Computational Topology in Reconstruction, Mesh Generation, and Data Analysis

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Outline

- Topological concepts:

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  - Topological spaces

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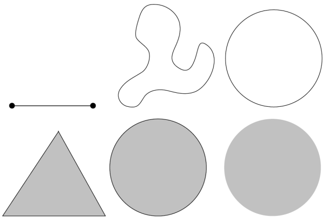
  - Topological data analysis

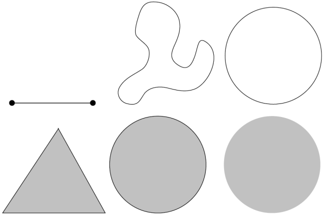
Topologcal spaces

- A point set with open subsets closed under union and finite intersections



Topologcal spaces

- A point set with open subsets closed under union and finite intersections
- d-ball B^d $\{x \in \mathbb{R}^d \mid ||x|| \leq 1\}$
- d-sphere S^d $\{x \in \mathbb{R}^d \mid ||x|| = 1\}$

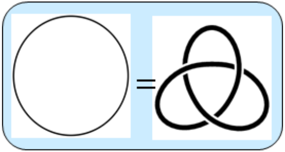


- k-manifold: neighborhoods 'homeomorphic' to open k-ball
  - 2-sphere, torus, double torus are 2-manifolds

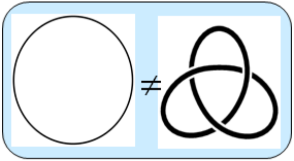
Maps

- Homeomorphism $h : T_1 \rightarrow T_2$ where h is continuous, bijective and has continuous inverse
- Isotopy : continuous deformation that maintains homeomorphism
- homotopy equivalence: map linked to continuous deformation only

Homeomorphic

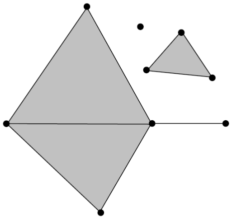


No isotopy



Simplicial complex

- Abstract
  - $V(K)$: vertex set, k-simplex: $(k+1)$-subset $\sigma \subseteq V(K)$
  - Complex
    $K = \{\sigma \| \; \sigma' \subseteq \sigma \Longrightarrow \sigma' \in K\}$



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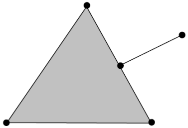

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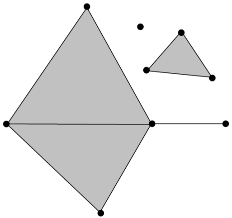
- Geometric
  - k-simplex: $k+1$-point convex hull
  - Complex K:
    - (i) $t \in K$ if t is a face of $t' \in K$
    - (ii) $t_1, t_2 \in K \Rightarrow t_1 \cap t_2$ is a face of both



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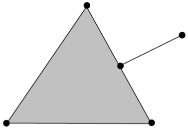

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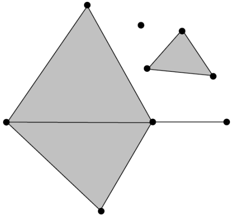
- Triangulation: K is a triangulation of a topological space T if $T \approx |K|$



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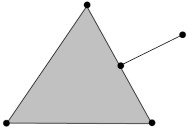

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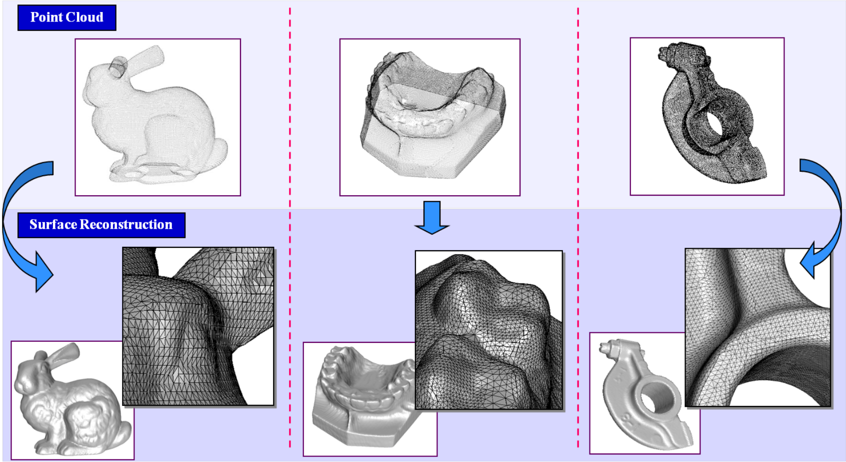
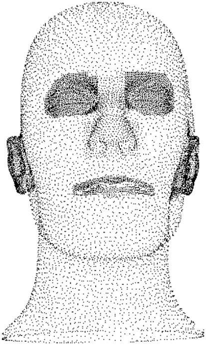
Surface Reconstruction

