Emerging Challenges in Computational Topology

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Abstract

Here we present the results of the NSF-funded Workshop on Computational Topology, which met on June 11 and 12 in Miami Beach, Florida. This report identifies important problems involving both computation and topology.

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1 Introduction and Background

Over the past 15 years, computational geometry has become a very productive area, with applications in fields such as graphics, robotics, and computer-aided design. Computational geometry, however, primarily focuses on discrete problems involving point sets, polygons, and polyhedra, and uses combinatorial techniques to solve these problems. It is now time for computational geometry to broaden its scope in order to meet the challenges set forth in the President's Information Technology Advisory Committee (PITAC) Report [PIT98] and the Information Technology for the Twenty-First Century (IT^2) Initiative [IT99], specifically the need for accelerated progress in information visualization, advanced scientific and engineering computation, and computational algorithms and methods.

There is a need to extend computational geometry—with its emphasis on provable correctness, efficiency, and robustness—to continuous domains, curved surfaces, and higher dimensions. Such an extension brings computational geometry into contact with classical topology, just as earlier research led to inextricable connections with combinatorial geometry—to the great benefit of both fields.

We intend the name *computational topology* to encompass both algorithmic questions in topology (for example, recognizing knots) and topological questions in algorithms (for example, whether a discrete construction preserves the topology of the underlying continuous domain).

Research into computational topology has started already [Veg97], and is at present being undertaken separately by topology, computational geometry, and computer graphics communities, among others. Each of these fields has developed its own favored approaches to shape representation, manipulation, and analysis. Algorithms are often specific to certain data representations, and the underlying questions common to all approaches have not been given adequate attention.

The Workshop on Computational Topology, June 11–12, 1999, in Miami Beach, Florida, brought together researchers involved in aspects of computational topology. The purposes of this interdisciplinary workshop were to set goals for computational topology, identify important problems and areas, and describe key techniques common to many areas.

2 Goals

Geometric computing is a fundamental element in several of the areas highlighted in the IT^2 initiative: information visualization [IT99, section 2, “Fundamental Information Technology Research”, under human-computer interaction and information management], advanced science and engineering computation [IT99, section 3, “Advanced Computing for Science, Engineering, and the Nation”], and computational and algorithmic methods [IT99, section 3, under computer science and enabling technology]. Scientific and engineering computing often simulates physical objects and their interactions, on scales that vary from the atomic to the astronomical. Modeling the shapes of these objects, and the space surrounding them, is a difficult part of these computations. Information visualization also involves shapes and motions, as well as sophisticated graphics rendering techniques. Each of these two areas, as well as many others, would benefit from advances in generic computational and algorithmic methods.

Some of the most difficult and least understood issues in geometric computing involve topology. Up until now, work on topological issues has been scattered among a number of fields, and its level of mathematical sophistication has been rather uneven. This report argues that a conscious focus on computational topology will accelerate progress in geometric computing.

Topology separates global shape properties from local geometric attributes, and provides a precise language for discussing these properties. Such a language is essential for composing software programs, such as connecting a mesh generator to a computational fluid dynamics simulation. Mathematical abstraction can also unify similar concepts from different fields. For example, basic questions of robot reachability or molecular docking become similar topological questions in the appropriate configuration or conformation spaces. Finally, by separating shape manipulation from application-specific operations, we expect to improve reliability of geometric computing in many domains, just as other large software systems (for example, operating systems and internet routing) have gained reliability through layered design.

3 Areas and Problems

We have identified five main areas in which computational topology can lead to advances in simulation and visualization.

- **Shape acquisition.** The entry of the shapes of physical objects into the computer is becoming increasingly automated. Part of this process is developing algorithms that turn a set of measurements or readings into a topologically valid shape representation.
- **Shape representation.** Many different computer representations of shape are in use. Describing the relationships between them, converting from one to the other, and developing new representations, all require topological ideas and methods.
- **Physical simulation.** For scientific and engineering computations, shape representations are typically meshed into small pieces. Many of the issues that have arisen in mesh generation are topological.
- **Configuration spaces.** Configuration or conformation spaces represent the possible motions of objects moving among obstacles, mechanical devices, or molecules. These spaces are usually high-dimensional and non-Euclidean, and hence raise some rather deep topological questions.
- **Topological computation.** Some recent advances in topology itself involve algorithms and computation. Better software for geometric computing will help advance this approach to topology, while new techniques and representations developed for topological problems will contribute to the advancement of geometric computing.

In each area we have selected a few problems for more detailed discussion.

3.1 Shape Acquisition

Computer representations of shapes can either be designed by a person using CAD tools or acquired from an existing physical object. The latter approach offers advantages of speed and faithfulness to an original, which is of course crucial in applications such as medical imaging. Automatic acquisition of shapes poses a wealth of geometric and topological challenges.

Shape Reconstruction from Scattered Points. Modern laser scanners can measure a large number of points on the surface of a physical object in a matter of seconds. The most basic computational problem is the reconstruction of the "most reasonable" geometric shape that generated the point sample. We find algorithmic solutions to this problem in both the computational geometry and the computer graphics literature. Consistent with the dominant cultures in these two areas, solutions suggested in computational geometry are discrete in nature, while solutions in computer graphics are based on numerical ideas.

The early work of Hoppe et al. [HDMS92] in computer graphics drew wide attention to the reconstruction problem. The authors give an algorithm that works for point data sampled everywhere densely on the surface of the object. A basic step in the reconstruction estimates the surface normal at a point using a best-fit plane determined by near neighbors. This idea is inherently differential and limits the algorithm to shapes and data for which locally linear approximations provide useful information. Raindrop Geomagic, Inc. takes a more global approach in its software Wrap, which reconstructs a shape using the 3-dimensional Delaunay triangulation of the sampled points [Rai96]. Amenta and Bern describe another algorithm using the Delaunay triangulation together with differential ideas [AB98], and go on to prove that their algorithm gives a geometrically close and topologically correct output, under certain assumptions about the point sample. This result rationalizes the reconstruction process and focuses attention on the more difficult cases in which the assumptions are violated, for example, surfaces with creases or boundaries, and sample points with noise.

