Geometry and Topology from Point Cloud Data

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Problems

- Two and Three dimensions:

Problems

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  - Curve and surface reconstruction

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- High dimensions:

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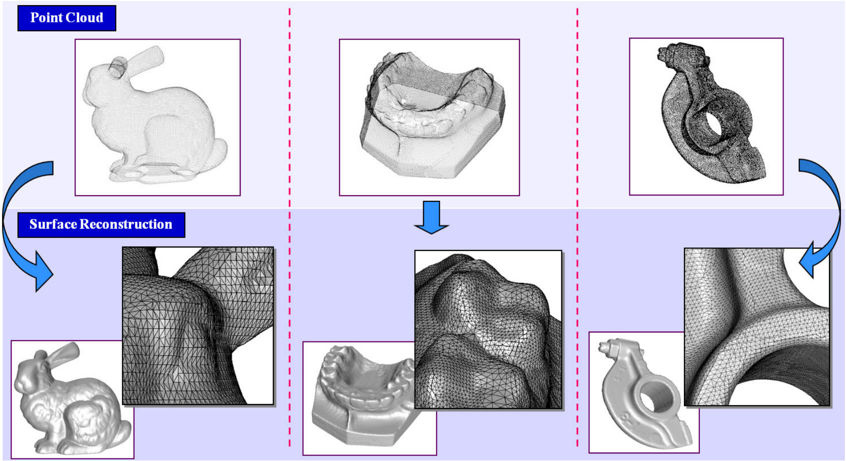
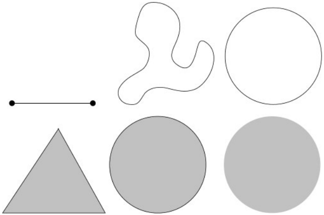
- Two and Three dimensions:
  - Curve and surface reconstruction
- High dimensions:
  - Manifold reconstruction
  - Homological attributes computation

Surface Reconstruction



Basic Topology

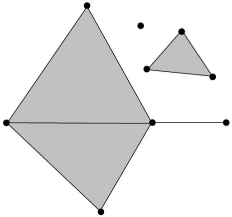
- d-ball B^d $\{x \in \mathbb{R}^d \mid ||x|| \leq 1\}$
- d-sphere S^d $\{x \in \mathbb{R}^d \mid ||x|| = 1\}$
- Homeomorphism $h : T_1 \rightarrow T_2$ where h is continuous, bijective and has continuous inverse



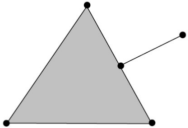
- k-manifold: neighborhoods homeomorphic to open k-ball
    - 2-sphere, torus, double torus are 2-manifolds
- k-manifold with boundary: interior points, boundary points
    - B^d is a d-manifold with boundary where $bd(B^d) = S^{d-1}$

Basic Topology

- Smooth Manifolds
- Triangulation
  - k-simplex
  - Simplicial complex K:
    (i) $t \in K$ if t is a face of $t' \in K$
    (ii) $t_1, t_2 \in K \Rightarrow t_1 \cap t_2$ is a face of both
  - K is a triangulation of a topological space T if $T \approx |K|$

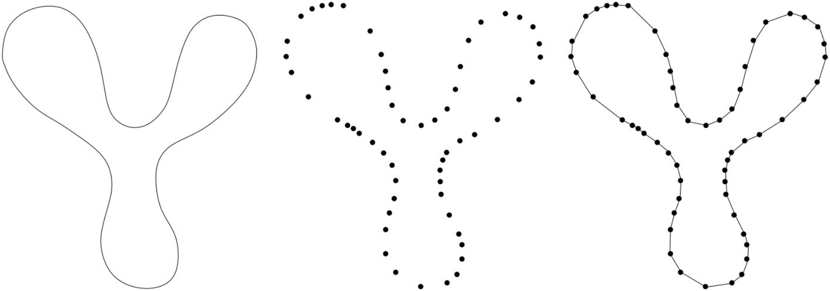


simplicial complex



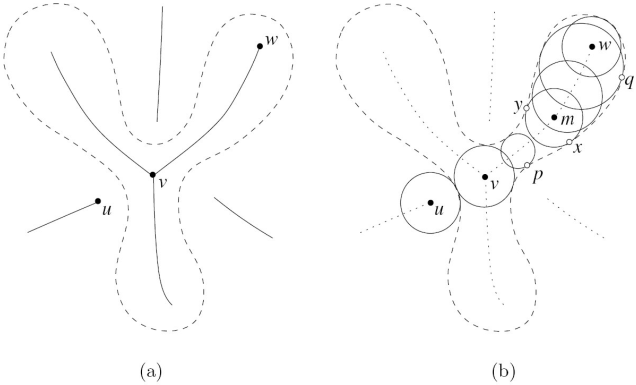
not a simplicial complex

Sampling



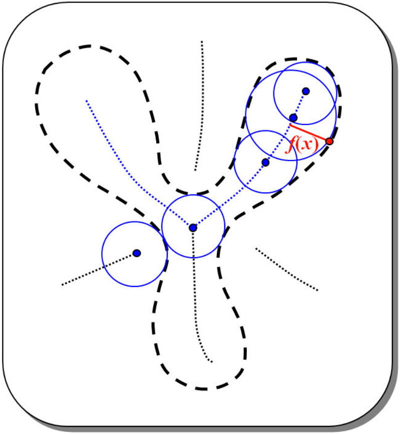
A curve in the plane   A sample   Reconstructed curve

Medial Axis



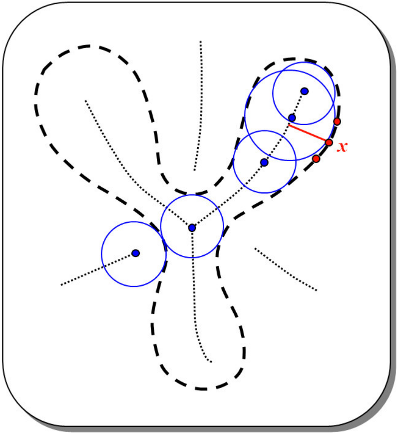
(a) A subset of the medial axis of the curve (b) medial ball centered at v touches the curve in three points, whereas the ones with centers u and w touch it in only one point and coincide with the curvature ball.

