Computational Topology for Data Analysis: Notes from Book by

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Topic 1: Basic Topology

Topology–mainly algebraic topology, is the fundamental mathematical subject that topological data analysis bases on. In this topic note, we introduce some of the very basics of this subject which include definitions and examples of topological spaces and different kinds of maps on them such as homeomorphisms and homotopy.

1.1 Topological space

The basic object in a topological space is a ground set whose elements are called points. A topology on these points specifies how they are *connected* by listing out what points constitute a neighborhood–the so-called an *open set*. The expression "rubber-sheet topology" commonly associated with the term 'topology' exemplifies this idea of neighborhoods. If we bend and stretch a sheet of rubber, it changes shape but always preserves the neighborhoods in terms of the points.

We first introduce basic notions from point set topology. These notions are prerequisites for more sophisticated topological ideas—manifolds, homeomorphism, isotopy, and other maps—used later to study algorithms for topological data analysis. Homeomorphisms, for example, offer a rigorous way to state that an operation preserves the topology of a domain, and isotopy offers a rigorous way to state that the domain can be deformed into a shape without ever colliding with itself.

Perhaps, it is more intuitive to understand the concept of topology in presence of a metric because then we can use the metric balls such as Euclidean balls in an Euclidean space to define neighborhoods–the open sets. Topological spaces provide a way to abstract out this idea without a metric or point coordinates, so they are more general than metric spaces. In place of a metric, we encode the connectedness of a point set by supplying a list of all of the open sets. This list is called a *system* of subsets of the point set. The point set and its system together describe a topological space.

Definition 1 (Topological space)**.** A *topological space* is a point set $\mathbb{T}$ endowed with a *system of subsets* T, which is a set of subsets of $\mathbb{T}$ that satisfies the following conditions.

- $\emptyset, \mathbb{T} \in T$.
- For every $U \subseteq T$, the union of the subsets in U is in T.
- For every finite $U \subseteq T$, the common intersection of the subsets in U is in T.

The system T is called a *topology* on $\mathbb{T}$. The sets in T are called the *open sets* in $\mathbb{T}$. A *neighborhood* of a point $p \in \mathbb{T}$ is an open set containing p.

First, we give examples of topological spaces to illustrate the definition above. These examples have the set $\mathbb{T}$ to be finite.

Example 1. *Let* $\mathbb{T} = \{0, 1, 3, 5, 7\}$*. Then,* $T = \{\emptyset, \{0\}, \{1\}, \{5\}, \{1, 5\}, \{0, 1\}, \{0, 1, 5\}, \{0, 1, 3, 5, 7\}\}$ *is a topology because* $\emptyset$ *and* $\mathbb{T}$ *is in* T *required by the first axiom, union of any sets in* T *is in* T *required by the second axiom, and intersection of any two sets is also in* T *required by the third axiom. However,* $T = \{\emptyset, \{0\}, \{1\}, \{1, 5\}, \{0, 1, 5\}, \{0, 1, 3, 5, 7\}\}$ *is not a topology because the set* $\{0, 1\} = \{0\} \cup \{1\}$ *is missing.*

Example 2. *Let $\mathbb{T} = \{u, v, w\}$. The power set $2^{\mathbb{T}} = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}$ is a topology. For any ground set $\mathbb{T}$, the power set is always a topolgy on it which is called the* discrete topology.

One may take a subset of the power set as a ground set and define a topology as the next example shows. We will recognize later the ground set here correspond to simplices in a simplicial complex and the 'stars' of simplices generate all open sets of a topology.

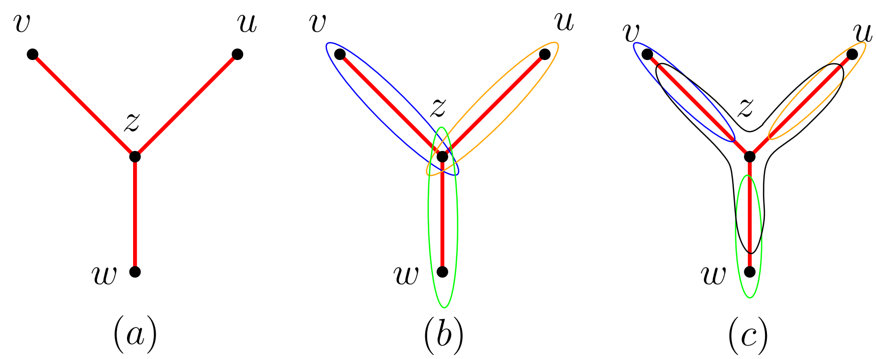


Figure 1.1: Example 3: (a) a graph as a topological space, stars of the vertices as open sets, (b) a closed cover with three elements, (c) an open cover with four elements.

