Zigzag persistence

Sunday, January 31, 2021 9:01 AM

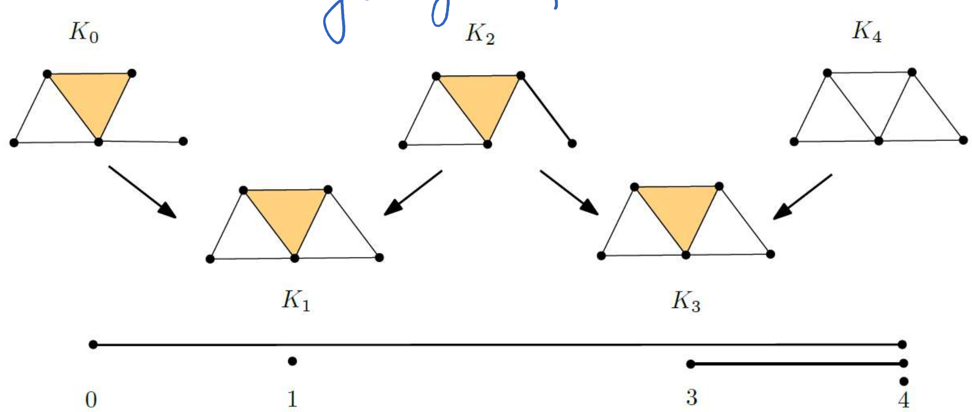
$\mathcal{F}: K_0 \leftrightarrow K_1 \leftrightarrow K_2 \leftrightarrow \cdots \leftrightarrow K_n$

- The arrows can be in both directions

  $K_i \hookrightarrow K_{i+1}$: forward arrow

  $K_i \hookleftarrow K_{i+1}$: backward arrow

- Zigzag filtration



$K_0 \hookrightarrow K_1 \leftrightarrow K_2 \hookrightarrow K_3 \hookleftarrow K_4$

- In general,

  $\mathcal{F}: X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_n$

  $\mathcal{F}$ is a zigzag space or simplicial filtration

  $X_i = \mathbb{T}_i$ $\quad$ $X_i = K_i$

• Zigzag persistence module:

$$H_pF : H_p(X_0) \xleftrightarrow{\varphi_0} H_p(X_1) \xleftrightarrow{\varphi_1} H_p(X_2) \xleftrightarrow{\varphi_2} \cdots \xleftrightarrow{\varphi_{n-1}} H_p(X_n)$$

Definition 89 (Quiver)**.** A quiver $Q = (N, E)$ is a directed graph which can be finite or infinite. A representation $\mathbb{V}(Q)$ of Q is an assignment of a vector space V_i to every node $N_i \in N$ and a linear map $v_{ij} : V_i \to V_j$ for every directed edge $(N_i, N_j) \in E$. Figure 4.5 illustrates representations of two quivers.

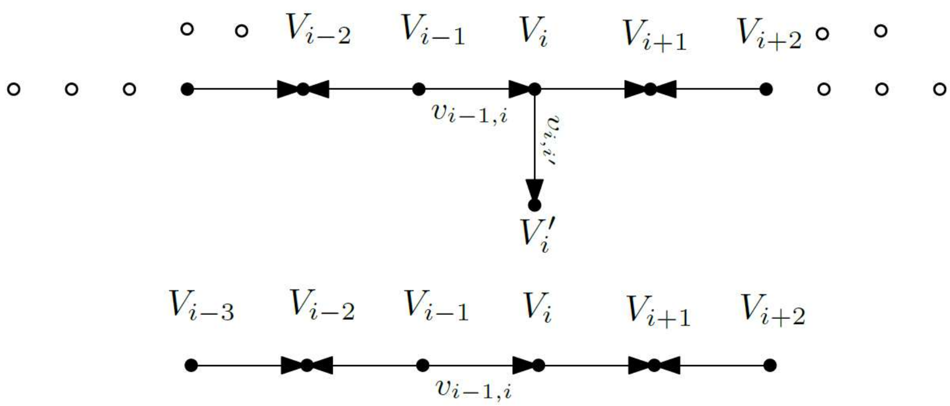


Figure 4.5: A representation of a quiver (top); a representation of an A_n-type quiver (bottom).