Manifold and Space Learning. Mathematically, it makes perfect sense to generalize the reconstruction problem from $\mathbb{R}^3$ to higher-dimensional Euclidean space. Perhaps somewhat surprisingly, this generalization makes sense also from the viewpoint of applications, including speech recognition, weather forecasting, and economic prediction. Many natural phenomena can be sampled by individual measurements, where each measurement can be interpreted as a point in $\mathbb{R}^d$, for some fixed dimension d.

The reconstruction problem in $\mathbb{R}^d$ is more difficult than in $\mathbb{R}^3$ not only because d usually exceeds 3, but also because we have no a priori knowledge about the intrinsic dimension of the shape that we wish to reconstruct. It might have mixed or fractal, or even altogether ambiguous, dimension. Since the input data is a discrete set of points, which by definition has dimension zero, the question itself is highly ambiguous and the answer depends on the scale at which we view the data. The idea of scale dependent variation as applied in the definition of fractal or Hausdorff dimension [Mat95] thus suggests itself. It appears in the work of Jones [Jon90], where local dimension is estimated through the variation of linear best-fits in a hierarchy of nested neighborhoods. It is also manifest in the work of Edelsbrunner and Mücke [EM94], who define alpha shapes as a family of reconstructed shapes parametrized by scale. One of the challenging problems in this context is the study of the interaction between noise and scale.

Reconstruction from Slices. In many applications the input data includes additional information that can help in reconstructing the shape. Examples are estimates of surface normals provided by the scanner or information encoded in the sampling sequence. A classic version of the reconstruction problem in the latter category presents the data in *slices*, each slice consisting of one or more polygons given by a cyclic sequence of vertices. Usually, the slicing planes are parallel, in which case the reconstruction reduces to connecting each pair of contiguous slices.

A problematic aspect of this approach to shape reconstruction is the nonsymmetric treatment of coordinate directions. In other applications, however, nonsymmetric treatment seems warranted or even necessary. For example, the animation of a moving planar shape can be viewed as sweeping a surface in $\mathbb{R}^2$ times time. Branching occurs at critical points, which correspond to moments in time where the shape changes its topology. The relevant mathematics here is Morse theory, which studies the combinatorial and differential structure of critical points [Mil63].

A related problem is morphing: given two surfaces in $\mathbb{R}^3$, construct a continuous deformation of one surface into the other. The deformation may be a homotopy or a cobordism. Again we can view this as a problem in Morse theory, only one dimension higher than before. This view is adopted in [CEF98], where canonical deformations are used in the construction of "shape spaces". Such spaces could be useful in building databases of shapes, such as drug compounds, anatomical structures, and mechanical tools.

Crystal Structures from X-ray Data. The standard tool for determining conformations of atoms and molecules in crystals is X-ray crystallography. Missing phase information must be inferred to convert observed Bragg diffraction intensity data into a phased Fourier amplitude set. This process continues to become more routine even for macromolecules such as proteins and viruses. A Fourier transform of the amplitude set then produces an electron density function over $\mathbb{R}^3$. If the observed intensity set extends far enough, the peaks of the the density function provide starting atomic coordinates suitable for least squares refinement, however macromolecular density maps usually do not have atomic resolution.

It is again convenient to describe the situation using the language of Morse theory. The density is viewed as the height function of a 3-manifold in $\mathbb{R}^4$, with four types of critical points: "peaks", "passes", "pales", and "pits". The reconstruction of the atomic or molecular configuration may be complicated by the presence of excessive noise, thermal motion, positional and occupancy disorder, or lack of atomic resolution. Morse theory interpretations of crystallographic density functions are being carried out for a variety of crystal structures ranging from macromolecules with less than atomic resolution [FCGL97], to ultra precise small molecule structures for which quantum mechanical perturbations such as lone pair density peaks in the middle of covalent bonds are detectable [Bad94]. When thermal motion is the primary focus, neutron Bragg diffraction data often are used, from which nuclear density rather than electon density maps are produced and thus do not include quantum perturbations.

Advances in computational topology can contribute to the above and to other problems such as classification schemes for crystal structures using Heegaard level surfaces between passes and pales [Joh99], and certain related minimal surfaces [LN99]. Delaunay-based reconstruction might provide useful tools to address the problem of "topological noise" such as spurious peaks and passes. Morse theory can also play a role in efficient algorithms for finding structures in electron density data [CSA99].

3.2 Shape Representation

Data structures for representing shapes have emerged independently in many different fields. These representations include unstructured collections of polygons ("polygon soup"), polyhedral models, subdivision surfaces, spline surfaces, implicit surfaces, skin surfaces, and alpha shapes. Generally speaking, these methods are at best adequate within their own fields, and not well-suited for connecting across fields. At least in the CAD area, there is growing awareness that future systems must be more mathematically sophisticated than today's systems. Rida Farouki writes, "At the heart of this problem lie some deep mathematical issues, concerned with the computation, representation, and manipulation of complex geometries" [Far99]. Shape representation appears to be an ideal area for collaboration between mathematicians and computer scientists.

Conversion Between Different Representations. A number of *ad hoc* methods exist for converting between different types of representations, usually to and from polyhedral models. These methods typically use geometric criteria to evaluate the conversion, for example, guaranteeing that the original surface is pointwise not more than a small tolerance distant from the polygonal mesh. Geometry alone is insufficient, however, as it does not guarantee topological properties such as "watertightness". Including topological criteria in the evaluation will lead to more correct conversion programs.

Topology Preserving Simplification. The process of replacing a polygonal surface with a simpler one, while essential to many hierarchical representations, is notorious for introducing topological errors which can be fatal for later operations. A popular method, edge contraction [HDD$^+$93], can be applied to general simplicial complexes, but is not in general guaranteed to produce a complex homeomorphic to the original. Dey et al. [DEGN98], however, have proved that the complex after contraction is homeomorphic to the original if the neighborhood of the contracted edge satisfies a *link condition*. For 2- and 3-manifolds, the link condition defines the contractable edges.

Even if the output of a simplification process is homeomorphic to the input, however, there is no guarantee that the output is correctly embedded. Self-intersections are often introduced, for instance, a problem sometimes know as "bubbling". One step towards guaranteeing a correct embedding was a paper by Varshney et al. [CVM$^+$96], in which a simplified 2-manifold is fitted into a shell around the original. Much more work, both in providing mathematically verifiable guarantees and in developing efficient algorithms, is required.