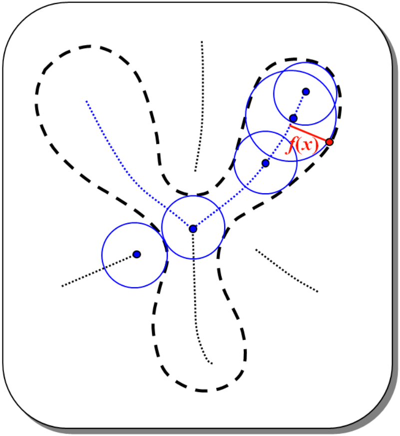


Sampling

- Sample $P \subset \Sigma \subset \mathbb{R}^3$

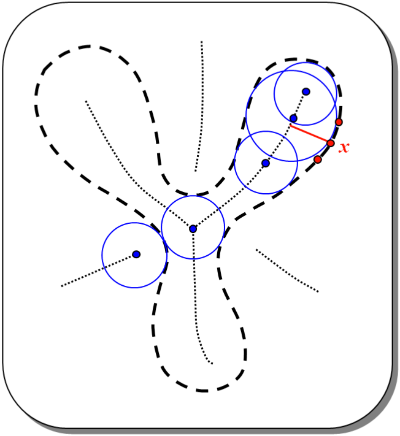


Local Feature Size



- $\mathrm{Lfs}(x)$ is the distance to medial axis

ε-sample (Amenta-Bern-Eppstein 98)



- Each x has a sample within $\varepsilon \mathrm{Lfs}(x)$ distance

Crust and Cocone Guarantees

Theorem (Crust: Amenta-Bern 1999)

Any point $x \in \Sigma$ is within $O(\varepsilon)\mathrm{Lfs}(x)$ distance from a point in the output. Conversely, any point of the output surface has a point $x \in \Sigma$ within $O(\varepsilon)\mathrm{Lfs}(x)$ distance for $\varepsilon < 0.06$.

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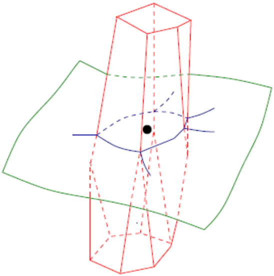
Theorem (Cocone: Amenta-Choi-Dey-Leekha 2000)

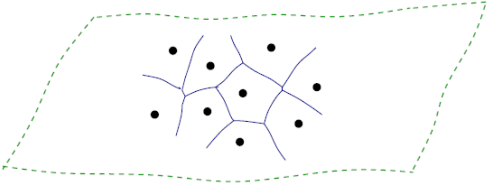
The output surface computed by $\textsc{Cocone}$ *from an* $\varepsilon -$ *sample is homeomorphic to the sampled surface for* $\varepsilon < 0.06$.

Restricted Voronoi/Delaunay

Definition

Restricted Voronoi: $\mathrm{Vor}\, P|_\Sigma$: Intersection of $\mathrm{Vor}\,(P)$ with the surface/manifold Σ.

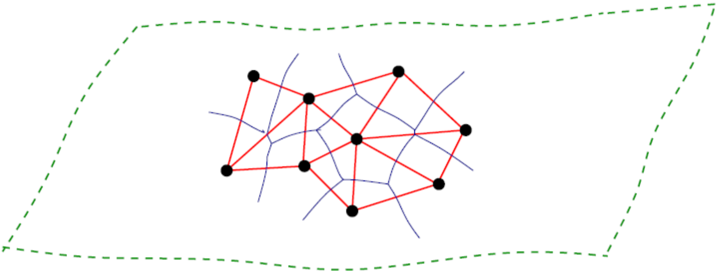




Restricted Voronoi/Delaunay

Definition

Restricted Delaunay: $\mathrm{Del}\, P|_\Sigma$: dual of $\mathrm{Vor}\, P|_\Sigma$



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Closed Ball property (Edelsbrunner, Shah 94)

If restricted Voronoi cell is a closed ball in each dimension, then $\mathrm{Del}\, P|_{\Sigma}$ *is homeomorphic to* Σ.

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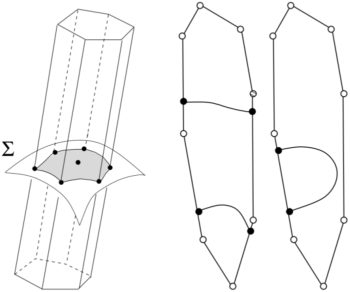

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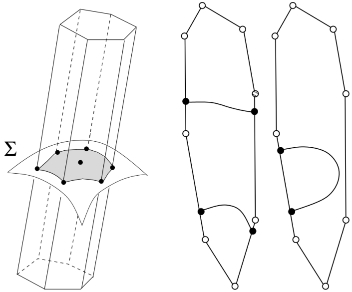
Theorem

For a sufficiently small ε *if* P *is an* ε*-sample of* Σ*, then* (P, Σ) *satisfies the closed ball property, and hence* $\mathrm{Del}\, P|_{\Sigma} \approx \Sigma$.