• Interval module:

Definition 90 (Interval module)**.** An *interval module* $\mathcal{I}_{[b,d]}$ also called an *interval* or a *bar* over an index set $0, 1, \ldots, n$ with field $\mathbf{k}$ is a sequence of vector spaces

$$\mathcal{I}_{[b,d]} : I_0 \leftrightarrow I_1 \cdots \leftrightarrow I_n$$

where $I_k = \mathbf{k}$ for $b \leq k \leq d$ and $\mathbf{0}$ otherwise with the maps $\mathbf{k} \leftarrow \mathbf{k}$ and $\mathbf{k} \to \mathbf{k}$ being identities.

Theorem: Every quiver representation $\mathbb{V}(Q)$ of A_n-type decomposes into intervals

$$\mathbb{V}(Q) \cong \bigoplus_i \mathcal{J}_{[b_i,d_i]}$$

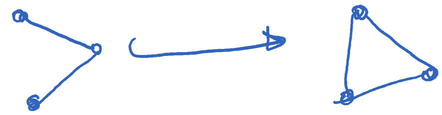
- We can define $[b,d]$ intervals for
  $H_p\mathcal{F}$: zigzag persistence modules
  Then, $\mathrm{Dgm}_p(\mathcal{F})$ is defined with
  intervals $[b_i, d_i]$

Algorithm for simplex-wise zigzag filtration

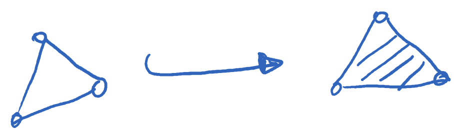
$$\mathcal{F} : \emptyset = K_0 \overset{\varphi_0}{\longleftrightarrow} K_1 \overset{\varphi_1}{\longleftrightarrow} \cdots\cdots \overset{\varphi_{n-1}}{\longleftrightarrow} K_n$$

- More complicated than standard persistence
- An algorithm based on maintaining representative cycles for bars is presented in the note (book)
- The algorithm processes the filtration from left to right. We have five Cases
- φ_i = isomorphic (nothing to do....)

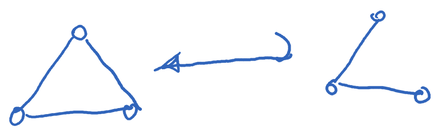
ϕ_i is forward and injective : Simplex insertion and a new cycle is born



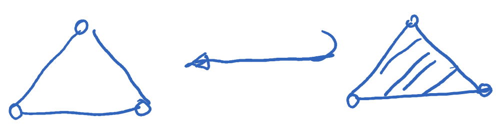
ϕ_i is forward and is surjective : simplex insertion and a cycle (class) is killed



ϕ_i is backward and is injective : Simplex deletion and a cycle (class) is destroyed



ϕ_i is backward and is surjective : Simplex deletion and a cycle (class) is created.



We need to maintain representative cycles consistently which is done by the algorithm.

Zigzag tower

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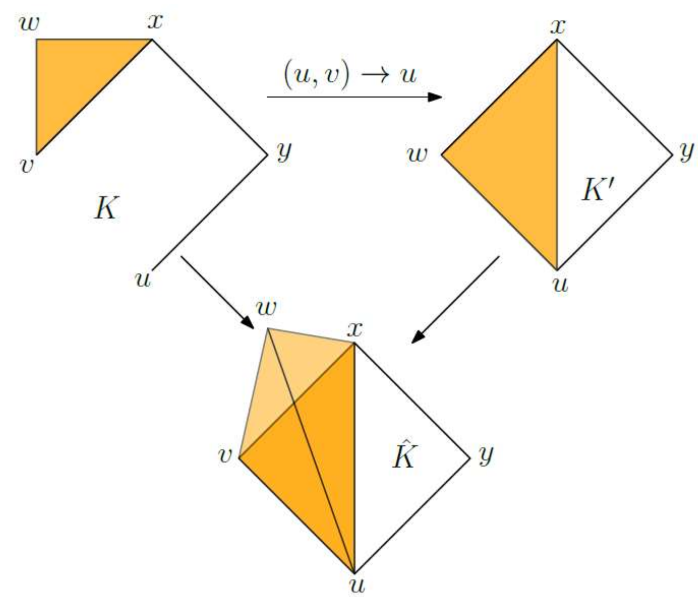
Zigzag Tower