Smoothness and Nonsmoothness. Smooth surface representations are commonly divided into implicit and explicit representations. Implicit surfaces can be defined by blending parametrized surfaces such as splines, or as level sets of scalar functions. The advantages of implicit surfaces include high degree of smoothness for arbitrary topology, ease of raytracing, and ease of combining several objects by blending. Disadvantages include difficulties with parameterization and conversion to polygonal meshes. Moreover, implicit surfaces may have singularities, which can be difficult to detect and control.

The most common explicit representation—ubiquitous in CAD—is that of nonuniform rational B-spline (NURBS) patches. A more systematic approach, which offers the advantage of multiresolution control, involves subdivision surfaces [ZSD$^+$99]. Both of these methods, however, produce surfaces with defects, for example, flat spots or areas of relatively low smoothness near extraordinary points. Whether or not a point is extraordinary depends on the local topology within the representing mesh, and has nothing to do with its geometric location. The changed amount of smoothness is thus an artifact of the representation, and should ideally not exist. An important challenge in smooth surfaces is to ensure integral measures of visual smoothness (fairness). Variational surfaces aim to handle such measures directly.

In the other direction, there is also need for representations that can handle singularities such as boundaries, creases, and corners. With standard polyhedral models, there is no distinction between the creases resulting from discretization and those that represent true surface features. Spline patches give rise to creases of high algebraic degree that cannot be manipulated directly, and implicit surfaces rarely allow any control of singularities.

Multiscale Representations. Multiscale representations, whether implicit or explicit, hold out the promise of efficiency even for very complex geometries. We identify the main challenge as developing representations that allow controlled topology changes between levels, while supporting a variety of efficient multiscale operations, such as animation, editing, and "signal processing".

Implicit multiscale representations (level sets) have been used in volume rendering. Volume data are themselves represented on regular or adaptively refined (octree) grids; thus it is natural to use classic functional multiscale representations such as wavelets [WE97, Wes94], yet it is also possible to construct unstructured mesh hierarchies on volume data [WJ95]. While allowing topological changes at different levels of the hierarchy, implicit representations offer little control of such changes. On the other hand, at least in the case of volume data represented on regular grids, signal processing techniques can be used to handle some of the topological problems [Wes94].

Some of the current explicit methods [SZL92, RB93, GH97] do allow topological changes, but the control over such changes is relatively limited and the hierarchies created by these methods are unsuitable for many purposes, for example, it may be difficult or impossible to parameterize finer levels of hierarchies over the coarse levels. The recent work of El-Sana and Varshney [ESV98] based on alpha shapes [EM94] aims to perform topological simplification in a more controlled manner.

Qualitative Geometry and Multiscale Topology. For the final highlighted problem, we move from shape representation to shape analysis. Topological invariants (see Section 3.5) such as Betti numbers are insensitive to scale, and do not distinguish between tiny holes and large ones. Moreover, features such as pockets, valleys, and ridges—which are sometimes crucial in applications—are not usually treated as topological features at all. Nevertheless, topological spaces naturally associated with a given surface can be used to capture scale-dependent and qualitative geometric features.

For example, the lengths of shortest linking curves [DG98], closed curves through or around a hole, can be used to distinguish small from large holes, and the areas of *compressing disks*, which "seal off" a hole, can be used to distinguish long narrow pipes from direct openings. The topology of offset or "neighborhood" surfaces is an appropriate tool for classifying depressions in a surface: a sinkhole with a small opening will seal off as the neighborhood grows, whereas a shallow puddle will not. Edelsbrunner has already used this idea to design an algorithm to detect pockets in molecular surfaces [EFL98], but further investigations are necessary to answer questions on the border of geometry and topology.

3.3 Physical Simulation

Scientific computing has traditionally been concerned with numerical issues such as the convergence of discrete approximations to partial differential equations (PDEs), the stability of integration methods for time-dependent systems, and the computational efficiency of software implementations of these numerical methods. Of central importance has been ultimate use of these techniques in the solution of complex problems in science and engineering such as the modeling of combustion systems, aerodynamics, structural mechanics, molecular dynamics, and problems from a large number of other application areas. However, as these applications have become more complex, the local convergence properties of numerical methods have not proven to be sufficient to ensure either correctness or robustness. There are a number of research areas where topological and differential methods could be integrated with existing numerical techniques in scientific computing to help resolve these difficulties.

Hexahedral Mesh Generation. For many scientific applications, the preferred discretization is a *hexahedral mesh* partitioning the domain into cuboids. A common approach to hexahedral mesh generation involves extending a quadrilateral mesh on the domain surface to a three-dimensional volume mesh of the entire domain [BP97]. Even though several software implementations of this approach exist [TM95], it is not yet known whether this extension can always be done. An obvious necessary condition for the existence of a hexahedral mesh is that there be an even number of boundary quadrilaterals; this is also sufficient to guarantee the existence of a *topological* mesh, meaning one in which hexahedral faces may be slightly nonplanar, for domains forming a simple topological structure [Mit96,Thu93] or having a bipartite boundary [Epp99]. However, it is not clear whether similarly simple conditions can guarantee the existence of a polyhedral mesh or whether additional algebraic conditions on the surface must be imposed.

Another important issue in automatic mesh generation is element quality; poorly shaped elements (flat or skinny, especially skinny in the "wrong direction") are directly responsible for poorly conditioned matrices [Fri72] and hence slow and inaccurate numerical computations [BR78]. For triangular, tetrahedral, and quadrilateral meshes, the solution to poor quality elements has been the introduction of "provably good" meshing methods that guarantee to produce a mesh with all elements having good quality according to various metrics [BE97, BEG94]. However for hexahedral meshes, little is known about quality metrics and even less is known about provably good meshing. Recent work using the Jacobian matrix norm as a quality metric for hexahedral elements has shown promise for finite-element calculations [Knu99].