Local Feature Size



- $f(x)$ is the distance to medial axis

ε-sample (Amenta-Bern-Eppstein 98)



- Each x has a sample within $\varepsilon f(x)$ distance

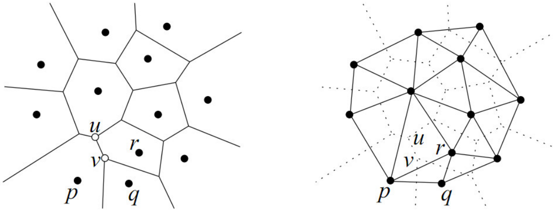
Voronoi Diagram & Delaunay Triangulation

Definition

Voronoi diagram $\mathrm{Vor}\, P$: collection of Voronoi cells $\{V_p\}$ and its faces
$V_p = \{x \in \mathbb{R}^3 \mid ||x - p|| \leq ||x - q|| \text{ for all } q \in P\}$

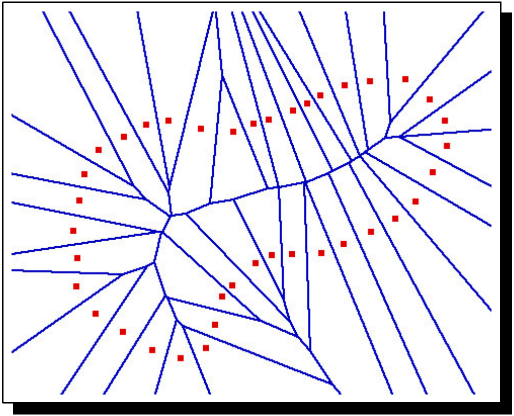
Definition

Delaunay triangulation $\mathrm{Del}\, P$: dual of $\mathrm{Vor}\, P$, a simplicial complex



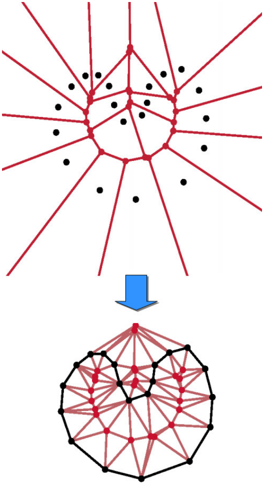
Voronoi diagram and Delaunay triangulation of a point set in the plane

Curve samples and Voronoi



Curve Reconstruction Algorithms

- Crust algorithm
  (Amenta-Bern-Eppstein 98)



Curve Reconstruction Algorithms

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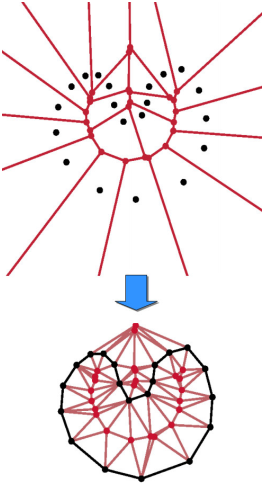
- Nearest neighbor algorithm
  (Dey-Kumar 99)



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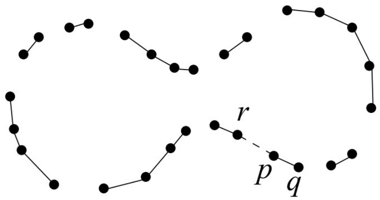
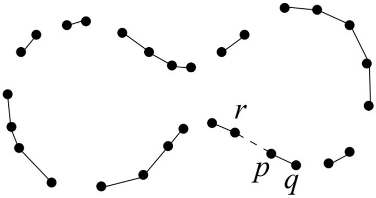


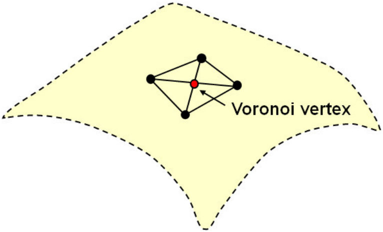
Curve Reconstruction Algorithms

- Crust algorithm (Amenta-Bern-Eppstein 98)
- Nearest neighbor algorithm (Dey-Kumar 99)
- many variations (DMR99,Gie00,GS00,FR01,AM02..)