Example 3. *Let $\mathbb{T} = \{u, v, w, z, (u, z), (v, z), (w, z)\}$; this can be viewed as a graph with four vertices and three edges as shown in Figure 1.1. Let*

- *$T_1 = \{\{(u, z)\}, \{(v, z)\}, \{(w, z)\}\}$ and*
- *$T_2 = \{\{(u, z), u\}, \{(v, z), v\}, \{(w, z), w\}, \{(u, z), (v, z), (w, z), z\}\}$.*

Then, $T = \{2^{T_1 \cup T_2}\}$ is a topology because it satisfies all three axioms. All open sets of T are generated by union of elements in $B = T_1 \cup T_2$ and there is no smaller set with this property. Such a set B is called a basis of T. We will see later in the next topic on simplicial complexes that these are open stars *of vertices and edges.*

We now present some more definitions that will be useful later.

Definition 2 (Closure, Closed sets)**.** A set Q is closed if its complement $\mathbb{T} \setminus Q$ is open. The closure $\mathrm{Cl}\, Q$ of a set $Q \subseteq T$ is the smallest closed set containing Q.

In Example 1, the set $\{3, 5, 7\}$ is closed because its complement $\{0, 1\}$ in $\mathbb{T}$ is open. The closure of the open set $\{0\}$ is $\{0, 3, 7\}$ because it is the smallest closed set (complement of open set $\{1, 5\}$) containing 0. In Example 2, all sets are both open and closed. In Example 3, the set $\{u, z, (u, z)\}$ is closed, but the set $\{z, (u, z)\}$ is neither open nor closed. Interestingly, observe that $\{z\}$ is closed. The closure of the open set $\{u, (u, z)\}$ is $\{u, z, (u, z)\}$. In all examples, the sets $\emptyset$ and $\mathbb{T}$ are both open and closed.

Definition 3. Given a topological space $(\mathbb{T}, T)$, the interior $\mathrm{Int}\, A$ of a subset $A \subseteq \mathbb{T}$ is the union of all open subsets of A. The boundary of A is $\mathrm{Bd}\ A = \mathrm{Cl}\, A \setminus \mathrm{Int}\, A$.

The interior of the set $\{3, 5, 7\}$ in Example 1 is $\{5\}$ and its boundary is $\{3, 7\}$.

Definition 4 (Connected)**.** A topological space $(\mathbb{T}, T)$ is disconnected if there are two disjoint non-empty open sets $U, V \in T$ so that $\mathbb{T} = U \cup V$. A topological space is connected if its not disconnected.

The topological space in Example 1 is connected. However, the topological subspace induced by the subset $\{0, 1, 5\}$ is disconnected because it can be obtained as the union of two disjoint open sets $\{0, 1\}$ and $\{5\}$. The topological space in Example 3 is also connected, but the subspace induced by the subset $\{(u, z), (v, z), (w, z)\}$ is disconnected.

Definition 5 (Cover and compactness)**.** An open (closed) cover of a topological space $(\mathbb{T}, T)$ is a collection C of open (closed) sets so that $\mathbb{T} = \bigcup_{c \in C} c$. The topological space $(\mathbb{T}, T)$ is called compact if every open cover C of it has a finite subcover, that is, there exists $C' \subseteq C$ so that $\mathbb{T} = \bigcup_{c \in C'} c$ and C' is finite.