Anisotropic Mesh Generation. In many applications the underlying physics is not isotropic and, as a result, standard mesh generation methods and element quality metrics are not appropriate. This is the case, for example, in modeling the fluid flow in a boundary layer, or in groundwater flow calculations where the porosity is highly nonisotropic because of geological features such as faults and layering of strata. For these problems element aspect ratios of 1000:1 are sometimes necesssary; however, the generation of these meshes is often *ad hoc*. For isotropic problems the shape optimization of elements based on measures generated from local metrics computed from the Hessian of element error functions has proved useful [Rip92] and optimal for the finite-element approximation of given functions [Sim94]. A promising area of future research is the extension of these results to anisotropic problems. For example, one would like to characterize the existence of canonical triangulations (perhaps something akin to Delaunay triangulation) given the Riemannian metrics generated by the error function estimates.

Moving Meshes. In problems such as casting and molding, the domain changes with time, and it is convenient to adapt the existing mesh rather than recomputing an entirely new mesh. Moving mesh problems also arise in Lagrangian discretization strategies for time-dependent PDEs. The challenge problem here is the identification and correction of topological changes as the mesh changes over time.

Visualization. Large-scale simulations can generate terabytes of numerical data. The analysis and interpretation of this voluminous data has become an increasingly important research problem. One promising approach extracts features such as vortex lines or sheets [SZF$^+$93]. The topology and qualitative geometry of these features can be of great interest. Examples include identification of voids and pockets in molecular surfaces [Bad94, EFL98], and simulation of high-temperature superconductors, in which magnetic field lines "tangle" with impurities in the material [GKL$^+$96, JP93].

3.4 Configuration Spaces

The notion of *configuration space* (also called *parametric space* or *realization space*) is used in numerous areas, including robotics, graphics, molecular biology, computer vision, and databases for representing the space of all possible states of a system characterized by many degrees of freedom. Instead of defining configuration spaces in general, we will illustrate the concept by giving an example in robotics.

In *robot motion planning*, the problem is to compute a collision-free motion between two given placements—or *configurations*—of a given robot among a set of obstacles. A configuration is typically described as a list of real parameters, and the set of all possible configurations is called the configuration space. *Free configuration space* $\mathcal{F}$ is the subset of the configuration space at which the robot does not intersect any obstacle. The robot can move from an initial configuration to a final configuration without intersecting any obstacle if and only if these two configurations lie in the same connected component of free configuration space. Planning a collision-free motion thus maps to planning the motion of a point in $\mathcal{F}$. In other words, the motion-planning problems map to connectivity questions, or related topological questions, in $\mathcal{F}$. Many other problems can be couched in terms of configuration spaces. Important examples include *assembly planning* and *molecular docking* [HKL97, HLW97, Lat91]. The topology of configuration spaces is little understood, except in very rudimentary cases, such as that of an object under rigid motion.

Representation and Computation. Most interesting configuration spaces are *semialgebraic sets*, finite Boolean combinations of solution sets of polynomial inequalities and equalities. The question of representing and computing a semi-algebraic set has received much attention in the last two decades. Since the topology of a semi-algebraic set can be quite intricate, developing a suitable representation is a challenging (and not fully solved) problem. A common technique to represent a semialgebraic set is to partition it into semialgebraic sets of constant description complexity, each of which is homeomorphic to $\mathbb{R}^j$ for some j [Bri93, Lie91, SS83]. Some commonly used general decomposition schemes are Collin's decomposition [ACM84, Col75] and vertical decomposition [CEGS89]. Because of efficiency considerations, we want to minimize the number of cells in the decomposition. A major open question in this area is to compute a decomposition of minimum size.

In motion planning, we are interested in computing a single connected component of $\mathcal{F}$. (It is not even obvious that a connected component of a semialgebraic set is also semi-algebraic; this was proved only recently [BPR98].) What is the combinatorial or topological complexity of such a component? Recently, Basu proved tight bounds on the sum of Betti numbers and used it to prove a sharp bound on the combinatorial complexity of a single component [Bas98]. However, no efficient algorithm is known for computing a single cell. A related open problem is to develop an efficient stratification scheme for a single component of a semialgebraic set.

In some applications even more challenging problems arise. If the obstacles are moving as well as the robot, then we need to update a road-map or stratification dynamically. Also, flexible objects such as elastic bands, rope, or cloth cannot be properly represented with a finite number of degrees of freedom. How can we represent configuration spaces of such objects? The key here may be to capture the notion that different configurations have an energy associated with them, and that only low-energy configurations are of interest. Are there good ways to parametrize these low-energy configurations and to plan motions among them?

Approximation. Computing exact high-dimensional configuration spaces is impractical. Thus it is reasonable to ask for approximate representations. Much of the difficulty in approximating a high-dimensional configuration space is in understanding and simplifying the topology of the space. Although several algorithms are known for simplifying the geometry of a surface, little is known about simplifying topology.

Recently, Monte Carlo algorithms have been developed for representing a higher dimensional semialgebraic set by a 1-dimensional network [KLMR95, KŠL96]. Intuitively, this network is an approximate representation of the *road map* (a network of 1-dimensional curves that captures the connectivity information of $\mathcal{F}$). These methods sample points in $\mathcal{F}$ and connect them by an edge if they can be connected by a direct path inside $\mathcal{F}$. So far very simple strategies have been developed for choosing random points. These methods work well when $\mathcal{F}$ is simple, but better sampling techniques are needed to handle planning problems involving narrow corridors or other difficult areas, in such a way that the connectivity of the sampled configuration space is preserved.

Decomposition. Dimension reduction is one approach to developing faster algorithms for problems in high dimensions. One possibility for motion planning is to search for solutions in one or more projections of the configuration space and then lift the solution back to the original space. For example, suppose we want to plan a motion for two disks in the plane amid obstacles. The four-dimensional free space of this system can be computed by decomposing the two-dimensional free space of each disk into simple cells, and then lifting these cells into $\mathbb{R}^4$. Proving that such a strategy succeeds requires several sophisticated techniques from algebraic topology, including Mayer-Vietoris sequences [AdBvdS$^+$98, FWY86, HW86a, HW86b]. A characterization of the sitations in which the configuration space can be decomposed and finding the "optimal" decomposition of the configuration space are two interesting open problems in this area.