$\mathcal{K}: K_0 \xleftrightarrow{f_0} K_1 \xleftrightarrow{f_1} \cdots \xleftrightarrow{f_{n-1}} K_n$

f_i : simplicial (assume elementary)

- We convert the tower to a filtration
- First convert elementary collapse to a composition of elementary inclusions



- $\hat{K} = K \cup (u * \overline{St}\, v)$ (coning $*$)

- $K, K' \subseteq \hat{K}$

- $$f_* : H_p(K) \longrightarrow H_p(K')$$

$$\|$$

$$H_p(K) \xrightarrow{i_*} H_p(\hat{K}) \xleftarrow[\simeq]{i'_*} H_p(K')$$

$$f_* = (i'_*)^{-1} \circ i_*$$

(1) $H_pK : H_p(K_0) \overset{f_{0*}}{\longleftrightarrow} H_p(K_1) \overset{f_{1*}}{\longleftrightarrow} H_p(K_2) \; \cdots\cdots \; \overset{f_{n-1*}}{\longleftrightarrow} H_p(K_n)$

$$\begin{array}{ccccc|ccccc}
\mathsf{H}_*(K_i) & \xrightarrow{f_{i*}} & \mathsf{H}_*(K_{i+1}) & \xleftarrow{=} & \mathsf{H}_*(K_{i+1}) & \mathsf{H}_*(K_i) & \xrightarrow{=} & \mathsf{H}_*(K_i) & \xleftarrow{f_{i*}} & \mathsf{H}_*(K_{i+1}) \\
\downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} & \downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} \\
\mathsf{H}_*(K_i) & \xrightarrow{\iota_{i*}} & \mathsf{H}_*(\hat{K}_i) & \xleftarrow{\simeq} & \mathsf{H}_*(K_{i+1}) & \mathsf{H}_*(K_i) & \xrightarrow{\simeq} & \mathsf{H}_*(\hat{K}_{i+1}) & \xleftarrow{\iota_{i*}} & \mathsf{H}_*(K_{i+1})
\end{array}$$

$f_i : K_i \longrightarrow K_{i+1}$ $\qquad$ $f_i : K_i \longleftarrow K_{i+1}$

- $\hat{K}_i = K_{i+1}$ if f_i inclusion
- $\hat{K}_{i+1} = K_i$ if f_i inclusion

- We convert H_pK to

(2) $$\mathsf{H}(K_0) \xrightarrow{g_0} \mathsf{H}(S_0) \xleftarrow{h_0} \mathsf{H}(K_1) \xrightarrow{g_1} \mathsf{H}(S_1) \xleftarrow{h_1} \mathsf{H}(K_2) \xrightarrow{g_2} \cdots \xleftarrow{h_{n-1}} \mathsf{H}(K_n)$$

where $g_i = f_i$, h_i = equality, $S_i = K_{i+1}$ if f_i forward

g_i = equality, $h_i = f_i$, $S_i = K_i$ if f_i backward

- Isomorphic module

(3) $$\mathsf{H}_*(K_0) \longrightarrow \mathsf{H}_*(T_0) \longleftarrow \mathsf{H}_*(K_1) \longrightarrow \mathsf{H}_*(T_1) \longleftarrow \mathsf{H}_*(K_2) \longrightarrow \cdots \longleftarrow \mathsf{H}_*(K_n)$$

(3) $H_*(K_0) \longrightarrow H_*(T_0) \longleftarrow H_*(K_1) \longrightarrow H_*(T_1) \longleftarrow H_*(K_2) \longrightarrow \cdots \longleftarrow H_*(K_n)$