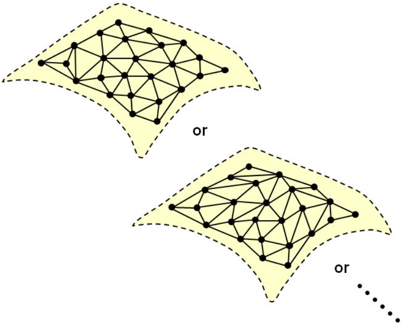


Difficulties in 3D

- Voronoi vertices can come close to the surface . . . slivers are nasty



- There is no unique 'correct' surface for reference



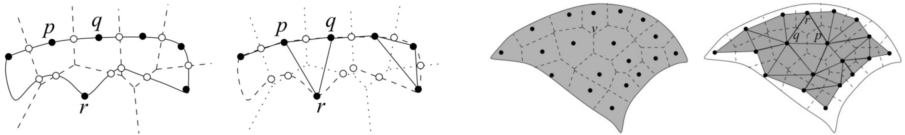
Restricted Voronoi/Delaunay

Definition

Restricted Voronoi: $\mathrm{Vor}\, P|_\Sigma = \{f_P|_\Sigma = f \cap \Sigma \mid f \in \mathrm{Vor}\, P\}$

Definition

Restricted Delaunay: $\mathrm{Del}\, P|_\Sigma = \{\sigma \mid V_\sigma \cap \Sigma \neq \emptyset\}$



Topology

Closed Ball property (Edelsbrunner, Shah 94)

If restricted Voronoi cell is a closed ball *in each dimension, then* $\mathrm{Del}\, P|_{\Sigma}$ *is homeomorphic to* Σ*.*

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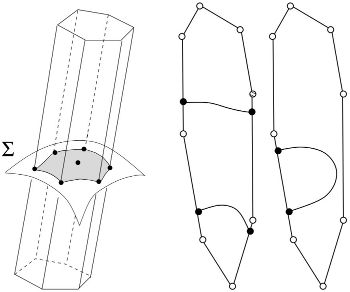


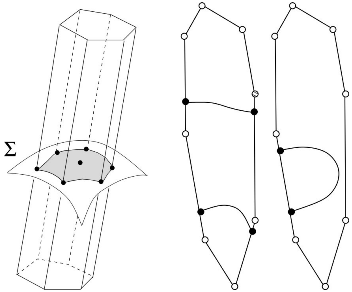
Topology

Closed Ball property (Edelsbrunner, Shah 94)

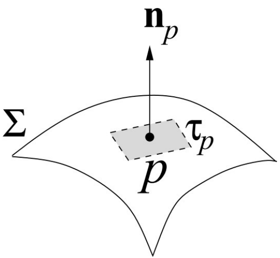
If restricted Voronoi cell is a closed ball *in each dimension, then* $\mathrm{Del}\, P|_{\Sigma}$ *is homeomorphic to* Σ.

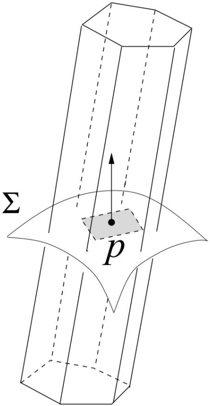
Theorem

For a sufficiently small ε *if* P *is an* ε*-sample of* Σ*, then* (P, Σ) *satisfies the closed ball property, and hence* $\mathrm{Del}\, P|_{\Sigma} \approx \Sigma$.

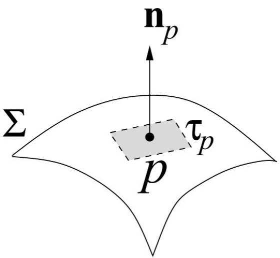


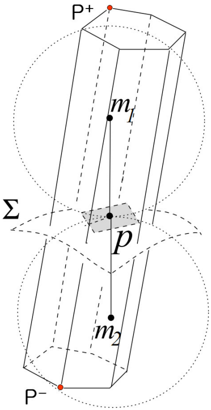
Normals and Voronoi Cells 3D (Amenta-Bern 98)





Long Voronoi cells and Poles

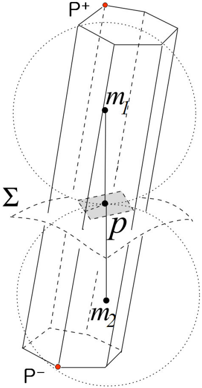




Normal Approximation

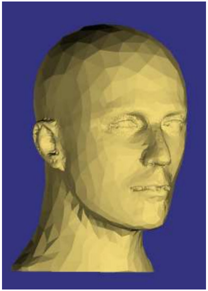
Lemma (Pole Vector)

$\angle((p^+ - p), \mathbf{n}_p) = 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$



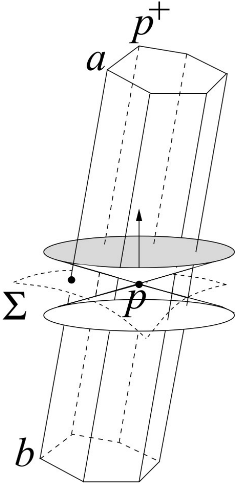
Crust in 3D (Amenta-Bern 98)

- Compute Voronoi diagram $\mathrm{Vor}\, P$
- Recompute the Voronoi diagram after introducing poles
- Filter crust triangles from Delaunay
- Filter by normals
- Extract manifold



Cocone

- $\mathbf{v}p = p^+ - p$ is the pole vector
- Space spanned by vectors within the Voronoi cell making angle $> \frac{3\pi}{8}$ with $\mathbf{v}_p$ or $-\mathbf{v}_p$



Cocone Algorithm

$\textsc{Cocone}(P)$

```
compute Vor P;
T = ∅;
for each p ∈ P do
    T_p = CandidateTriangles(V_p);
    T := T ∪ T_p;
end for
M := ExtractManifold(T);
output M
```