In Figure 1.1(b), the cover consisting of $\{\{u, z, (u, z)\}, \{v, z, (v, z)\}, \{w, z, (w, z)\}$ is a closed cover whereas the cover consisting of $\{\{u, (u, z)\}, \{v, (v, z)\}, \{w, (w, z)\}, \{z, (u, z), (v, z), (w.z)\}$ in Figure 1.1(c) is an open cover. Any topological space with finite point set $\mathbb{T}$ is compact because all of its covers are finite. Thus, all topological spaces in the discussed examples are compact. We will see example of non-compact topological spaces where the ground set is infinite.

Definition 6 (Subspace topology)**.** For every point set $\mathbb{U} \subseteq \mathbb{T}$, the topology T induces a *subspace topology* on $\mathbb{U}$, namely the system of open subsets $U = \{P \cap \mathbb{U} : P \in T\}$. The point set $\mathbb{U}$ endowed with the system U is said to be a *topological subspace* of $\mathbb{T}$.

In Example 1, consider the subset $\mathbb{U} = \{1, 5, 7\}$. It has the subspace topology

$$U = \{\emptyset, \{1\}, \{5\}, \{1, 5\}, \{1, 5, 7\}\}.$$

In Example 3, the subset $\mathbb{U} = \{u, (u, z), (v, z)\}$ has the subspace topology

$$\{\emptyset, \{u, (u, z)\}, \{(u, z)\}, \{(v, z)\}, \{(u, z), (v, z)\}, \{u, (u, z), (v, z)\}\}.$$

In the above examples, the ground set $\mathbb{T}$ is finite. It can be infinite in general and topology may have uncountably infinitely many open sets containing uncountably infinitely many points. We introduce the next concept of *quotient topology* assuming $\mathbb{T}$ is infinite. Given a space $(\mathbb{T}, T)$ and an equivalence relation $\sim$ on elements in $\mathbb{T}$, one can define a topology induced by the original topology T on the quotient set $\mathbb{T}/\sim$ whose elements are equivalence classes $[x]$ for every point $x \in \mathbb{T}$.

Definition 7 (Quotient topology)**.** Given a topological space $(\mathbb{T}, T)$ and an equivalence relation $\sim$ defined on the set $\mathbb{T}$, a quotient space $(\mathbb{S}, S)$ induced by $\sim$ is defined by the set $\mathbb{S} = \mathbb{T}/\sim$ and the quotient topology S where

$$S := \{U \subseteq \mathbb{S} \,|\, \{x : [x] \in U\} \in T\}.$$

We will see the use of quotient topology when we study Reeb graphs.

Infinite topological spaces may seem baffling from a computational point of view, because they may have uncountably infinitely many open sets containing uncountably infinitely many points. The easiest way to define such a topological space is to inherit the open sets from a metric space. A topology on a metric space excludes information that is not topologically essential. For instance, the act of stretching a rubber sheet changes the distances between points and thereby changes the metric, but it does not change the open sets or the topology of the rubber sheet. In the next section, we construct such a topology on a metric space and examine it from the concept of limit points.

1.2 Metric space topology

Metric spaces are a special type of topological space commonly encountered in practice. Such a space admits a *metric* that specifies the scalar *distance* between every pair of points satisfying certain axioms.

Definition 8 (Metric space)**.** A metric space is a pair $(\mathbb{T}, \mathsf{d})$ where $\mathbb{T}$ is a set and d is a distance function $\mathsf{d} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfying the following properties:

- $\mathsf{d}(p, q) = 0$ if and only if $p = q \; \forall p \in \mathbb{T}$;
- $\mathsf{d}(p, q) = \mathsf{d}(q, p) \; \forall p, q \in \mathbb{T}$;
- $\mathsf{d}(p, q) \leq \mathsf{d}(p, r) + \mathsf{d}(r, q) \; \forall p, q, r \in \mathbb{T}$.

In a metric space $\mathbb{T}$, an open *metric ball* with center c and radius r is defined to be the point set $B_o(c, r) = \{p \in \mathbb{T} : \mathsf{d}(p, c) < r\}$. Metric balls define a topology on a metric space.

Definition 9 (Metric space topology)**.** Given a metric space $\mathbb{T}$, all metric balls $\{B_o(c, r) \mid c \in \mathbb{T} \text{ and } 0 < r \leq \infty\}$ and their union constituting the open sets define a topology on $\mathbb{T}$.