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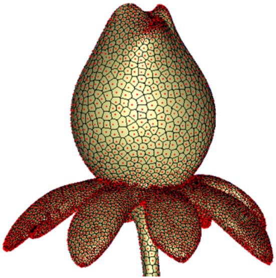

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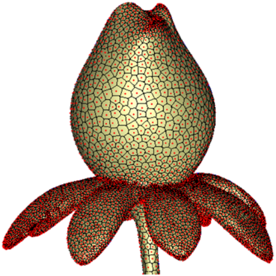
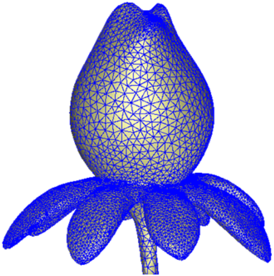
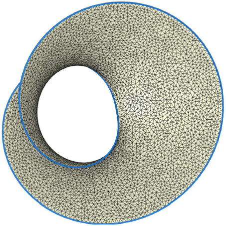




Boundaries



- Ambiguity in reconstruction
- Non-homeomorphic Restricted Delaunay [DLRW09]
- Non-orientabilty

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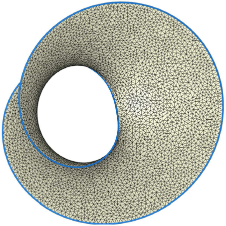

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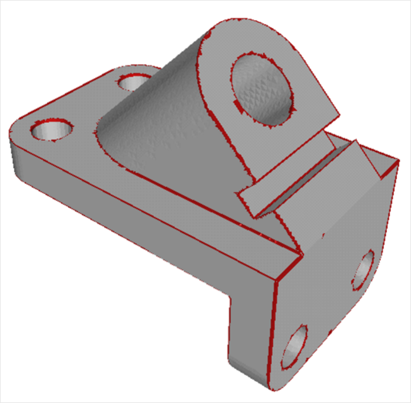
Theorem (D.-Li-Ramos-Wenger 2009)

Given a sufficiently dense sample of a smooth compact surface Σ with boundary one can compute a Delaunay mesh isotopic to Σ.

Open: Reconstructing nonsmooth surfaces

- Guarantee of homeomorphism is open



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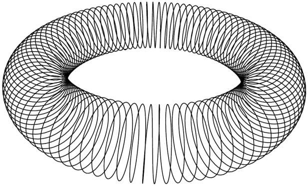

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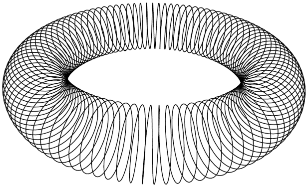

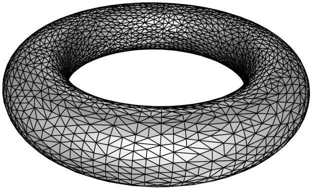

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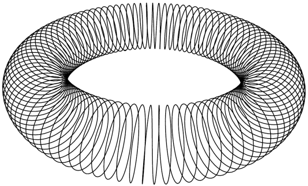

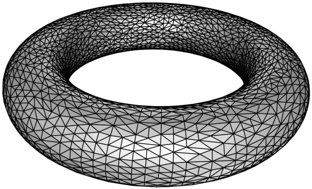

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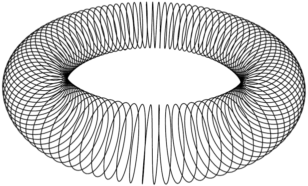
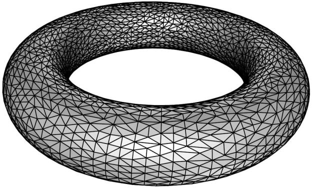
  - Slivers are arbitrarily oriented [CDR05] $\Rightarrow \mathrm{Del}\, P|_\Sigma \not\approx \Sigma$ no matter how dense P is.





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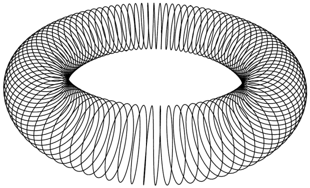
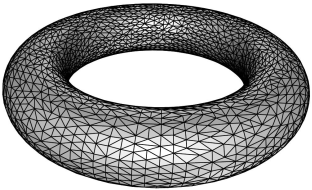
- Delaunay triangulation becomes harder





Reconstruction

Theorem (Cheng-Dey-Ramos 2005)

Given an (ε, δ)-sample P of a smooth manifold $\Sigma \subset \mathbb{R}^d$ for appropriate $\varepsilon, \delta > 0$, there is a weight assignment of P so that $\mathrm{Del}\, \hat{P}|_{\Sigma} \approx \Sigma$ which can be computed efficiently.

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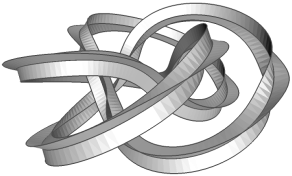
Theorem (Chazal-Lieutier 2006)

Given an ε-noisy sample P of manifold $\Sigma \subset \mathbb{R}^d$, there exists $r_p \leq \rho(\Sigma)$ for each $p \in P$ so that the union of balls $B(p, r_p)$ is homotopy equivalent to Σ.

