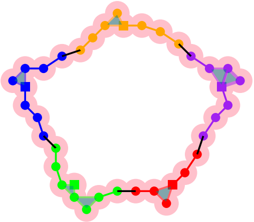

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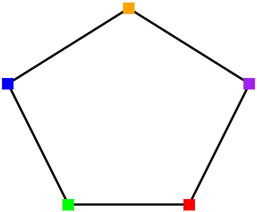

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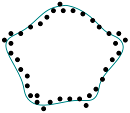

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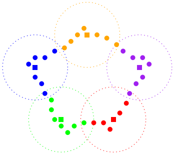

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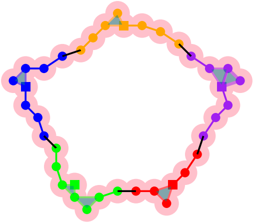

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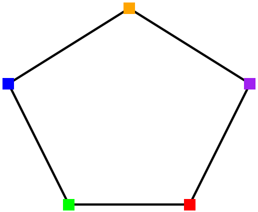
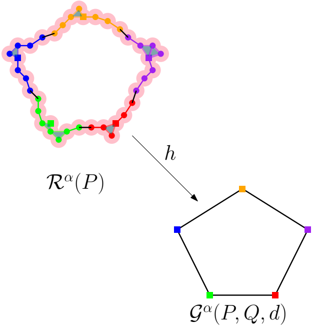


Contiguous maps between Rips and GIC



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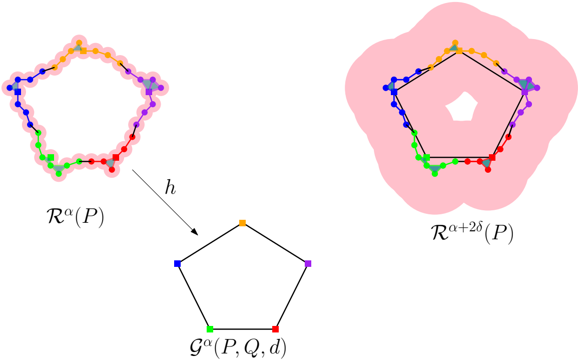

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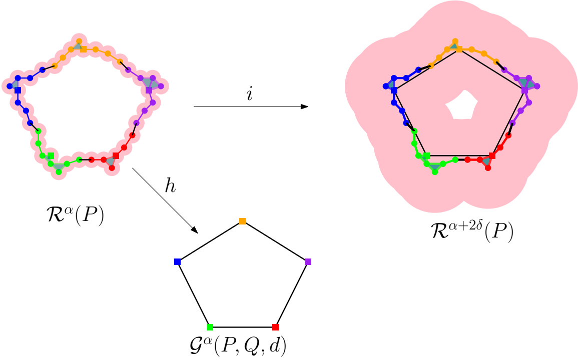

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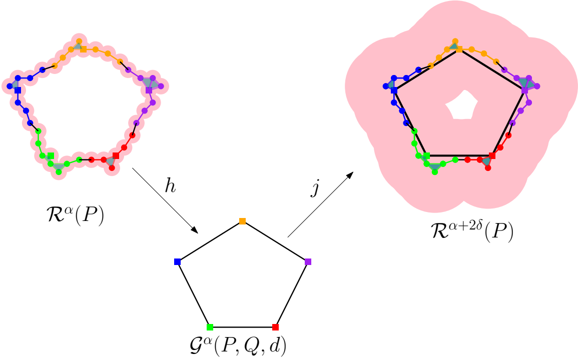

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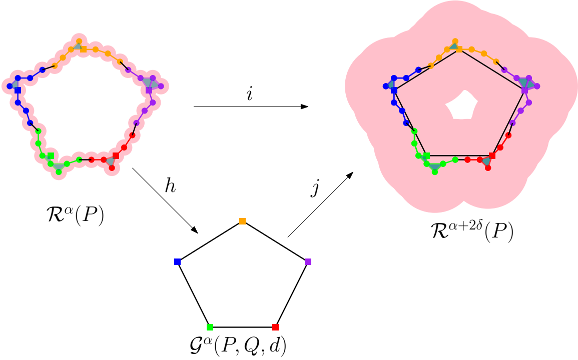