$T_i = \hat{K}_i$ if f_i forward

$T_i = \hat{K}_{i+1}$ if f_i backward

$$\begin{tikzcd}
(2) & H_*(K_0) \arrow[r, "g_0"] \arrow[d, "="] & H_*(S_0) \arrow[d, "\simeq"] & H_*(K_1) \arrow[l, "h_0"'] \arrow[r, "g_1"] \arrow[d, "="] & H_*(S_1) \arrow[d, "\simeq"] & H_*(K_2) \arrow[l, "h_1"'] \arrow[r, "g_2"] \arrow[d, "="] & \cdots\cdots & H_*(K_n) \arrow[l] \arrow[d, "="] \\
(3) & H_*(K_0) \arrow[r] & H_*(T_0) & H_*(K_1) \arrow[l] \arrow[r] & H_*(T_1) & H_*(K_2) \arrow[l] \arrow[r] & \cdots\cdots & H_*(K_n) \arrow[l]
\end{tikzcd}$$

- $\mathbb{K}: K_0 \overset{f_0}{\longleftrightarrow} K_1 \overset{f_1}{\longleftrightarrow} \cdots\cdots \overset{f_{n-1}}{\longleftrightarrow} K_n$

  $\mathcal{F}: K_0 \hookrightarrow T_0 \hookleftarrow K_1 \hookrightarrow T_1 \hookleftarrow K_2 \cdots \hookleftarrow K_n$

  $T_i = \hat{K}_i$ if f_i forward

  $= \hat{K}_{i+1}$ if f_i backward

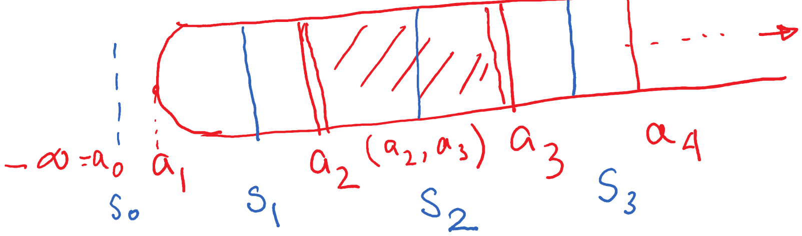
- So, we can apply zigzag algorithm for filtration $\mathcal{F}$ to get barcode for $\mathbb{K}$

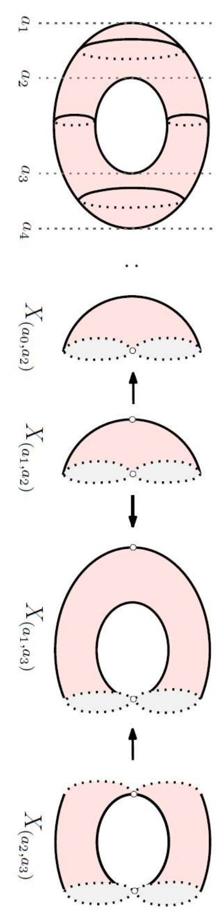
Level set zigzag persistence

- Let $f: X \to \mathbb{R}$ is Morse type function
- Morse type means f has the property:
  - X_s : each level set has finite dimensional homology
  - f has finitely many homological critical values where homology changes
  - $-\infty = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$ critical values

    (a_i, a_{i+1}) : regular interval

    $s_i \in (a_i, a_{i+1})$ regular value





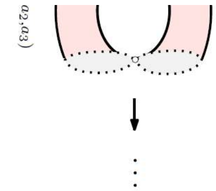
- Level set zigzag filtration

$$X_{[s_0,s_1]} \hookleftarrow \cdots \hookrightarrow X_{[s_{i-1},s_i]} \hookleftarrow X_{=s_i} \hookrightarrow X_{[s_i,s_{i+1}]} \hookleftarrow \cdots \hookrightarrow X_{[s_{n-1},s_n]}$$