Reconstructing Compacts



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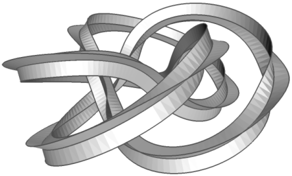

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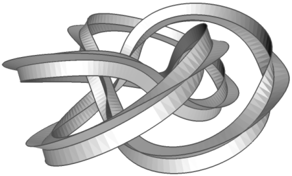

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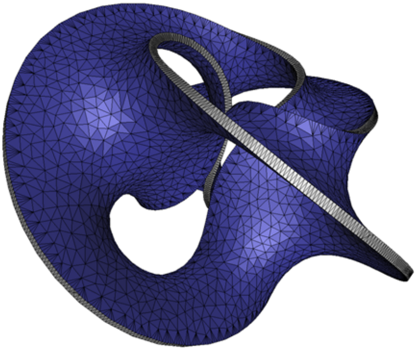
Theorem (Chazal-Cohen-S.-Lieutier 2006)

Given an (ε, μ)-sample P of a compact $K \subset \mathbb{R}^d$ for appropriate $\varepsilon, \mu > 0$, there is an α so that union of balls $B(p, \alpha)$ is homotopy equivalent to K^η for arbitrarily small η.

Surface and volume mesh



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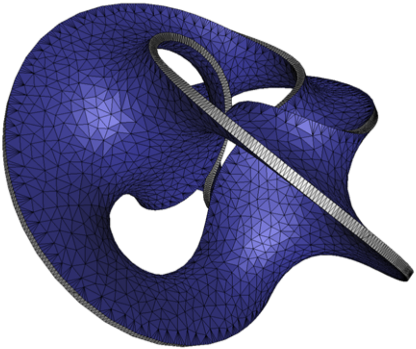

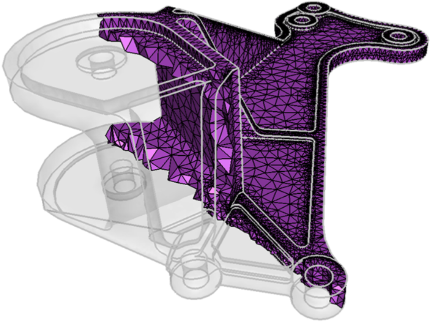

Delaunay refinement

- Pioneered by Chew89,Ruppert92,Shewchuk98

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- Burden is to show termination (by packing argument)

Sampling Theorem

Theorem (Amenta-Bern 98, Cheng-D.-Edelsbrunner-Sullivan 01)

If $P \subset$ Sigma is a discrete ε-sample of a smooth surface Σ, then for $\varepsilon < 0.09$, $\mathrm{Del}\, P|_\Sigma$ satisfies:

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- *Hausdorff distance between Σ and $\mathrm{Del}\, P|_\Sigma$ is $O(\varepsilon^2)$ of LFS.*

Sampling Theorem Modified

Theorem (Boissonnat-Oudot 05)

If $P \in \Sigma$ is such that each Voronoi edge-surface intersection x lies within $\varepsilon \mathrm{Lfs}(x)$ from a sample, then for $\varepsilon < 0.09$, $\mathrm{Del}\, P|_\Sigma$ satisfies:

- *It is homeomorphic to Σ*
- *Each triangle has normal aligning within $O(\varepsilon)$ angle to the surface normals*
- *Hausdorff distance between Σ and $\mathrm{Del}\, P|_\Sigma$ is $O(\varepsilon^2)$ of LFS.*

Basic Delaunay Refinement

1. Initialize points $P \subset \Sigma$, compute $\mathrm{Del}\, P$
2. If some **condition** is not satisfied, insert a point $c \in \Sigma$ into P and repeat
3. Return $\mathrm{Del}\, P|_{\Sigma}$

Surface Delaunay Refinement

1. Initialize points $P \subset \Sigma$, compute $\mathrm{Del}\, P$
2. If some Voronoi edge intersects Σ at x with $d(x, P) > \varepsilon LFS(x)$, insert x in P and repeat
3. Return $\mathrm{Del}\, P|_\Sigma$

Difficulty

- How to compute $\mathrm{Lfs}(x)$?
- Can be approximated by computing approximatemedial axis–needs a dense sample.

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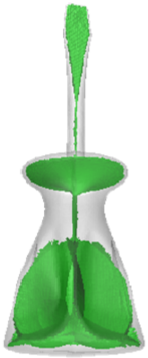

A Solution

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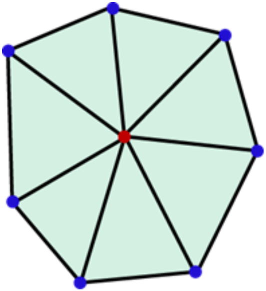