All definitions for general topological spaces apply to metric spaces with the above defined topology. However, there are alternative definitions using the concept of limit points which may be more intuitive. More details can be found in the book.

In the Euclidean space $\mathbb{R}^d$ we can use the Euclidean distance to define its metric space topology. On the surface of the coffee mug, we could choose the Euclidean distance too; alternatively, we could choose the *geodesic distance*, namely the length of the shortest path from p to q on the mug's surface.

Example 4 (Euclidean ball)**.** *In $\mathbb{R}^d$, the Euclidean d-ball with center c and radius r, denoted $B(c, r)$, is the point set $B(c, r) = \{p \in \mathbb{R}^d : \mathsf{d}(p, c) \leq r\}$. A 1-ball is an edge, and a 2-ball is called a disk. A unit ball is a ball with radius 1. The boundary of the d-ball is called the Euclidean $(d-1)$-sphere and denoted $S(c, r) = \{p \in \mathbb{R}^d : \mathsf{d}(p, c) = r\}$. The name expresses the fact that we consider it a $(d-1)$-dimensional point set—to be precise, a $(d-1)$-dimensional manifold—even though it is embedded in d-dimensional space. For example, a circle is a 1-sphere, and a layman's "sphere" in $\mathbb{R}^3$ is a 2-sphere. If we remove the boundary from a ball, we have the open Euclidean d-ball $B_o(c, r) = \{p \in \mathbb{R}^d : \mathsf{d}(p, c) < r\}$.*

The topological spaces that are subspaces of a metric space such as $\mathbb{R}^d$ inherit their topology as a subspace topology. Examples of topological subspaces are the Euclidean d-ball $\mathbb{B}^d$, Euclidean d-sphere $\mathbb{S}^d$, open Euclidean d-ball $\mathbb{B}^d_o$, and Euclidean halfball $\mathbb{H}^d$, where

$$\begin{aligned}
\mathbb{B}^d &= \{x \in \mathbb{R}^d \,:\, \|x\| \leq 1\}, \\
\mathbb{S}^d &= \{x \in \mathbb{R}^{d+1} \,:\, \|x\| = 1\}, \\
\mathbb{B}^d_o &= \{x \in \mathbb{R}^d \,:\, \|x\| < 1\}, \\
\mathbb{H}^d &= \{x \in \mathbb{R}^d \,:\, \|x\| < 1 \text{ and } x_d \geq 0\}.
\end{aligned}$$

1.3 Maps, homeomorphisms, and homotopies

Two topological spaces are considered to be the same if the points that comprise them are connected the same way. For example, the boundary of a cube can be deformed into a sphere without cutting or gluing it. They have the same topology. We formalize this notion of topological equality via functions that map the points of one space to points of the other and preserves how they are connected.

First, a function from one space to another that preserves the open sets is called a *continuous function* or a *map*. Continuity is just a step on the way to topological equivalence, because a continuous function can map many points to a single point in the target space, or map no points to a given point in the target space. If the former does not happen—that is, if the function is injective—the function is called an *embedding* of the domain into the target space. True equivalence is marked by a *homeomorphism*, a bijective function from one space to another that possesses both continuity and a continuous inverse, so that open sets are preserved in both directions.

Definition 10 (Continuous function; Map)**.** A function $f : \mathbb{T} \to \mathbb{U}$ from the topological space $\mathbb{T}$ to another topological space $\mathbb{U}$ is *continuous* if for every open set $Q \subseteq \mathbb{U}$, $f^{-1}(Q)$ is open. Continuous functions are also called *maps*.

Definition 11 (Embedding)**.** A map $g : \mathbb{T} \to \mathbb{U}$ is an *embedding* of $\mathbb{T}$ into $\mathbb{U}$ if g is injective.

A topological space can be *embedded* into a Euclidean space by assigning coordinates to its points such that the assignment is continuous and injective. For example, a drawing of a square is an embedding of $\mathbb{S}^1$ into $\mathbb{R}^2$. Not every topological space has an embedding into a Euclidean space, or even into a metric space—there are spaces that cannot be represented by any metric.

Next we define homeomorphism that connects two spaces that have essentially the same topology.