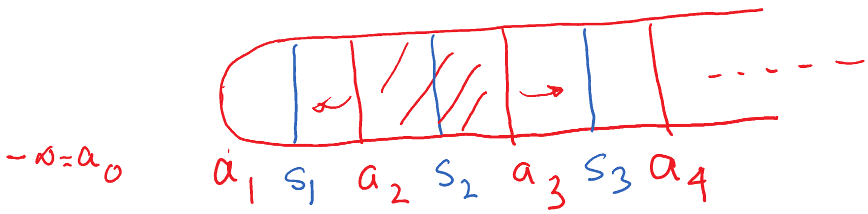
$$X_{[s_0,s_1]} \hookleftarrow X_{=s_1} \hookrightarrow X_{[s_1,s_2]} \hookleftarrow X_{=s_2} \hookrightarrow X_{[s_2,s_3]} \cdots\cdots$$

- To get rid of dependence on choices of s_i

$$\mathbb{X}: X \hookleftarrow \cdots \hookrightarrow X \hookleftarrow X \hookrightarrow X \hookleftarrow \cdots \hookrightarrow X$$



$$\mathbb{X} : X_{(a_0,a_2)} \hookleftarrow \cdots \hookrightarrow X_{(a_{i-1},a_{i+1})} \hookleftarrow X_{(a_i,a_{i+1})} \hookrightarrow X_{(a_i,a_{i+2})} \hookleftarrow \cdots \hookrightarrow X_{(a_{n-1},a_{n+1})}$$



$$\begin{array}{ccccc} X_{(a_1,a_3)} & \hookleftarrow & X_{(a_2,a_3)} & \hookrightarrow & X_{(a_2,a_4)} \\ \wr\| & & \wr\| & & \wr\| \\ X_{[s_1,s_2]} & & X_{=s_2} & & X_{[s_2,s_3]} \end{array}$$

- Zigzag module

$$\mathsf{H}_p\mathbb{X} : \mathsf{H}_p(X_{(a_0,a_2)}) \leftarrow \cdots \rightarrow \mathsf{H}_p(X_{(a_{i-1},a_{i+1})}) \leftarrow \mathsf{H}_p(X_{(a_i,a_{i+1})}) \rightarrow \mathsf{H}_p(X_{(a_i,a_{i+2})}) \leftarrow \cdots \rightarrow \mathsf{H}_p(X_{(a_{n-1},a_{n+1})})$$

Critical space   regular space

- We can talk about interval (bar) decomposition
  - * if the end is in critical space $X_{(a_{i-1}, a_{i+1})}$ then it is a_i
  - * if the end is in regular space $X_{(a_i, a_{i+1})}$ Then it is s_i

- Convert intervals:

  $[a_i, a_j]$ closed-closed interval

• Convert intervals:

$[a_i, a_j] \longleftrightarrow [a_i, a_j]$ closed-closed interval

$[a_i, s_j] \longleftrightarrow [a_i, a_{j+1})$ closed-open interval

$[s_i, a_j] \longleftrightarrow (a_i, a_j]$ open-closed interval

$[s_i, s_j] \longleftrightarrow (a_i, a_{j+1})$ open-open interval

* We want to compute these four types of bars for PL function $f : |K| \to \mathbb{R}$.

• Let $X_{(i,j)} := X_{(a_i, a_j)}$

$$\mathcal{X} : X_{(0,2)} \hookleftarrow \cdots \hookrightarrow X_{(i-1,i+1)} \hookleftarrow X_{(i,i+1)} \hookrightarrow X_{(i,i+2)} \hookleftarrow \cdots \hookrightarrow X_{(n-1,n+1)}$$

$$K_{(i,j)} = \{\sigma \mid f(v) \in (a_i, a_j)\ \forall v \in \mathrm{Vert}(\sigma)\}$$

$$\mathcal{K} : K_{(0,2)} \hookleftarrow \cdots \hookrightarrow K_{(i-1,i+1)} \hookleftarrow K_{(i,i+1)} \hookrightarrow K_{(i,i+2)} \hookleftarrow \cdots \hookrightarrow K_{(n-1,n+1)}$$

Under some compatibity condition which can be enforced by subdivisions $\mathcal{X}$ & $\mathcal{K}$ give isomorphic modules.