3.5 Topological Computation

The study of algorithms for topological problems has grown quite popular in recent years; it is one of the few growing branches of topology. In the last few years, there have been several workshops and the founding of an on-line community—www.computop.org. Much of the recent effort has focused on classifying the inherent complexity of topological problems. Typically, planar problems are easy (polynomially solvable), problems in $\mathbb{R}^3$ are hard (exponentially solvable and thought to be NP-complete), and problems in $\mathbb{R}^4$ and higher dimensions are known to be undecidable.

Unknot Recognition. A knot is said to be unknotted if it can be deformed to a (geometric) circle without passing through itself. In the early 1960s, Haken used a combinatorial representation of surfaces, called *normal surfaces*, in an algorithm for deciding if a knot is unknotted [Hak61]. A recent collaboration between mathematicians and computer scientists showed that this algorithm will take at most exponential time in the number of crossings of the knot [HLP99]. It is still open, however, whether this problem is NP-complete or can be solved in subexponential time.

Knot and Link Equivalence. Two knots are equivalent if one can be deformed into the other without passing through itself. Knot equivalence is known to be decidable [Hem92], but the algorithm is extremely complicated and the computational complexity is as yet unknown. An important related question asks whether two links (collections of intertwined knots) are equivalent. No algorithm is yet known for this problem.

Three-Sphere Recognition. The development of almost normal surfaces, a generalization of normal surfaces, has led to the Rubinstein-Thompson algorithm for deciding if a manifold is the 3-sphere [Tho94]. Recent work of Casson shows that this algorithm will take at most exponential time.

Shellings. A *shelling* of a cell complex is an ordering of the cells such that if cells are added one by one in that order the topological type remains invariant. While interesting for their own sake, shellings also provide a very useful calculational tool. Hence it is an important algorithmic problem to determine if a cell complex is shellable and, if not, modify it so that it is. Current algorithms [Let99] are not yet practical, and improvements are needed.

Hyperbolic Geometry. Three-dimensional manifolds with a hyperbolic structure have many useful properties, allowing extremely efficient and powerful topological calculations. The computer software package SnapPea [Wee], written by Weeks, implements many of these calculations and has proven an exteremely useful tool for low-dimensional topology. There are still many open questions in algorithmic hyperbolic geometry, for example, whether it is possible to decide if a manifold has a hyperbolic structure.

Topological Invariants. Another area of interest, with a number of practical applications outside mathematics, is the calculation of topological invariants. Many physical objects can change geometry more easily than they can change topology. Examples range from molecules to alphabetic characters to geological formations. For these objects, topological invariants offer a more meaningful description than geometric measures.

The most useful topological invariants involve *homology*, which defines a sequence of groups describing the "connectedness" of a topological space. For example, the *Betti numbers* of an object embedded in $\mathbb{R}^3$ are respectively the number of connected components separated by gaps, the number of circles surrounding tunnels, and the number of shells surrounding voids. Technically, the Betti numbers are the ranks of the free parts of the homology groups. For more abstract topological spaces, not embedded in $\mathbb{R}^3$, the relevant invariants include *torsion coefficients* as well.

For 2-manifolds without boundary, the homology can be computed quite easily by computing Euler characteristics and orientability. The case of 3-complexes requires more sophistication, but computational geometers have devised quite efficient algorithms for the case of 3-complexes embedded in $\mathbb{R}^3$ [DE95, DG98]. However, these algorithms use the three-dimensional embedding heavily and it is not yet clear whether they can be extended to general complexes. These problems are not just of mathematical interest: nonmanifold 2-complexes are used quite often in modeling shock fronts, crack propagation, or domains made of two different materials. In dimension beyond 3, there is yet no algorithm that would be practical for large complexes.

From a practical point of view it may often be impossible to determine the topology of an object completely, and estimation of topological invariants may be appropriate. In materials science, structural properties of composite materials such as concrete or high-impact plastic appear to be related to the Betti numbers of randomly selected cross sections.

Finally, in addition to studying the shape of objects in space, topological computations may prove useful in studying the shape of space itself! Research is currently underway using astronomical data to investigate the geometry and topology of the universe. One approach uses maps of cosmic background radiation to piece together the global structure of the universe [CW98, Wee98].

4 Techniques

We can already identify a number of techniques that computational topology could bring to bear on the applications described above. We list them in order from general scientific principles down to specific algorithmic methods.

- **Mathematical Viewpoint.** Topology separates global shape properties from local geometric attributes, and provides a precise language for discussing these properties. Such a language is essential for composing software applications, such as connecting a mesh generator to a computational fluid dynamics simulation. Mathematical abstraction can also unify similar concepts from different fields. For example, basic questions of robot reachability or molecular docking become topological questions in the appropriate configuration or conformation spaces.

- **Asymptotic Analysis.** The signature technique of theoretical computer science is asymptotic worst-case and average-case analysis of algorithms. This type of analysis, while sometimes overemphasized as an end in itself, is helpful in providing a common yardstick to measure progress and encourage future work. Although proving upper bounds on algorithm performance is usually a matter of concrete analysis, topological ideas such as Betti numbers can be useful in proving lower bounds [Yao94].

- **Exact Geometric Computation.** This technique draws on algebraic number theory to ensure the topological correctness of geometric computations. In principle, this technique solves most of the numerical robustness problems in such applications as CAD modeling and computational simulation.

- **Differential Methods.** Many techniques from differential geometry, such as Morse theory for studying singularities, are essential in analyzing surfaces and models in diverse applications such as medical imaging, crystallography, and molecular modeling.

- **Topological Methods in Discrete Geometry.** Topological results such as the Borsuk-Ulam theorem [Bor33], that any continuous antipodal function on a sphere must have a zero, have commonly been used in discrete geometry to prove the existence of geometric configurations such as ham sandwich cuts and centerpoints [Bjö95, Živ97]. However, such methods do not generally lead to efficient algorithms for finding such configurations [Ko95], so further research on effective existence proofs may be warranted.

- **Multiscale Synthesis and Analysis.** Multiresolution techniques have already assumed great importance in the synthesis of computer graphics models and in numerical methods for physical simulation. Multiscale techniques are fast becoming equally important in visualization and analysis of unstructured "natural" data. One example [Ler, Jon90] uses techniques drawn from geometric measure theory and harmonic analysis for approximating a set with best fit planes at different resolutions. This approach segments a point set or image into subsets of different geometric structure; by combining continuous and discrete analysis, it produces results even for noisy data.