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1. Initialize points $P \subset \Sigma$, compute $\mathrm{Del}\, P$
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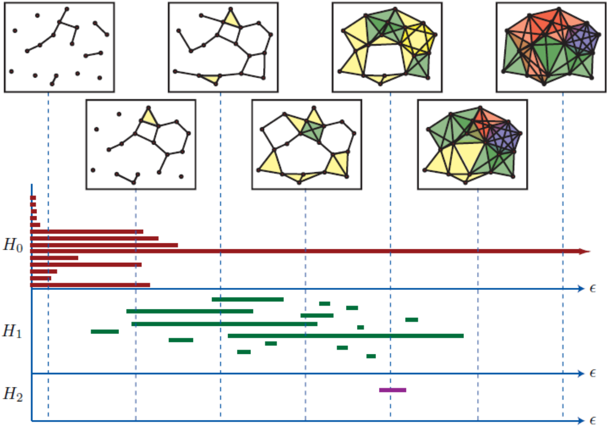
    3. *Hausdorff distance between Σ and $\mathrm{Del}\, P|_{\Sigma}$ is $O(\lambda^2)$ of LFS.*

Data Analysis by Persistent Homology

- Persistent homology [Edelsbrunner-Letscher-Zomorodian 00], [Zomorodian-Carlsson 02]



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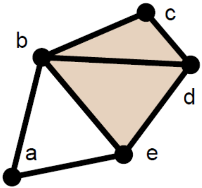

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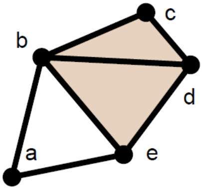

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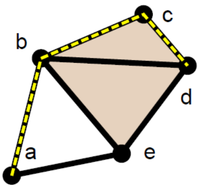

1-chain $ab + bc + cd$ $(a_i \in \mathbb{Z}_2)$

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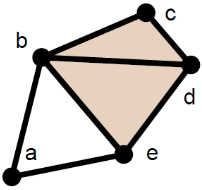

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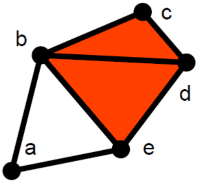

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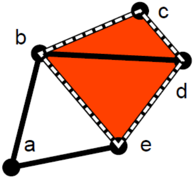

1-boundary $bc+cd+db+bd+de+eb = bc+cd+de+eb = \partial_2(bcd+bde)$ (under $\mathbb{Z}_2$)

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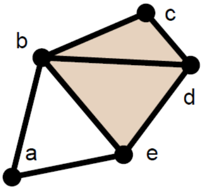

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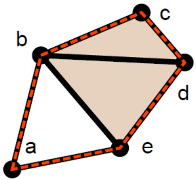

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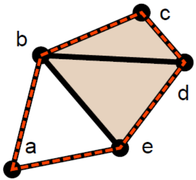

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- Each p-boundary is a p-cycle: $\partial_p \circ \partial_{p+1} = 0$

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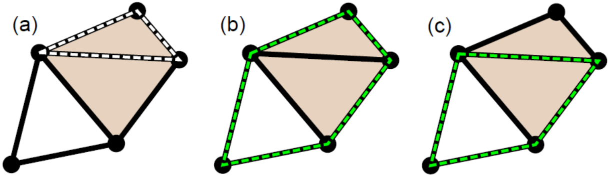

(a) trivial (null-homologous) cycle; (b), (c) nontrivial homologous cycles

Complexes

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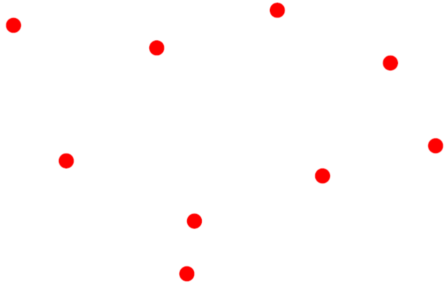
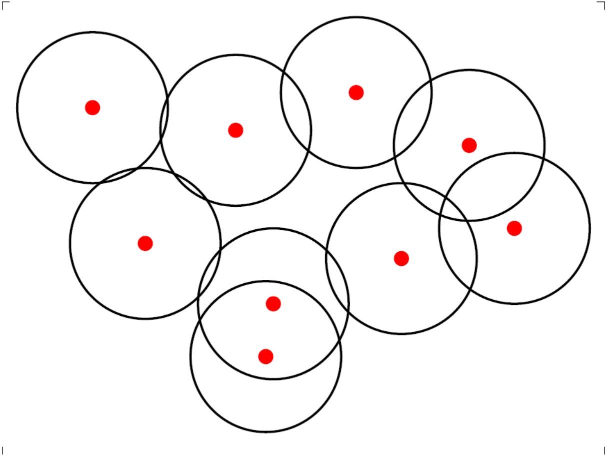
Proposition

For any finite set $P \subset \mathbb{R}^d$ and any $r \geq 0$, $\mathcal{C}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \mathcal{C}^{2r}(P)$