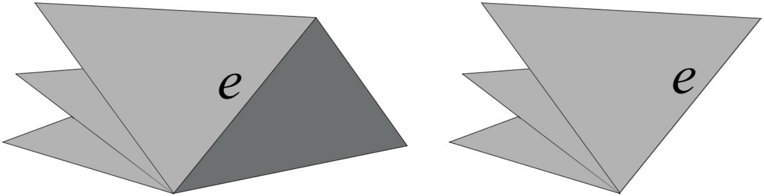
Candidate Triangle Properties

The following properties hold for sufficiently small ε ($\varepsilon < 0.06$)

- Candidate triangles include the restricted Delaunay triangles
- Their circumradii are small $O(\varepsilon)f(p)$
- Their normals make only $O(\varepsilon)$ angle with the surface normals at the vertices
- Candidate triangles include restricted Delaunay triangles

Manifold Extraction: Prune and Walk

Remove **Sharp** edges with their triangles



Walk outside or inside the remaining triangles

Homeomorphism

Let M be the triangulated surface obtained after the manifold extraction.

Define $h : \mathbb{R}^3 \to \Sigma$ where $h(q)$ is the closest point on Σ. h is well defined except at the medial axis points.

Lemma (Homeomorphism)

The restriction of h to M, $h : M \to \Sigma$, is a homeomorphism.

Cocone Guarantees

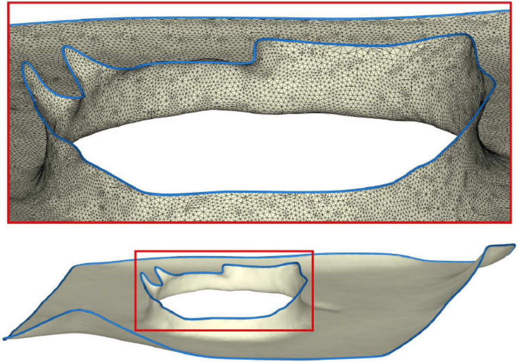
Theorem

Any point $x \in \Sigma$ is within $O(\varepsilon)f(x)$ distance from a point in the output. Conversely, any point of the output surface has a point $x \in \Sigma$ within $O(\varepsilon)f(x)$ distance for $\varepsilon < 0.06$.

Theorem (Amenta-Choi-Dey-Leekha)

The output surface computed by COCONE *from an ε − sample is homeomorphic to the sampled surface for $\varepsilon < 0.06$.*

Boundaries



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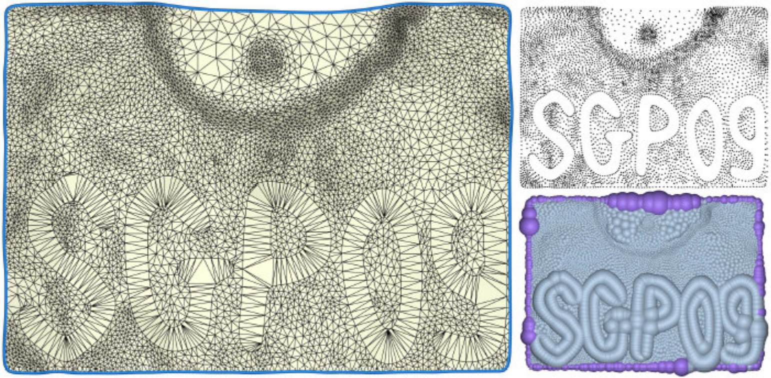
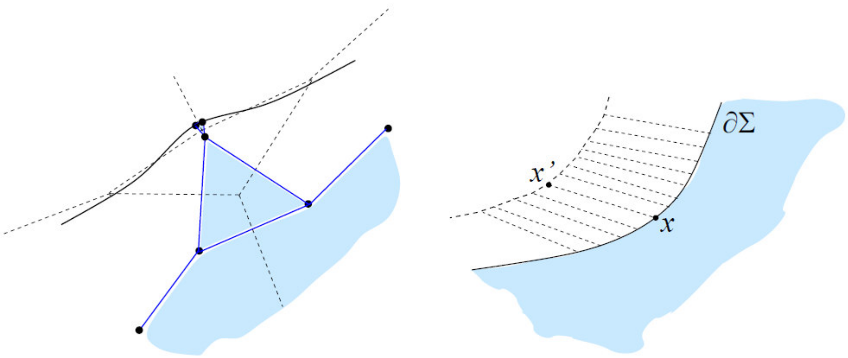


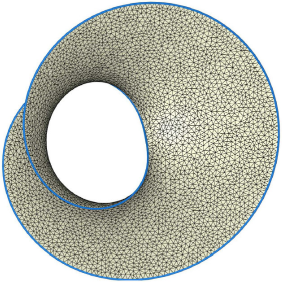
- Ambiguity in reconstruction

Boundaries



- Non-homeomorphic Restricted Delaunay [DLRW09]

Boundaries



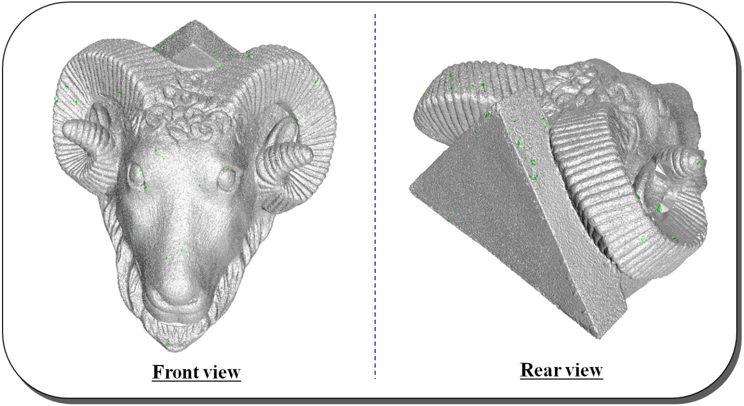
- Non-orientabilty

Boundaries

Theorem (Dey-Li-Ramos-Wenger 2009)