- **Normal Surfaces.** Invented by Kneser [Kne29], normal surfaces were the basis for Haken's knot algorithm [Hak61], and have been used in numerous algorithmic and finiteness results in topology [Has97, JS98]. Instead of representing a curve or surface with an explicit mesh or parameterization, normal surface theory describes how that curve or surface intersects a given mesh of the ambient space. This yields a very efficient representation for densely folded curves and surfaces, which has potential in applications where such curves and surfaces occur. Moreover, normal surface theory provides a natural "addition" operation for surfaces, which is useful for their manipulation.

5 Recommendations

- **Research Community.** There is need to build a computational topology research community including computer scientists, engineers, and mathematicians. Such a community could be held together by organizing workshops and conference special sessions, and by maintaining Web sites, bibliographies, and collections of open problems. Techniques from topology have already been used in geometric computing and vice versa. We want to strengthen and formalize this link.

- **Online Clearinghouse.** To encourage a sense of community, we should establish a clearinghouse of research projects, papers, software, and informal communications between workers in this area. The web site already present at www.computop.org could possibly provide a location for this collection.

- **Research Funding.** Grant opportunities are needed to encourage further work in these areas, either as a separate initiative or continued funding from the relevant areas within NSF.

- **Continuation of Workshop.** It seems premature to establish a journal or annual conference series in this area, but at the least there should be another workshop on computational topology. This year's workshop was an invitation-only, direction-finding session; what is needed now is a forum for collecting new work in the area and fostering continued interdisciplinary collaboration. Perhaps such an event could be held in conjunction with the annual ACM Symposium on Computational Geometry, to be held next year in Hong Kong.

Bibliography

[AB98] Nina Amenta and Marshall W. Bern. Surface reconstruction by Voronoi filtering. In *Proc. 14th Symp. Computational Geometry*, pages 39–48. ACM, June 1998.

[ACM84] Dennis S. Arnon, George E. Collins, and Scott McCallum. Cylindrical algebraic decomposition 2: The adjacency algorithm for the plane. *SIAM J. Comput.*, 13(4):878–889, November 1984.

[AdBvdS+98] Boris Aronov, Mark T. de Berg, A. Frank van der Stappen, Petr Švestka, and Jules Vleugels. Motion planning for multiple robots. In *Proc. 14th Symp. Computational Geometry*, pages 374–382. ACM, June 1998.

[Bad94] Richard F. W. Bader. *Atoms in Molecules: A Quantum Theory*. Clarenden Press, Oxford, England, 1994.

[Bas98] Saugata Basu. On the combinatorial and topological complexity of a single cell. In *Proc. 39th Symp. Foundations of Computer Science (FOCS '98)*, pages 606–616. IEEE, November 1998.

[BE97] Marshall W. Bern and David Eppstein. Quadrilateral meshing by circle packing. In *Proc. 6th Int. Meshing Roundtable*, pages 7–20, October 1997, cs.CG/9908016.

[BEG94] Marshall W. Bern, David Eppstein, and John R. Gilbert. Provably good mesh generation. *J. Computing & Systems Sciences*, 48(3):384–409, June 1994.

[Bjö95] Anders Björner. Topological methods. In Ronald L. Graham, Martin Grötschel, and László Lovász, editors, *Handbook of Combinatorics*, chapter 34, pages 1819–1872. North-Holland, Amsterdam, 1995.

[Bor33] Karol Borsuk. Drei Sätze über die n-dimensionale euklidische Sphäre. *Fund. Math.*, 20:177–190, 1933.

[BP97] Marshall W. Bern and Paul E. Plassmann. Mesh generation. Technical Report CSE-97-019, Pennsylvania State U., Dept. of Computer Science & Engineering, 220 Pond Laboratory, University Park, PA, 16802-6106, USA, 1997.

[BPR98] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. Complexity of computing semi-algebraic descriptions of the connected components of a semi-algebraic set. Unpublished manuscript, 1998.

[BR78] Ivo Babuška and Werner C. Rheinboldt. Error estimates for adaptive finite element computations. *SIAM J. Numerical Analysis*, 15(4):736–754, August 1978.

[Bri93] Erik Brisson. Representing geometric structures in d dimensions: topology and order. *Discrete Comput. Geom.*, 9(1):387–426, 1993.

[CEF98] Ho-Lun Cheng, Herbert Edelsbrunner, and Ping Fu. Shape space from deformation. In *Proc. 6th Pacific Conf. Comput. Graphics Appl.*, pages 104–113, Singapore, 1998.

[CEGS89] Bernard Chazelle, Herbert Edelsbrunner, Leonidas J. Guibas, and Micha Sharir. A singly-exponential stratification scheme for real semi-algebraic varieties and its applications. In *Proc. 16th Int. Coll. Automata, Languages and Programming (ICALP '89)*, Lecture Notes in Computer Science, no. 372, pages 179–192, Berlin, July 1989. Springer-Verlag.

[Col75] George E. Collins. Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition. In Helmut Brakhage, editor, *Proc. 2nd GI Conf. Automata Theory and Formal Languages*, Lecture Notes in Computer Science, no. 33, pages 134–183, Berlin, 1975. Springer-Verlag.

[CSA99] Hamish Carr, Jack Snoeyink, and Ulrike Axen. Computing contour trees in all dimensions. Technical Report TR-99-09, Univ. of British Columbia, Dept. of Computer Science, August 1999.

[CVM+96] Jonathan Cohen, Amitabh Varshney, Dinesh Manocha, Greg Turk, Hans Weber, Pankaj K. Agarwal, Frederick P. Brooks, Jr., and William Wright. Simplification envelopes. In Holly Rushmeier, editor, *Proc. SIGGRAPH '96*, Computer Graphics Proceedings, Annual Conference Series, pages 119–128. Addison Wesley, August 1996.

[CW98] Neil J. Cornish and Jeffrey R. Weeks. Measuring the shape of the universe. *Notices Amer. Math. Soc.*, 45(11):1463–1471, 1998.

[DE95] Cecil Jose A. Delfinado and Herbert Edelsbrunner. An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere. *Computer Aided Geometric Design*, 12(7):771–784, 1995.