Definition 12 (Homeomorphism)**.** Let $\mathbb{T}$ and $\mathbb{U}$ be topological spaces. A *homeomorphism* is a bijective map $h : \mathbb{T} \to \mathbb{U}$ whose inverse is continuous too.

Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.

Homeomorphism induces an equivalence relation among topological spaces, which is why two homeomorphic topological spaces are called *topologically equivalent*. Figure 1.2 show pairs of topological spaces that are homeomorphic. A less obvious example is that the open d-ball $\mathbb{B}^d_o$ is homeomorphic to the Euclidean space $\mathbb{R}^d$, as demonstrated by the map $h(x) = \frac{1}{1-\|x\|}x$. The same map shows that the halfball $\mathbb{H}^d$ is homeomorphic to the Euclidean halfspace $\{x \in \mathbb{R}^d \,:\, x_d \geq 0\}$.

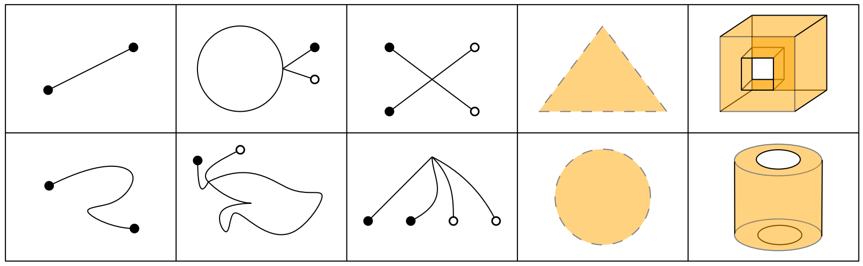
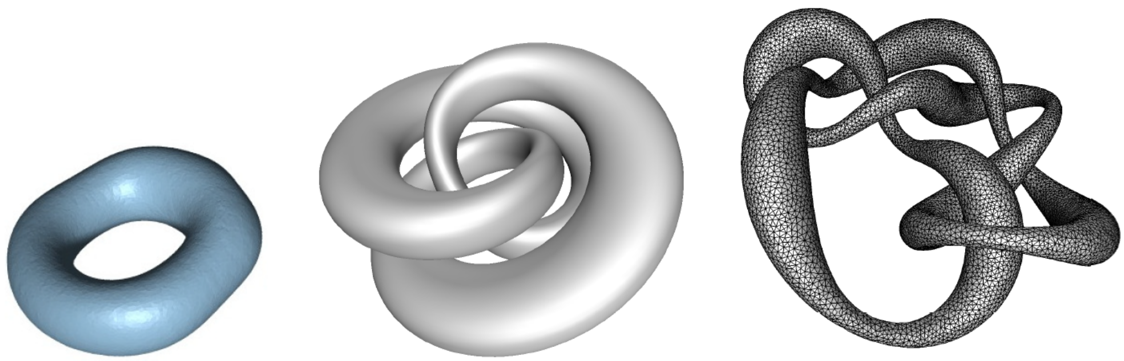


Figure 1.2: Each point set in this figure is homeomorphic to the point set above or below it, but not to any of the others. Open circles indicate points missing from the point set, as do the dashed edges in the point sets second from the right.



(a) (b) (c)

Figure 1.3: Two tori knotted differently, one triangulated and the other not. Both are homeomorphic to the standard unknotted torus on the left.

For maps between compact spaces, there is a weaker condition to be verified for homeomorphism because of the following property.

Proposition 1. *If $\mathbb{T}$ and $\mathbb{U}$ are compact metric spaces, every bijective map from $\mathbb{T}$ to $\mathbb{U}$ has a continuous inverse.*

One can take advantage of this fact to prove that certain functions are homeomorphisms by showing continuity only in the forward direction.

There is another notion of similarity among topological spaces that is weaker than homeomorphism, called *homotopy equivalence*. It relates spaces that can be continuously deformed to one another but may not be homeomorphic. For example, a ball can shrink to a point, but they are not homeomorphic; there is not even a bijective function from an infinite point set to a single point. However, homotopy preserves some aspects of connectedness, such as the number of connected components and the number of holes in a space. Thus a coffee cup is homotopy equivalent to a circle, but not to a ball or a point.