* We can expand each inclusion in $\mathcal{K}$ and make simplex-wise

$$\mathcal{F}: \cdots \hookrightarrow K_{(i-1,i+1)} \hookleftarrow \cdots \hookleftarrow K_{\ell-1} \hookleftarrow K_\ell \hookleftarrow \cdots \hookleftarrow K_{(i,i+1)} \hookrightarrow K_{(i,i+2)} \hookleftarrow \cdots$$

* We can compute bars for this simplex-wise zigzag filtration

$$\mathcal{F}: \varnothing = K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_{n-1} \leftrightarrow K_n$$

Barcode: For a bar $[b,d]$ of $\mathcal{F}$, if both K_b and K_d are in the expansion of a same complex, we ignore it. Otherwise:

(Case 1.) K_b is either a regular complex $K_{(i,i+1)}$ or in the expansion of $K_{(i-1,i+1)} \hookleftarrow K_{(i,i+1)}$: the complex K_b is a subcomplex of the critical complex $K_{(i-1,i+1)}$ which stands for the critical value a_i. So, the end b is mapped to a_i and made open because the class for the bar $[b,d]$ does not exist in $K_{(i-1,i+1)}$.
(Case 2.) K_b is either the critical complex $K_{(i,i+2)}$ or in the expansion of $K_{(i,i+1)} \hookrightarrow K_{(i,i+2)}$: the complex is a subcomplex of the critical complex $K_{(i,i+2)}$ which stands for the critical value a_{i+1}. So, the end b is mapped to a_{i+1} and is closed because the class for $[b,d]$ is alive in $K_{(i,i+2)}$.
(Case 3.) K_d is the critical complex $K_{(i-1,i+1)}$ or is in the expansion of the backward inclusion $K_{(i-1,i+1)} \hookleftarrow K_{(i,i+1)}$: the complex is a subcomplex of the critical complex $K_{(i-1,i+1)}$ which stands for the critical value a_i. So, the end d is mapped to a_i and made closed because the class for the bar $[b,d]$ exists in $K_{(i-1,i+1)}$.
(Case 4.) K_d is either the regular complex $K_{(i,i+1)}$ or in the expansion of $K_{(i,i+1)} \hookrightarrow K_{(i,i+2)}$: the complex is a subcomplex of the critical complex $K_{(i,i+2)}$ which stands for the critical value a_{i+1}. So, the end d is mapped to a_{i+1} and is open because the class for $[b,d]$ is not alive in $K_{(i,i+2)}$.

- Connection to Sublevel Set persistence
- $S_{[0,i]} = f^{-1}(-\infty, s_i)$, $s_i \in (a_i, a_{i+1})$

  $K_{[0,i]} = \{\sigma \mid f(v) \leq a_i \ \forall v \in \mathrm{Vert}(\sigma)\}$

$$\mathcal{X} : X_{[0,0]} \to X_{[0,1]} \to \cdots \to X_{[0,n]}$$

$$\mathcal{K} : K_{[0,0]} \to K_{[0,1]} \to K_{[0,2]} \cdots \to K_{[0,n]}$$

Expanding K, we get simplex-wise filtration and $\mathrm{Dgm}_p(K)$.

Theorem 46. *Let $\mathcal{K}$ and $\mathcal{K}'$ denote the filtrations for the sublevel sets and level sets respectively induced by a continuous function f on a topological space with critical values $a_0, a_1, \cdots, a_{n+1}$ where $a_0 = -\infty$ and $a_{n+1} = \infty$. For every $p \geq 0$,*

1. *$[a_i, a_j)$, $j \neq n+1$ is a bar for $\mathrm{Dgm}_p(\mathcal{K})$ iff it is so for $\mathrm{Dgm}_p(\mathcal{K}')$,*
2. *$[a_i, a_{n+1})$ is a bar for $\mathrm{Dgm}_p(\mathcal{K})$ iff either $[a_i, a_j]$ is a closed-closed bar for $\mathrm{Dgm}_p(\mathcal{K}')$ for some $a_j > a_i$, or (a_j, a_i) is an open-open bar for $\mathrm{Dgm}_{p-1}(\mathcal{K}')$ for some $a_j < a_i$.*