[DEGN98] Tamal K. Dey, Herbert Edelsbrunner, Sumanta Guha, and Dmitry V. Nekhayev. Topology preserving edge contraction. Technical Report RGI-Tech-98-018, Raindrop Geomagic Inc., Research Triangle Park, North Carolina, 1998.

[DG98] Tamal K. Dey and Sumanta Guha. Computing homology groups of simplicial complexes in $\mathbb{R}^3$. *J. ACM*, 45(2):266–287, March 1998.

[EFL98] Herbert Edelsbrunner, Michael A. Facello, and Jie Liang. On the definition and the construction of pockets in macromolecules. *Discrete Appl. Math.*, 88(1–3):83–102, November 1998.

[EM94] Herbert Edelsbrunner and Ernst P. Mücke. Three-dimensional alpha shapes. *ACM Trans. Graphics*, 13(1):43–72, January 1994.

[Epp99] David Eppstein. Linear complexity hexahedral mesh generation. *Computational Geometry Theory & Applications*, 12(1–2):3–16, February 1999, cs.CG/9809109.

[ESV98] Jihad El-Sana and Amitabh Varshney. Topology simplification for polygonal virtual environments. *IEEE Trans. Visualization and Computer Graphics*, 4(2):133–144, April–June 1998.

[Far99] Rida T. Farouki. Closing the gap between CAD model and downstream application. *SIAM News*, 32(5), June 1999.

[FCGL97] Suzanne Fortier, Anthony C. Chiverton, Janice I. Glasgow, and Laurence Leherte. Critical-point analysis in protein electron-density map interpretation. In Charles M. Carter, Jr. and Robert M. Sweet, editors, *Macromolecular Crystallography, Part B*, Methods in Enzymology, no. 277. Academic Press, 1997.

[Fri72] Isaac Fried. Condition of finite element matrices generated from nonuniform meshes. *AIAA Journal*, 10(2):219–221, April 1972.

[FWY86] Steven Fortune, Gordon T. Wilfong, and Chee Yap. Coordinated motion of two robot arms. In *Proc. Int. Conf. Robotics and Automation*, pages 1216–1223. IEEE, April 1986.

[GH97] Michael Garland and Paul S. Heckbert. Surface simplification using quadric error metrics. In Turner Whitted, editor, *Proc. SIGGRAPH '97*, Computer Graphics Proceedings, Annual Conference Series, pages 209–216. Addison Wesley, August 1997.

[GKL$^+$96] William D. Gropp, Hans G. Kaper, Gary K. Leaf, David M. Levine, Mario Palumbo, and Valerii M. Vinokur. Numerical simulation of vortex dynamics in type-II superconductors. *J. Comp. Phys.*, 123(2):254–266, 1996.

[Hak61] Wolfgang Haken. Theorie der Normalflächen: Ein Isotopickriterium für der Kreisknoten. *Acta Math.*, 105:245–375, 1961.

[Has97] Joel Hass. Algorithms for recognizing knots and 3-manifolds. xxx.lanl.gov e-print archive, December 1997, math.GT/9712269.

[HDD$^+$93] Hugues Hoppe, Tony DeRose, Tom Duchamp, John McDonald, and Werner Stützle. Mesh optimization. In *Proc. SIGGRAPH '93*, pages 19–26. ACM SIGGRAPH, August 1993.

[HDMS92] Hugues Hoppe, Tony DeRose, John McDonald, and Werner Stützle. Surface reconstruction from unorganized points. *Computer Graphics*, pages 71–78, 1992.

[Hem92] Geoffrey Hemion. *The Classification of Knots and 3-Dimensional Spaces*. Oxford Science Publications. Oxford Univ. Press, 1992.

[HKL97] Dan Halperin, Lydia E. Kavraki, and Jean-Claude Latombe. Robotics. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 41, pages 755–778. CRC Press, Boca Raton, FL, 1997.

[HLP99] Joel Hass, Jeffrey C. Lagarias, and Nicholas Pippenger. The computational complexity of knot and link problems. *J. ACM*, 42(2):185–211, 1999, math.GT/9807016.

[HLW97] Dan Halperin, Jean-Claude Latombe, and Randall H. Wilson. A general framework for assembly planning: The motion space approach. Unpublished manuscript, 1997.

[HW86a] John E. Hopcroft and Gordon T. Wilfong. Motion of objects in contact. *Int. J. Robotics Research*, 4(4):32–46, Winter 1986.

[HW86b] John E. Hopcroft and Gordon T. Wilfong. Reducing multiple object motion planning to graph searching. *SIAM J. Comput.*, 15(3):768–785, August 1986.

[IT99] Information Technology for the Twenty-First Century: A Bold Investment in America's Future. Natl. Coordination Office for Computing, Information, and Communications, 1999, `http://www.ccic.gov/pubs/it2-ip/`.

[Joh99] Carroll K. Johnson. Crystallographic topology II: Overview and work in progress. In Vasilios Alexiades and George Siopsis, editors, *Trends in Mathematical Physics*, AMS/IP Studies in Advanced Mathematics, no. 13, pages 307–344. Amer. Math. Soc., Providence, RI, 1999.

[Jon90] Peter W. Jones. Rectifiable sets and the traveling salesman problem. *Inventiones Mathematicae*, 102(1):1–15, 1990.

[JP93] Mark T. Jones and Paul E. Plassmann. Computation of equilibrium vortex structures for type-II superconductors. *Int. J. Supercomputer Applications*, 7(2):129–143, 1993.

[JS98] William Jaco and Eric Sedgwick. Decision problems in the space of Dehn fillings. xxx.lanl.gov e-print archive, November 1998, math.GT/9811031.

[KLMR95] Lydia E. Kavraki, Jean-Claude Latombe, Rajeev Motwani, and Prabhakar Raghavan. Randomized query processing in robot path planning. In *Proc. 27th Symp. Theory of Computing (STOC '95)*, pages 353–362. ACM, June 1995.

[Kne29] Hellmuth Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresbericht der Deut. Math. Verein.*, 38:248–260, 1929.