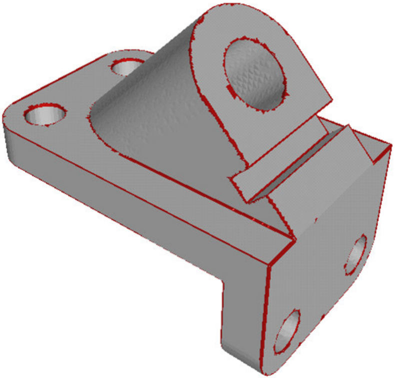
Let P be a sample of a smooth compact Σ with boundary where $d(x, P) \leq \varepsilon\rho$, $\rho = \inf_x \mathrm{lfs}(x)$. For sufficiently small $\varepsilon > 0$ and $6\varepsilon\rho \leq \alpha \leq 6\varepsilon\rho + O(\varepsilon\rho)$, $\textsc{Peel}(P, \alpha)$ computes a Delaunay mesh isotopic to Σ.

Noisy Data: Ram Head



- Hausdorff distance $d_H(P, \Sigma)$ is $\varepsilon f(p)$
- Theoretical guarantees [Dey-Goswami04, Amenta et al.05]

Nonsmoothness



- Guarantee of homeomorphism is open

High Dimensional PCD

- Curse of dimensionality (intrinsic vs. extrinsic)

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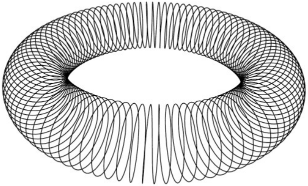

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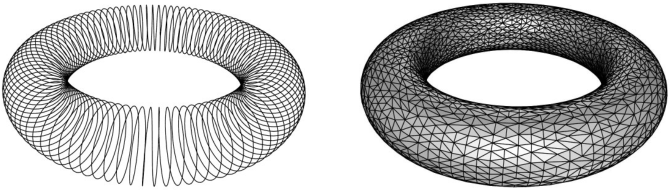

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- Curse of dimensionality (intrinsic vs. extrinsic)
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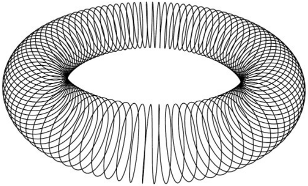
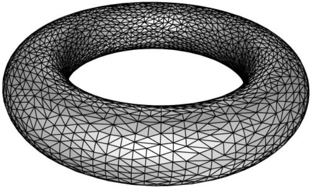
  - Use (ε, δ)-sampling





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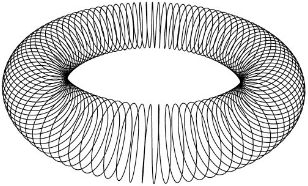
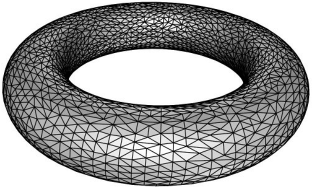
- Restricted Delaunay does not capture topology
  - Slivers are arbitrarily oriented [CDR05] $\Rightarrow \mathrm{Del}\, P|_\Sigma \not\approx \Sigma$ no matter how dense P is.





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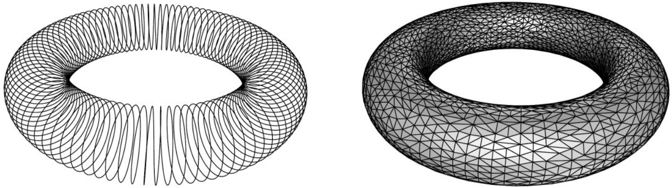
- Restricted Delaunay does not capture topology
  - Slivers are arbitrarily oriented [CDR05] $\Rightarrow \mathrm{Del}\, P|_{\Sigma} \not\approx \Sigma$ no matter how dense P is.
- Delaunay triangulation becomes harder



Reconstruction

Theorem (Cheng-Dey-Ramos 2005)

Given an (ε, δ)-sample P of a smooth manifold $\Sigma \subset \mathbb{R}^d$ for appropriate $\varepsilon, \delta > 0$, there is a weight assignment of P so that $\mathrm{Del}\, \hat{P}|_\Sigma \approx \Sigma$ which can be computed efficiently.

Reconstruction

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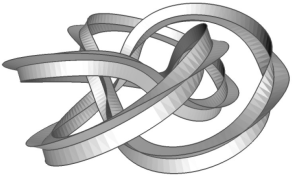
Given an (ε, δ)-sample P of a smooth manifold $\Sigma \subset \mathbb{R}^d$ for appropriate $\varepsilon, \delta > 0$, there is a weight assignment of P so that $\mathrm{Del}\, \hat{P}|_\Sigma \approx \Sigma$ which can be computed efficiently.