Point set P



Balls $B(p, r/2)$ for $p \in P$



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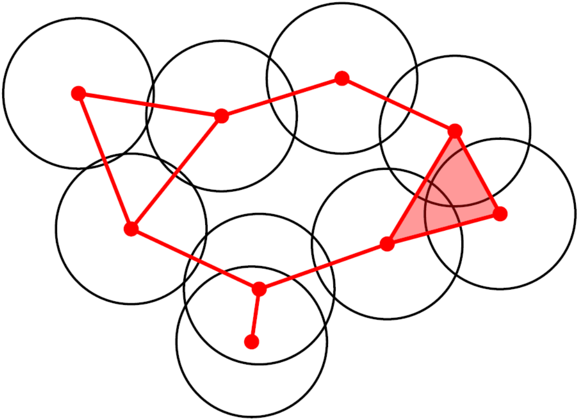
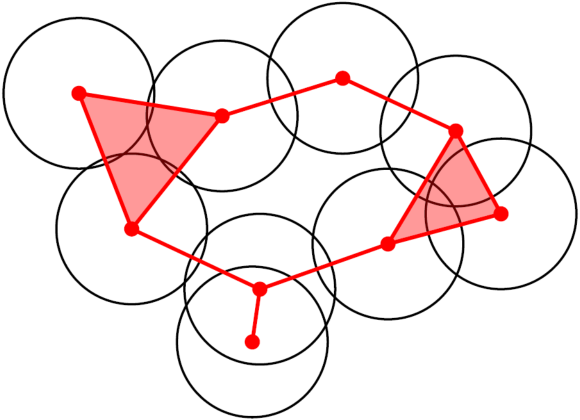
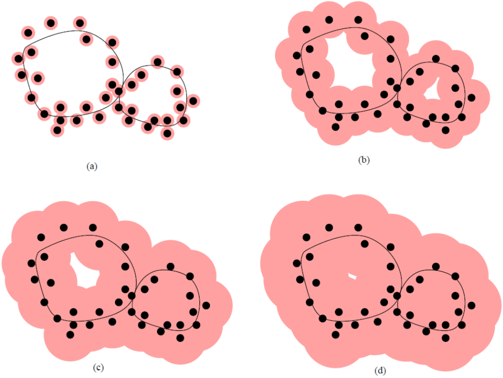


Rips complex $\mathcal{R}^r(P)$

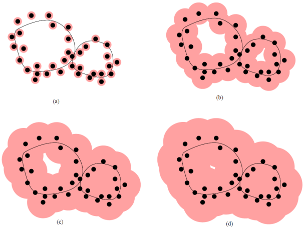


Topological persistence

- $r(x) = d(x, P)$: distance to point cloud P
- Sublevel sets $r^{-1}[0, a]$ are union of balls
- Evolution of the sublevel sets with increasing a–left hole persists longer
- Persistent homology quantizes this idea



Persistent Homology



- $f : \mathbb{T} \to \mathbb{R}$; $\mathbb{T}_a = f^{-1}(-\infty, a]$, the sublevel set
- $\mathbb{T}_a \subseteq \mathbb{T}_b$ for $a \leq b$ provides inclusion map $\iota : \mathbb{T}_a \to \mathbb{T}_b$
- Induced map $\iota_* : \mathrm{H}_p(\mathbb{T}_a) \to \mathrm{H}_p(\mathbb{T}_b)$ giving the sequence

$$0 \to \mathrm{H}_p(\mathbb{T}_{a_1}) \to \mathrm{H}_p(\mathbb{T}_{a_2}) \to \cdots \to \mathrm{H}_p(\mathbb{T}_{a_n}) \to \mathrm{H}_p(\mathbb{T})$$

- Persistent homology classes: Image of $f_p^{ij} : \mathrm{H}_p(\mathbb{T}_{a_i}) \to \mathrm{H}_p(\mathbb{T}_{a_j})$

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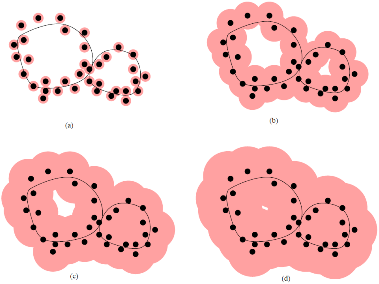

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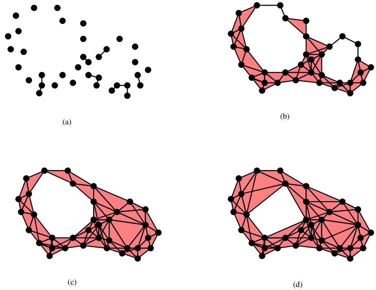

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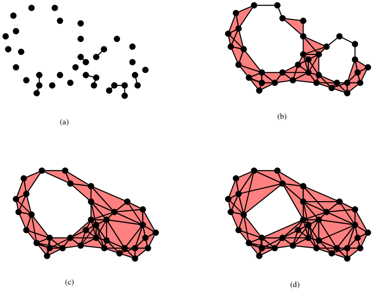
(a) (b) (c) (d)

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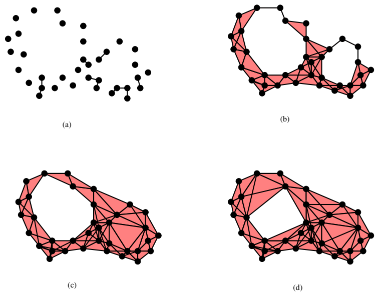