To get to homotopy equivalence, we first need the concept of homotopies, which generalize isotopies so that homeomorphism is not required.

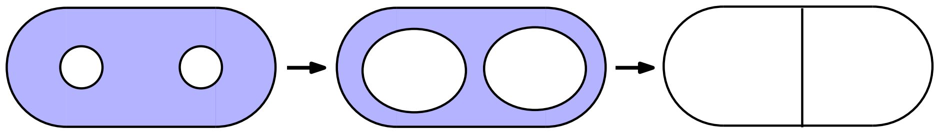


Figure 1.4: All three of these point sets are homotopy equivalent, because they are all deformation retracts of the leftmost point set.

Definition 13 (Homotopy)**.** Let $g : \mathbb{X} \to \mathbb{U}$ and $h : \mathbb{X} \to \mathbb{U}$ be maps. A *homotopy* is a map $H : \mathbb{X} \times [0, 1] \to \mathbb{U}$ such that $H(\cdot, 0) = g$ and $H(\cdot, 1) = h$. Two maps are *homotopic* if there is a homotopy connecting them.

For example, if $g : \mathbb{B}^3 \to \mathbb{R}^3$ is the identity map on a unit ball and $h : \mathbb{B}^3 \to \mathbb{R}^3$ maps every point in the ball to the origin, the fact that g and h are homotopic is demonstrated by the homotopy $H(x, t) = (1 - t) \cdot g(x)$; hence $H(\mathbb{B}^3, t)$ deforms continuously a ball at time zero to a point at time one. A key property of a homotopy is that, as H is continuous, at every time t the map $H(\cdot, t)$ is continuous.

It is more revealing to consider two maps that are not homotopic. Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be the identity map from the circle to itself, and let $h : \mathbb{S}^1 \to \mathbb{S}^1$ map every point on the circle to a single point $p \in \mathbb{S}^1$. Although it is easy to imagine contracting a circle to a point, that image is misleading: the map H is constrained by the requirement that every point on the circle at every time maps to a point on the circle. The circle can contract to a point only if we cut it somewhere, implying that H is not continuous.

Observe that whereas a homeomorphism is a topological relationship between two topological spaces $\mathbb{T}$ and $\mathbb{U}$, a homotopy or an isotopy (which is a special kind of homotopy) is a relationship between two maps, which indirectly establishes a relationship between two topological subspaces $g(\mathbb{X}) \subseteq \mathbb{U}$ and $h(\mathbb{X}) \subseteq \mathbb{U}$. That relationship is not necessarily an equivalence class, but the following relationship is.

Definition 14 (Homotopy equivalent)**.** Two topological spaces $\mathbb{T}$ and $\mathbb{U}$ are *homotopy equivalent* if there exist maps $g : \mathbb{T} \to \mathbb{U}$ and $h : \mathbb{U} \to \mathbb{T}$ such that $h \circ g$ is homotopic to the identity map $\iota_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ and $g \circ h$ is homotopic to the identity map $\iota_{\mathbb{U}} : \mathbb{U} \to \mathbb{U}$.

Whereas homeomorphic spaces have the same dimension, homotopy equivalent spaces sometimes do not. To see that the 2-ball is homotopy equivalent to a single point p, construct a map $h : \mathbb{B}^2 \to \{p\}$ and a map $g : \{p\} \to \mathbb{B}^2$ where $g(p)$ is any point q in $\mathbb{B}^2$. Observe that $h \circ g$ is the identity map on $\{p\}$, which is trivially homotopic to itself. In the other direction, $g \circ h : \mathbb{B}^2 \to \mathbb{B}^2$ sends every point in $\mathbb{B}^2$ to q. There is a homotopy connecting $g \circ h$ to the identity map $id_{\mathbb{B}^2}$, namely the map $H(x, t) = (1 - t)q + tx$.

The definition of homotopy equivalent is somewhat mysterious. A useful intuition for understanding it is the fact that two spaces $\mathbb{T}$ and $\mathbb{U}$ are homotopy equivalent if and only if there exists a third space $\mathbb{X}$ such that both $\mathbb{T}$ and $\mathbb{U}$ are *deformation retracts* of $\mathbb{X}$, illustrated in Figure 1.4.