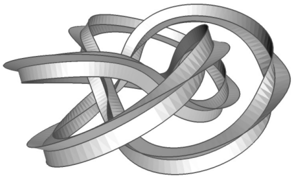
Theorem (Chazal-Lieutier 2006)

Given an ε-noisy sample P of manifold $\Sigma \subset \mathbb{R}^d$, there exists $r_p \leq \rho(\Sigma)$ for each $p \in P$ so that the union of balls $B(p, r_p)$ is homotopy equivalent to Σ.

Reconstructing Compacts

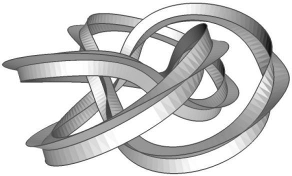


Reconstructing Compacts



- lfs vanishes, introduce μ-reach and define (ε, μ)-samples.

Reconstructing Compacts



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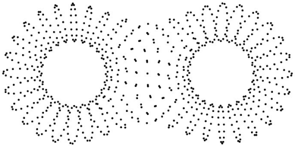
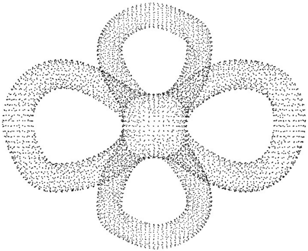
Theorem (Chazal-Cohen-S.-Lieutier 2006)

Given an (ε, μ)-sample P of a compact $K \subset \mathbb{R}^d$ for appropriate $\varepsilon, \mu > 0$, there is an α so that union of balls $B(p, \alpha)$ is homotopy equivalent to K^η for arbitrarily small η.

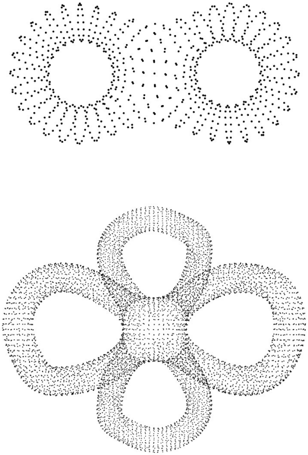
Homology from PCD





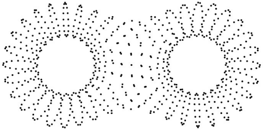
Point cloud

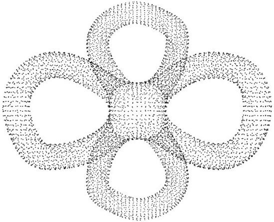
Homology from PCD



Point cloud   Loops

PCD→complex→homology





Point cloud

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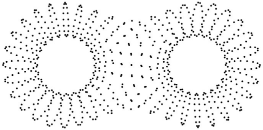
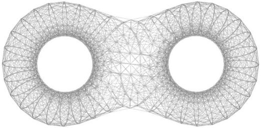
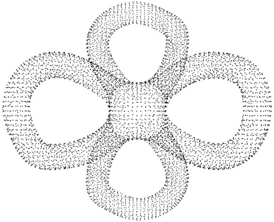
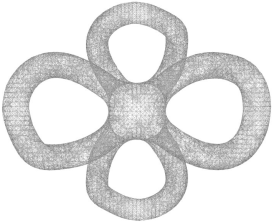




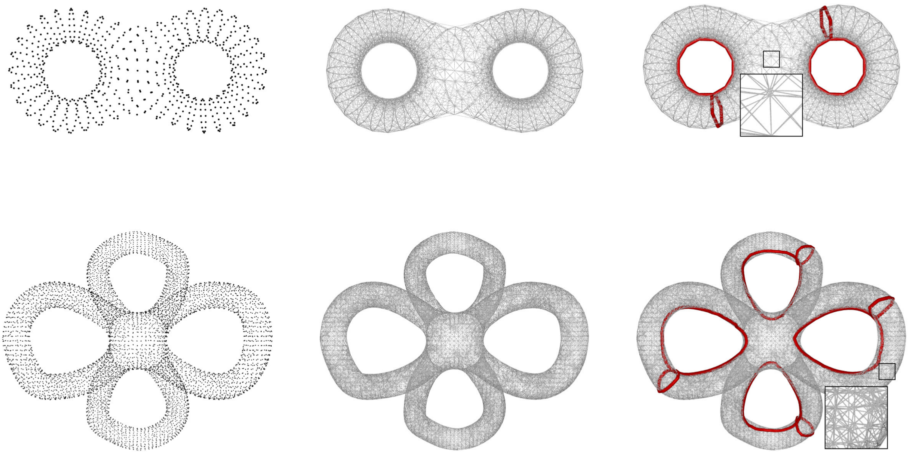




Point cloud

Rips complex

PCD→complex→homology



Point cloud | Rips complex | Loops

Boundary

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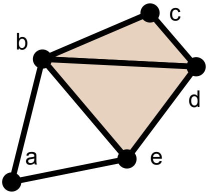

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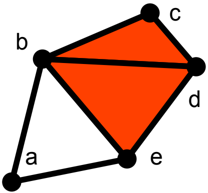

2-chain $bcd + bde$ (under $\mathbb{Z}_2$)

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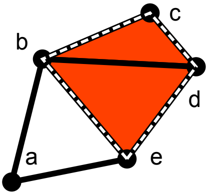

1-boundary $bc+cd+db+bd+de+eb = bc+cd+de+eb = \partial_2(bcd+bde)$ (under $\mathbb{Z}_2$)

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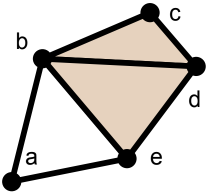