[Knu99] Patrick Knupp. Achieving finite element mesh quality via optimization of the Jacobian matrix norm and associated quantities II: A framework for volume mesh optimization. Technical Report SAND 99-0709J, Sandia National Laboratories, 1999.

[Ko95] Ker-I Ko. Computational complexity of fixed points and intersection points. *J. Complexity*, 11(2):265–292, June 1995.

[KŠL96] Lydia E. Kavraki, Petr Švestka, and Jean-Claude Latombe. Probabilistic roadmaps for path planning in high dimensional configuration spaces. *IEEE Trans. Robot. Autom.*, 12(4):566–580, August 1996.

[Lat91] Jean-Claude Latombe. *Robot Motion Planning*. Kluwer Int. Ser. Engineering and Computer Science, no. 124. Kluwer Academic Publishers, Boston, 1991.

[Ler] Gilad Lerman. Geometric transcriptions of sets and their applications to data analysis. In preparation.

[Let99] David Letscher. Shelling subdivisions of S^3. Preprint, 1999.

[Lie91] Pascal Lienhardt. Topological models for boundary representation: A comparison with n-dimensional generalized maps. *Computer Aided Design*, 23(1):59–82, January–February 1991.

[LN99] Stefano Leoni and Reinhard Nesper. Tilings on hyperbolic surfaces as representatives of chemical networks. In *Proc. XVIII IUCr Cong. & General Assembly, Glasgow, Scotland*, page 567, August 1999. Abstract P13.22.003.

[Mat95] Pertti Mattila. *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge Studies in Advanced Mathematics, no. 44. Cambridge Univ. Press, 1995.

[Mil63] John Milnor. *Morse Theory*. Annals Math. Studies. Princeton Univ. Press, 1963.

[Mit96] Scott A. Mitchell. A characterization of the quadrilateral meshes of a surface which admit a compatible hexahedral mesh of the enclosed volume. In *Proc. 13th Symp. Theoretical Aspects of Computer Science (STACS '96)*, Lecture Notes in Computer Science, no. 1046, pages 465–476, Berlin, 1996. Springer-Verlag.

[PIT98] President's Information Technology Advisory Committee Report to the President: Information Technology Research, Investing in Our Future. Natl. Coordination Office for Computing, Information, and Communications, 1998, `http://www.ccic.gov/ac/report/`.

[Rai96] Raindrop Geomagic Inc. Geomagic Wrap. Commercial software, 1996.

[RB93] Jarek Rossignac and Paul Borrel. Multi-resolution 3D approximation for rendering complex scenes. In Bianca Falcidieno and Toshiyasu L. Kunii, editors, *Proc. 2nd Conf. Modeling in Computer Graphics: Methods and Applications*, IFIP Ser. Computer Graphics, pages 453–465, Berlin, June 1993. Springer-Verlag.

[Rip92] Shmuel Rippa. Long and thin triangles can be good for linear interpolation. *SIAM J. Numer. Analysis*, 29(1):257–270, 1992.

[Sim94] R. Bruce Simpson. Anisotropic mesh transformations and optimal error control. *Applied Numer. Math.*, 14(1–3):183–198, 1994.

[SS83] Jacob T. Schwartz and Micha Sharir. On the "piano movers" problem II: General techniques for computing topological properties of real algebraic manifolds. *Advances in Applied Mathematics*, 4(3):298–351, 1983.

[SZF+93] Deborah Silver, Norman J. Zabusky, Victor Fernandez, Ming Gao, and Ravi Samtaney. Ellipsoidal quantification of evolving phenomena. In Nicholas M. Patrikalakis, editor, *Proc. Computer Graphics International '91: Scientific Visualization of Physical Phenomena*, pages 573–588, Berlin, 1993. Springer-Verlag.

[SZL92] William J. Schroeder, Jonathan A. Zarge, and William E. Lorensen. Decimation of triangle meshes. *Computer Graphics*, 26(2):65–70, July 1992. Proc. SIGGRAPH '92.

[Tho94] Abigail Thompson. Thin position and the recognition problem for S^3. *Math. Res. Lett.*, 1(5):613–630, 1994.

[Thu93] William P. Thurston. Hexahedral decomposition of polyhedra. Posting to sci.math, 25 October 1993, `http://www.ics.uci.edu/~eppstein/gina/Thurston-hexahedra.html`.

[TM95] Timothy J. Tautges and Scott A. Mitchell. Whisker weaving: invalid connectivity resolution and primal construction algorithm. In *Proc. 4th Int. Meshing Roundtable*, pages 115–128, 1995.

[Veg97] Gert Vegter. Computational topology. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 28, pages 517–536. CRC Press, Boca Raton, FL, 1997.

[WE97] Rüdiger Westermann and Thomas Ertl. A multiscale approach to integrated volume segmentation and rendering. *Computer Graphics Forum*, 16(3):C117–C129, September 1997. Proc. 18th Conf. Eur. Assoc. Computer Graphics (Eurographics '97).

[Wee] Jeffrey R. Weeks. SnapPea. http://thames.northnet.org/weeks/index/SnapPea.html.

[Wee98] Jeffrey R. Weeks. Reconstructing the global topology of the universe from the cosmic microwave background. *Classical Quantum Gravity*, 15(9):2599–2604, 1998.

[Wes94] Rüdiger Westermann. A multiresolution framework for volume rendering. In Arie Kaufman and Wolfgang Krueger, editors, *Proc. Symp. Volume Visualization*, pages 51–58. ACM SIGGRAPH, October 1994.

[WJ95] David M. Weinstein and Christopher R. Johnson. Hierarchical data structures for interactive volume visualization. Technical Report UUCS-95-012, Univ. of Utah, Dept. of Computer Science, 1995.

[Yao94] Andrew Chi-Chih Yao. Decision tree complexity and Betti numbers. In *Proc. 25th Symp. Theory of Computing*, pages 615–624. ACM, 1994.

[Živ97] Rade T. Živaljević. Topological methods. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 11, pages 209–224. CRC Press, Boca Raton, FL, 1997.

[ZSD+99] Denis Zorin, Peter Schröder, Tony DeRose, Jos Stam, Joe Warren, and Henrik Weimer. *Subdivision for Modeling and Animation*. SIGGRAPH '99 Courses, no. 37. ACM SIGGRAPH, 1999.