Definition 15 (Deformation retract)**.** Let $\mathbb{T}$ be a topological space, and let $\mathbb{U} \subset \mathbb{T}$ be a subspace. A *retraction* r of $\mathbb{T}$ to $\mathbb{U}$ is a map from $\mathbb{T}$ to $\mathbb{U}$ such that $r(x) = x$ for every $x \in \mathbb{U}$. The space $\mathbb{U}$ is a *deformation retract* of $\mathbb{T}$ if the identity map on $\mathbb{T}$ can be continuously deformed to a retraction with no motion of the points already in $\mathbb{U}$: specifically, there is a homotopy $R : \mathbb{T} \times [0,1] \to \mathbb{T}$ such that $R(\cdot, 0)$ is the identity map on $\mathbb{T}$, $R(\cdot, 1)$ is a retraction of $\mathbb{T}$ to $\mathbb{U}$, and $R(x,t) = x$ for every $x \in \mathbb{U}$ and every $t \in [0,1]$.

Fact 1. *If $\mathbb{U}$ is a deformation retract of $\mathbb{T}$, then $\mathbb{T}$ and $\mathbb{U}$ are homotopy equivalent.*

For example, any point on a line segment (open or closed) is a deformation retract of the line segment and is homotopy equivalent to it. The letter V is a deformation retract of the letter W, and also of a ball. Moreover, two spaces are homotopy equivalent if they are deformation retractions of a common space. The symbols ∅, ∞, and ⊸⊸ (viewed as one-dimensional point sets) are deformation retracts of a double doughnut—a doughnut with two holes. Therefore, they are homotopy equivalent to each other, although none of them is a deformation retract of any of the others. They are not homotopy equivalent to X, O, ⊕, ⊙, ⊚, a ball, nor a coffee cup.

1.4 Manifolds

A manifold is a set of points that is locally connected in a particular way. A 1-manifold has the structure of a string, possibly with its ends tied forming a loop. A 2-manifold (with boundary) has the structure of a piece of paper or rubber sheet, possibly with the boundaries glued together forming a closed surface—a category that includes disks, spheres, tori, and Möbius bands.

Definition 16 (Manifold)**.** A topological space M is an *m-manifold*, or simply *manifold*, if every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{B}^m_o$ or $\mathbb{H}^m$. The *dimension* of M is m.

A manifold can be viewed as a purely abstract topological space, or it can be embedded into a metric space or a Euclidean space. Even without an embedding, every manifold can be partitioned into boundary and interior points. Observe that these words mean very different things for a manifold than they do for a metric space or topological space.

Definition 17 (Boundary; interior)**.** The *interior* $\operatorname{Int} M$ of an m-manifold M is the set of points in M that have a neighborhood homeomorphic to $\mathbb{B}^m_o$. The *boundary* $\operatorname{Bd} M$ of M is the set of points $M \setminus \operatorname{Int} M$. The boundary $\operatorname{Bd} M$, if not empty, consists of the points that have a neighborhood homeomorphic to $\mathbb{H}^m$. If $\operatorname{Bd} M$ is the empty set, we say that M is *without boundary*.

A single point, a 0-ball, is a 0-manifold without boundary according to this definition. The closed disk $\mathbb{B}^2$ is a 2-manifold whose interior is the open disk $\mathbb{B}^2_o$ and whose boundary is the circle $\mathbb{S}^1$. The open disk $\mathbb{B}^2_o$ is a 2-manifold whose interior is $\mathbb{B}^2_o$ and whose boundary is the empty set. There is some subtlety here: when $\mathbb{B}^2_o$ is viewed as a point set in the space $\mathbb{R}^2$, its boundary is $\mathbb{S}^1$; but viewed as a manifold, its boundary is empty. The boundary of a manifold is *always* included in the manifold.

The open disk $\mathbb{B}^2_o$, the Euclidean space $\mathbb{R}^2$, the sphere $\mathbb{S}^2$, and the torus are all connected 2-manifolds without boundary. The first two are homeomorphic to each other, but the last two are topologically different from the others. The sphere and the torus in $\mathbb{R}^3$ are compact (bounded and closed with respect to $\mathbb{R}^3$ whereas $\mathbb{B}^2_o$ and $\mathbb{R}^2$ are not.