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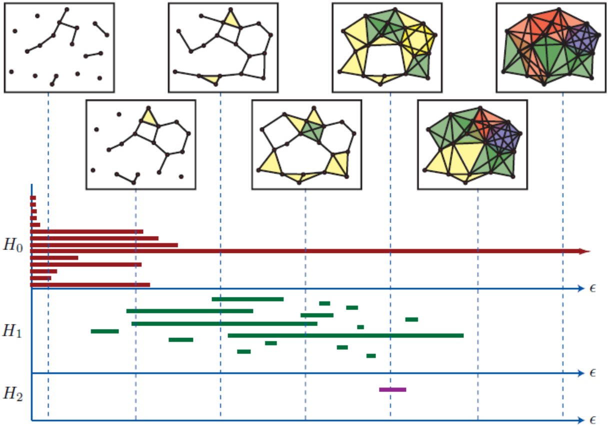
- Birth and Death of homology classes

Bar Codes

- birth-death and bar codes



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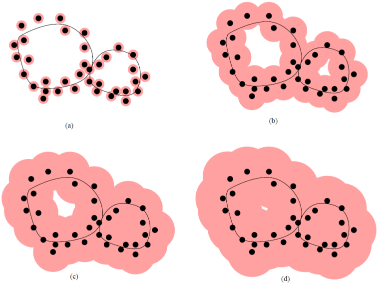

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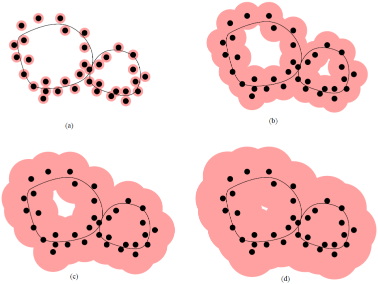
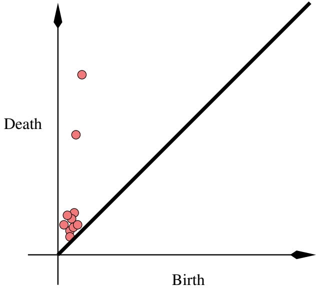
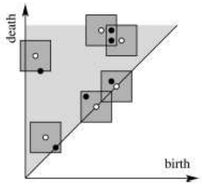




Stability of Persistence Diagram

- Bottleneck distance (C all bijections)

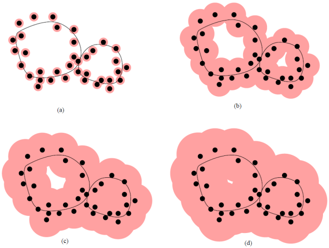
$$d_B(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) := \inf_{c \in C} \sup_{x \in \mathrm{Dgm}_p(f)} \|x - c(x)\|$$



Theorem (Cohen-Steiner, Edelsbrunner, Harer 06)

$$d_B(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \leq \|f - g\|_\infty$$

Back to Point Data



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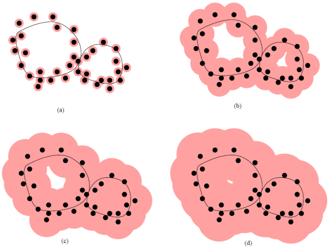

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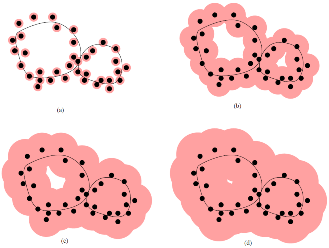

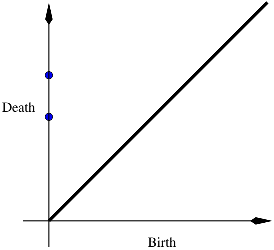

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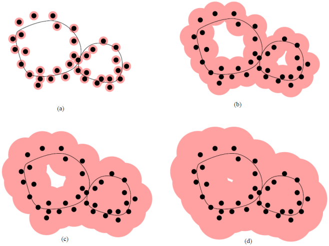

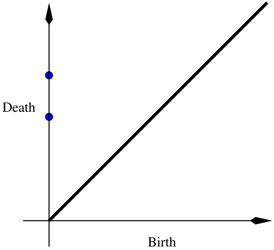

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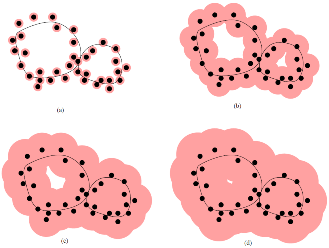

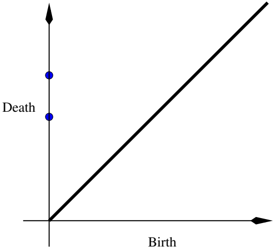

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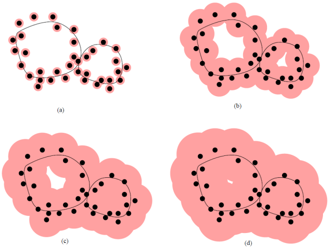
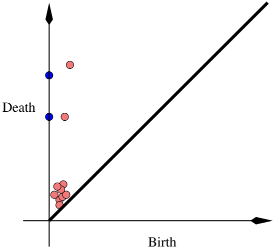
Back to Point Data