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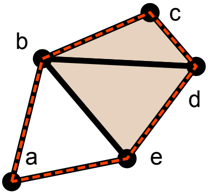

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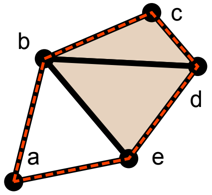

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- Each *p*-boundary is a *p*-cycle: $\partial_p \circ \partial_{p+1} = 0$

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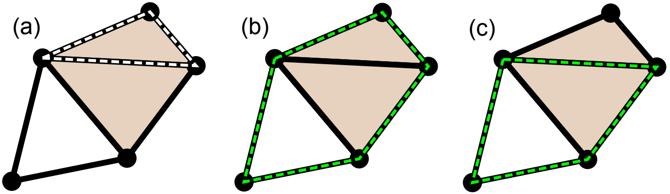

(a) trivial (null-homologous) cycle; (b), (c) nontrivial homologous cycles

Complexes

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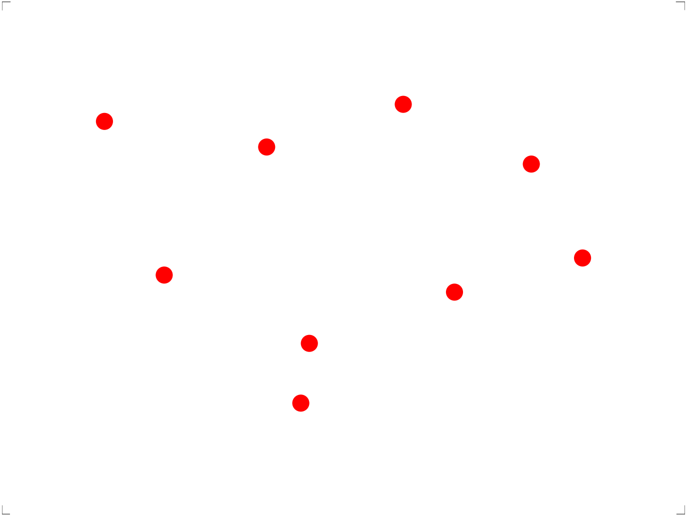
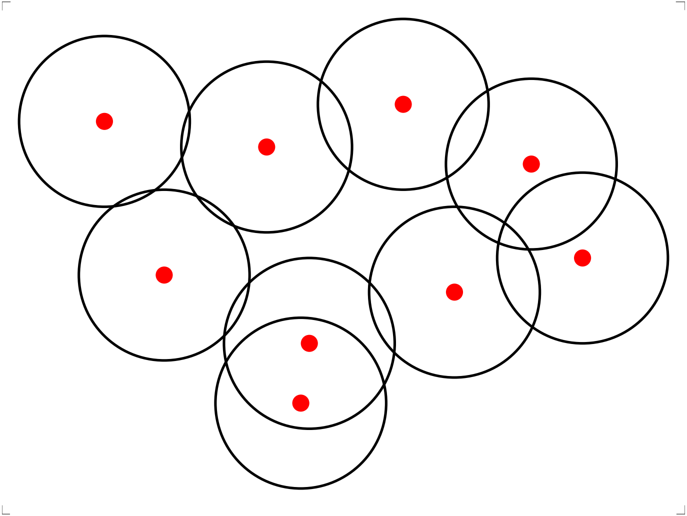
Proposition

For any finite set $P \subset \mathbb{R}^d$ and any $r \geq 0$, $\mathcal{C}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \mathcal{C}^{2r}(P)$

Point set P



Balls $B(p, r/2)$ for $p \in P$



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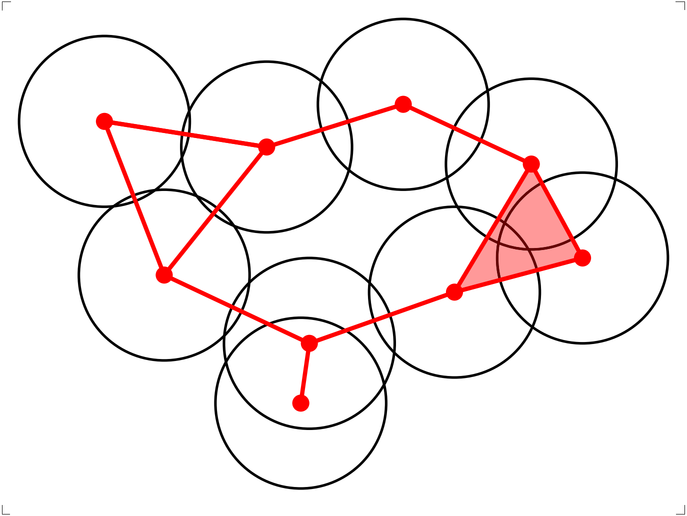
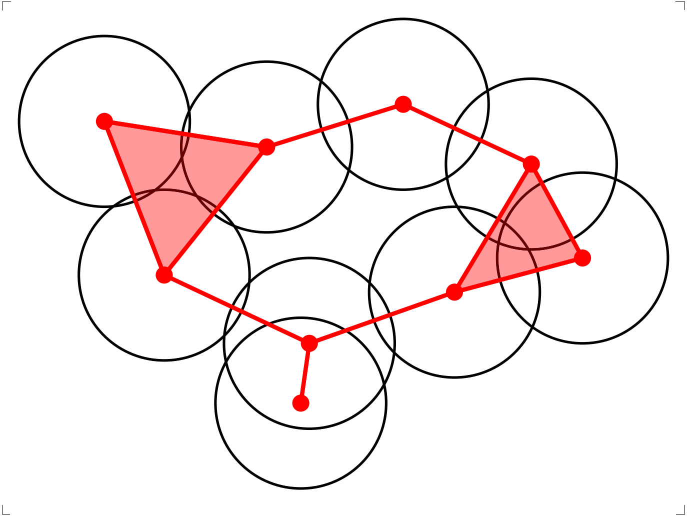