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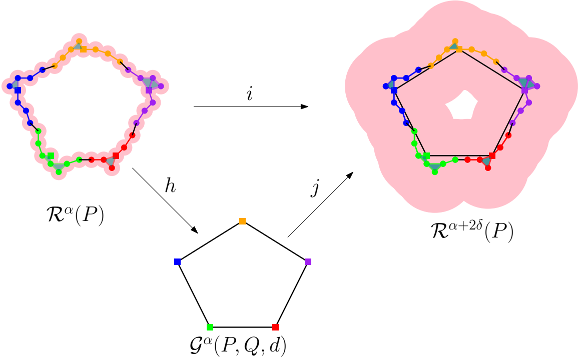

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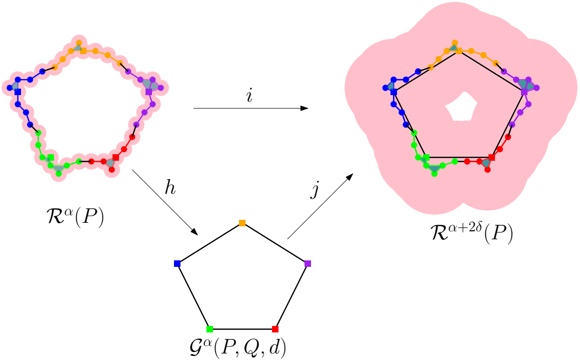

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$$(j \circ h)_* \ , \ i_* : \mathsf{H}(\mathcal{R}^\alpha(P)) \to \mathsf{H}(\mathcal{R}^{\alpha+2\delta}(P))$$

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$h_* : \mathsf{H}_1(\mathcal{R}^\alpha(P)) \to \mathsf{H}_1(\mathcal{G}^\alpha(P, Q))$ *isomorphism*

Isomorphism of h_*

Theorem

- P: an ϵ-sample of a surface with positive reach ρ in $\mathbb{R}^3$;
- Q: a δ-sparse δ-sample of (P, d_E);
- $\epsilon \leq \frac{1}{162}\rho$, $12\epsilon \leq \alpha \leq \frac{2}{27}\rho$, and $8\epsilon \leq \delta \leq \frac{2}{27}\rho$;

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Improved H_1 Inference

- *Homological loop feature size* for simplicial complex $\mathcal{K}$
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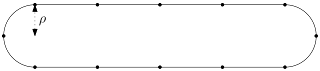

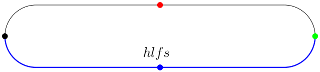

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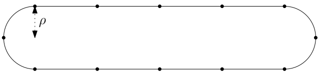
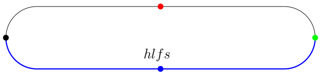
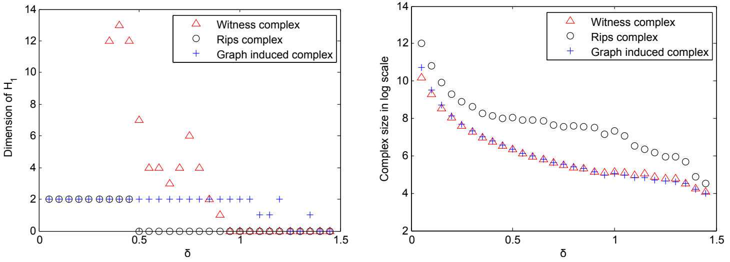




Theorem

If Q is a δ-sample of (P, d_G) for $\delta < \frac{1}{2}\mathrm{hlfs}(\mathcal{R}^\alpha(P)) - \frac{1}{2}\alpha$, then $h_* : \mathsf{H}_1(\mathcal{R}^\alpha(P)) \to \mathsf{H}_1(\mathcal{G}^\alpha(P, Q, d_G))$ is an isomorphism.

Experimental Result for Improved H_1 Inference

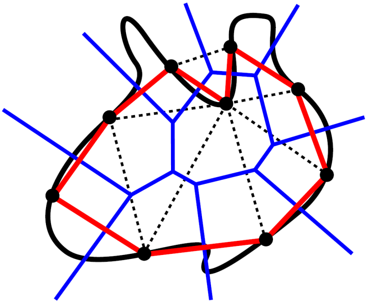


(a) (b)

- Klein bottle in $\mathbb{R}^4$ with $|P| = 40,000$;
- GIC size : 154 $(\delta = 1.0)$