A 2-manifold M is *non-orientable* if, starting from a point p, one can walk on one side of M and end up on the opposite side of M upon returning to p. Otherwise, M is *orientable*. Spheres and balls are orientable, whereas the *Möbius band* in Figure 1.5 is a non-orientable 2-manifold.

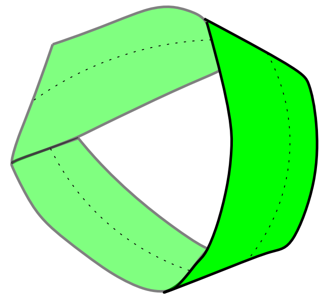
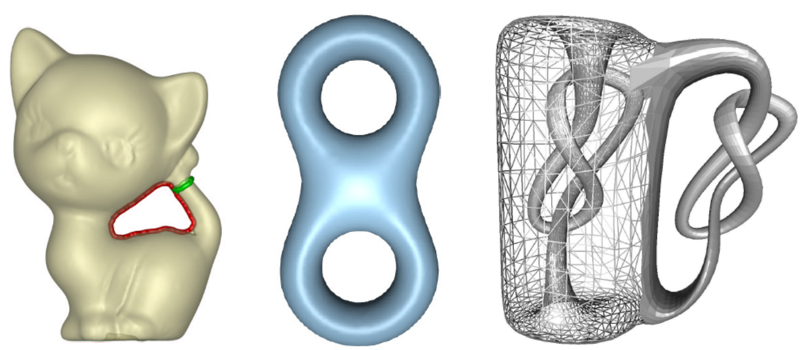


Figure 1.5: Möbius band.

A *surface* is a 2-manifold that is a subspace of $\mathbb{R}^d$. Any compact surface without boundary in $\mathbb{R}^3$ is an orientable 2-manifold. To be non-orientable, a compact surface must have a nonempty boundary (like the Möbius band) or be embedded in a 4- or higher-dimensional Euclidean space.

A surface can sometimes be disconnected by removing one or more *loops* (connected 1-manifolds without boundary) from it. The *genus* of an orientable and compact surface without boundary is g if $2g$ is the maximum number of loops that can be removed from the surface without disconnecting it; here the loops are permitted to intersect each other. For example, the sphere has genus zero as every loop cuts it into two balls. The torus has genus one: a circular cut around its neck and a second circular cut around its circumference, illustrated in Figure 1.6(a), allow it to unfold into a rectangle, which topologically is a disk. A third loop would cut it into two pieces. Figure 1.6(b) shows a 2-manifold without boundary of genus 2. Although a high-genus surface can have a very complex shape, all compact 2-manifolds in $\mathbb{R}^3$ that have the same genus and no boundary are homeomorphic to each other.



(a) (b) (c)

Figure 1.6: (a) Removal of the red and green loops opens up the torus into a topological disk, (b) A double torus: every surface without boundary in $\mathbb{R}^3$ resembles a sphere or a conjunction of one or more tori, (c) Double torus knotted.

1.5 Notes and Exercises

A good source on point set topology is Munkres [4]. The concepts of various maps and manifolds are well described in Hatcher [2]. Books by Guillemin and Pollack [1] and Milnor [3] are good sources for Morse theory on smooth manifolds and differential topology in general.

Exercises

1. A space is called Hausdorff if every two disjoint point sets have disjoint open sets containing them.

   (a) Give an example of a space that is not Hausdorff.

   (b) Give an example of a space that is Hausdorff.

   (c) Show the above examples on the same point set.

3. Show that any metric on a finite set induces a discrete topology.

4. Prove that the metric is a continuous function on the Cartesian space $\mathbb{T} \times \mathbb{T}$ of a metric space $\mathbb{T}$.

5. A space is called *normal* if it is Hausdorff and for any two disjoint closed sets X and Y, there are disjoint open sets $U_X \supset X$ and $U_Y \supset Y$. Show that any metric space and compact space are normal.

6. Deduce that homeomorphism is an equivalence relation. Show that the relation of homotopy among maps is an equivalence relation.

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