Rips complex $\mathcal{R}^r(P)$



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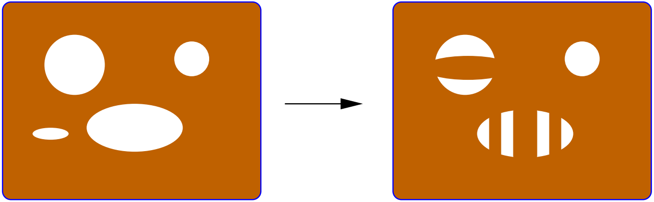

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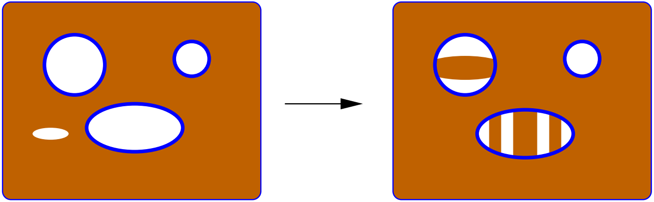

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Theorem (Chazal-Oudot 2008)

Rank of the image of i^* equals the rank of $\mathrm{H}_k(M)$ if P is dense sample of M and r is chosen appropriately.

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Step 2: *Persistent homology* can be computed by the persistence algorithm [Edelsbrunner-Letscher-Zomorodian 2000].

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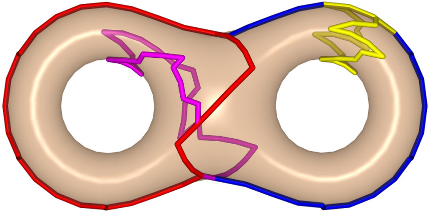

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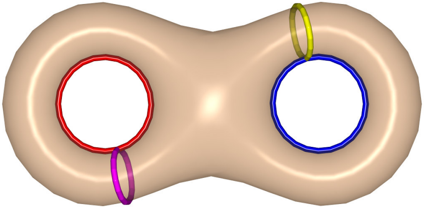

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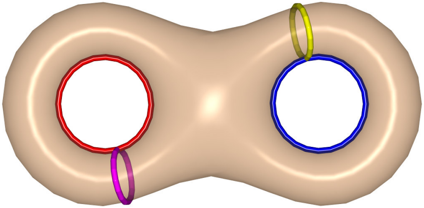

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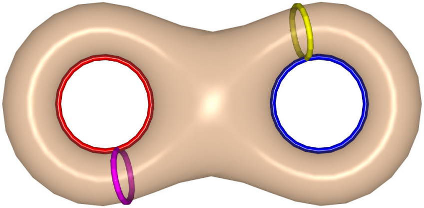

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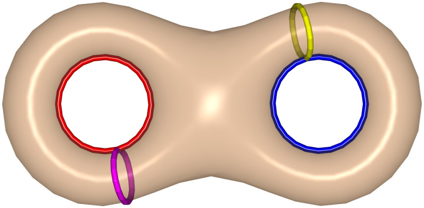

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Definition

A minimal set $\{[g_1], \ldots, [g_k]\}$ generating $\mathsf{H}_1(\mathcal{T})$ is called its basis
Here $k = \operatorname{rank} \mathsf{H}_1(\mathcal{T})$

Shortest Basis

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Definition

A shortest basis of $H_1(\mathcal{T})$ is a set of k loops with minimal length that generates $H_1(\mathcal{T})$

Optimal basis for simplicial complex

Theorem (Dey-Sun-Wang 2010)

Let $\mathcal{K}$ be a finite simplicial complex with non-negative weights on edges. A shortest basis for $\mathrm{H}_1(\mathcal{K})$ can be computed in $O(n^4)$ time where $n = |\mathcal{K}|$

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- Compute a shortest basis of $\mathrm{H}_1(\mathcal{K})$
- Argue that if P is *dense*, a subset of computed loops approximate a shortest basis of $\mathrm{H}_1(\mathcal{M})$ within constant factors

Approximation Theorem

Theorem (Dey-Sun-Wang 2010)

Let $\mathcal{M} \subset \mathbb{R}^d$ be a smooth, closed manifold with l as the length of a shortest basis of $\mathrm{H}_1(\mathcal{M})$ and $k = \operatorname{rank} \mathrm{H}_1(\mathcal{M})$. Given a set $P \subset \mathcal{M}$ of n points which is an ε-sample of $\mathcal{M}$ and $4\varepsilon \leq r \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(\mathcal{M}), \rho_c(\mathcal{M})\}$, one can compute a set of loops G in $O(nn_e^2 n_t)$ time where

$$\frac{1}{1 + \frac{4r^2}{3\rho^2(\mathcal{M})}} l \leq \mathrm{Len(G)} \leq (1 + \frac{4\varepsilon}{\mathrm{r}})\mathrm{l}.$$

Here n_e, n_t are the number of edges and triangles in $\mathcal{R}^{2r}(P)$

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  - persistence, Reeb graphs, Morse-Smale complexes, Laplace spectra...etc.

Thank You