Surface Reconstruction

- Crust [AB99] and Cocone [ACDL00]:
  - Compute a subcomplex $T \subset \mathrm{Del}\, P$;
  - Argue T contains the restricted Delaunay triangulation $\mathrm{Del}|_M\, P$;



  - Prune T to output a 2-manifold;

Surface Reconstruction by GIC in $\mathbb{R}^3$

- Restricted Delaunay triangulation $\mathrm{Del}|_M\, Q \subset \mathcal{G}(P, Q, d_E)$;

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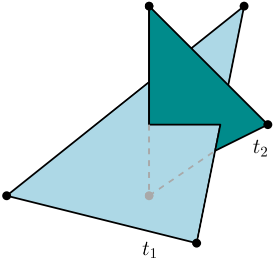

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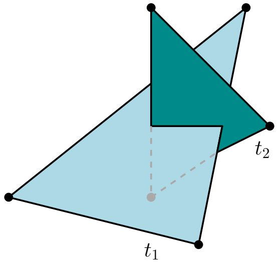


Cleaning

If V is the vertex set of t_1 and t_2 together, then at least one of t_1 and t_2 is not in $\mathrm{Del}\, V$. The triangle which is not in $\mathrm{Del}\, V$ cannot be in $\mathrm{Del}\, Q$ as well.

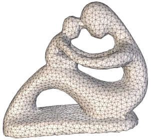
Theorem

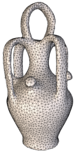
- P: ε-sample
- Q: δ-sparse, δ-sample of P
- $8\varepsilon \leq \delta \leq \frac{2}{27}\rho$, $\alpha \geq 8\varepsilon$

A triangulation $T \subseteq G^{\alpha}(P, Q, d_E)$ can be computed.

	FERTILITY		BOTIJO	
	mesh	GIC	mesh	GIC
0-dim	3007	3007	4659	4659
1-dim	9039	9817	14001	14709
2-dim	6026	6304	9334	10755
3-dim		139		718

- $|P| = 1,575,055$ for FERTILITY;
- $|P| = 1,049,892$ for BOTIJO;





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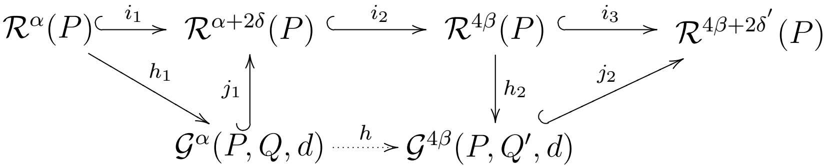

- Q δ-sparse δ-sample of P;
- Q' δ'-sparse δ'-sample of P with $\delta' > \delta$;
- Denote $\beta = \alpha + 2\delta$;

GIC for Compact Sets

- Above diagram gives sequence

$$\mathsf{H}_k(\mathcal{R}^{\alpha}(P)) \xrightarrow{h_{1*}} \mathsf{H}_k(\mathcal{G}^{\alpha}(P,Q,d)) \xrightarrow{j_{1*}} \mathsf{H}_k(\mathcal{R}^{\alpha+2\delta}(P))$$

$$\xrightarrow{i_{2*}} \mathsf{H}_k(\mathcal{R}^{4\beta}(P)) \xrightarrow{h_{2*}} \mathsf{H}_k(\mathcal{G}^{4\beta}(P,Q',d)) \xrightarrow{j_{2*}} \mathsf{H}_k(\mathcal{R}^{4\beta+2\delta'}(P))$$

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Theorem

- *P: an ϵ-sample of a compact set (X, d_E);*
- *Q: a δ-sparse δ-sample of (P, d) $(d = d_E \text{ or } d_G)$;*
- *Q': a δ'-sparse δ'-sample of (P, d) $(\delta' > \delta)$;*
- $0 < \epsilon < \frac{1}{9}\mathrm{wfs}(X)$, $2\epsilon \le \alpha \le \frac{1}{4}(\mathrm{wfs}(X) - \epsilon)$ *and* $(\alpha + 2\delta) + \frac{1}{2}\delta' \le \frac{1}{4}(\mathrm{wfs}(X) - \epsilon)$,

$$\Rightarrow \ \mathrm{im}\ h_* \cong \mathsf{H}_k(X^{\lambda}) \ (0 < \lambda < \mathrm{wfs}(X))$$

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  - ▷ Potential use in topological data analysis;

THANKS!