- $d_{\mathbb{T}}$ be the distance function from the space $\mathbb{T}$.
- d_P be the distance function from sample P
- $\|d_{\mathbb{T}} - d_P\|_\infty \leq d(\mathbb{T}, P) = \varepsilon$
- $d_B(\mathrm{Dgm}_p(d_{\mathbb{T}}), \mathrm{Dgm}(d_P)) \leq \varepsilon$

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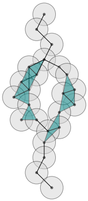

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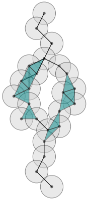

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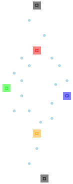

Q

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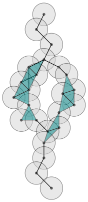

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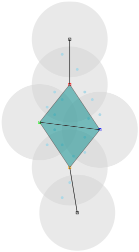

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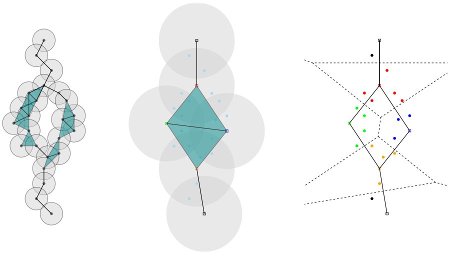

$\mathcal{R}^r(P)$ $\mathcal{R}^{r'}(Q)$ $\mathcal{G}^r(P,Q)$

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$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_n$$

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- Efficient algorithm for Zigzag simplicial maps?

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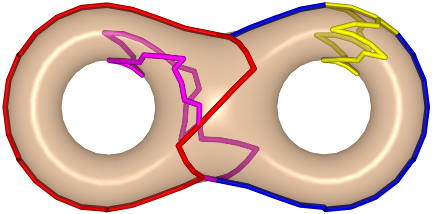

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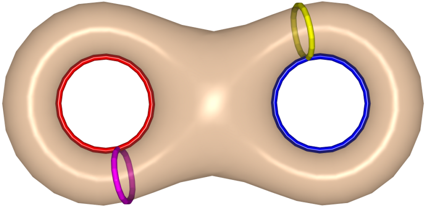

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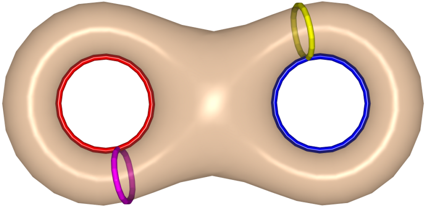

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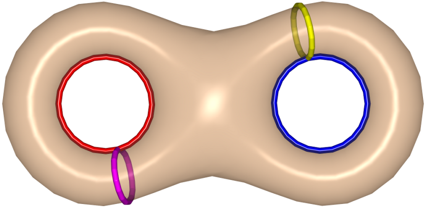

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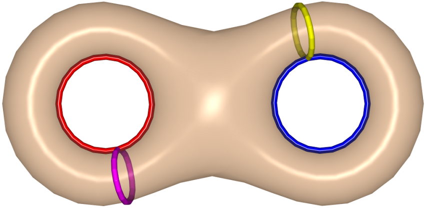

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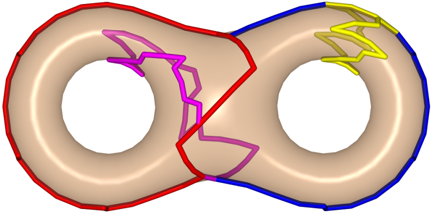

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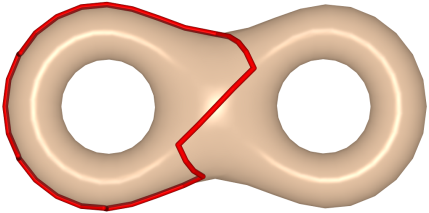

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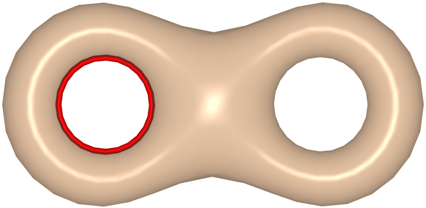

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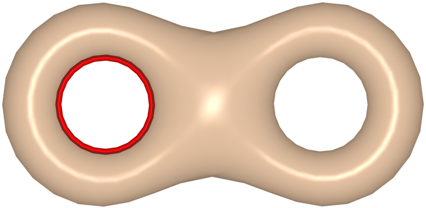

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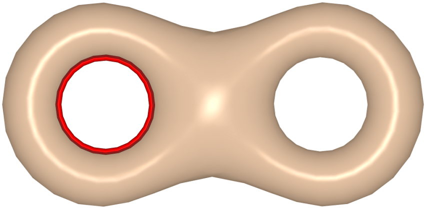

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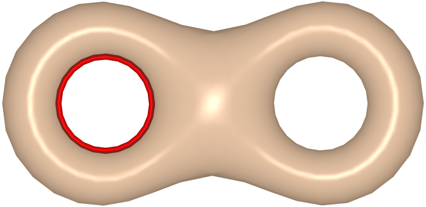

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Thank You