Homology Groups

Saturday, January 2, 2021 7:25 PM

Def (Groups): A set G with a binary op. '$+$'.
(i) $a, b \in G \Rightarrow a+b \in G$ (ii) $a+(b+c) = (a+b)+c$ (iii) identity 0, $a+0 = a \; \forall a \in G$ (iv) $\forall a \in G$, inverse $-a$ exists s.t. $a+(-a)=0$

- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_2$ are groups.

Def (homomorphism): $h : \underset{'+'}{G_1} \to \underset{'*'}{G_2}$ is a homomorphism if

$h(a+b) = h(a) * h(b)$.

$h : \mathbb{Z} \to \mathbb{Z}_2$, $h(z) = z \bmod 2$

$h(-2+3) = h(1) = 1$

$h(-2) = 0$, $h(3) = 1$, $h(-2)+h(3) = 1 = h(-2+3)$

Def (Coset): Let $H \subseteq G$ be a subgroup. and G be an abelian group. For $a \in G$, the coset

$aH = \{a+b \mid b \in H\}$

Quotient group $G/H = \{aH \mid a \in G\}$ with ops.

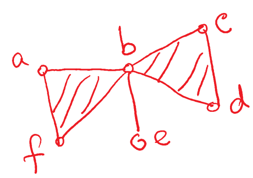
$aH + bH = (a+b)H$

Chains in K

Def (Chains): A p-chain is a formal sum of p-simplices with coefficients in a 'Ring'.

- Coefficient ring we take is $\mathbb{Z}_2$; $0+0=0$, $0+1=1$, $1+1=0$

Example:



2-chain: $abf + bcd$

1-chain: $ab + bc + be + 0.bd + 0.bf + af + cd$

- $c = \sum \alpha_i \sigma_i$, $\alpha_i \in (0,1)$ for $\mathbb{Z}_2$

- $c = \sum \alpha_i \sigma_i$, $c' = \sum \alpha_i' \sigma_i$

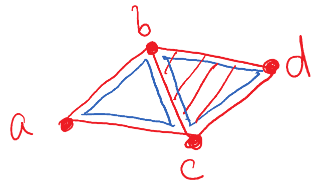
$$c + c' = \sum (\alpha_i + \alpha_i')\sigma_i$$

- $\underbrace{(ab + bc + bd)}_{c} + \underbrace{(ab + be + bd)}_{c'}$

$= (ab + ab) + (bd + bd) + (bc + be)$

$= (bc + be)$

Example



$c = ab + ac + bc$

$c' = bd + cd + bc$

$c + c' = ab + ac + cd + bd$

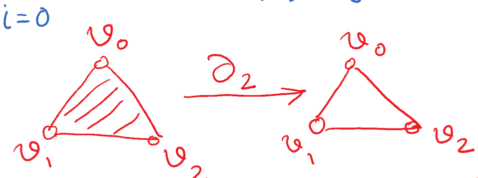
$$C + C = \sum (\alpha_i + \alpha_i)\sigma_i = \sum 0 \cdot \sigma_i = 0$$

p-chains with '+' over $\mathbb{Z}_2$ coefficient forms the p-chain group C_p.

Boundary and cycles

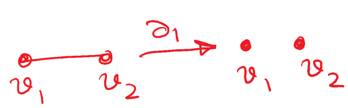
Def (boundary): $\sigma = \{v_0, v_1, \cdots v_p\}$ a p-simplex

$\partial_p \sigma = \sum_{i=0}^{p} \{v_0 \cdots \hat{v_i} \cdots v_p\}$ ($\hat{v_i}$ means v_i is omitted)



$\partial_2 (v_0 v_1 v_2) = v_1 v_2 + v_0 v_2 + v_0 v_1$

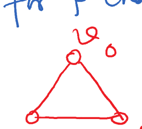
$\partial_1 (v_1 v_2) = v_2 + v_1$



Def (boundary operator): $\partial_p : C_p \to C_{p-1}$, p-chain group to (p-1)-chain group

$\partial_p c = \partial_p (\sum \alpha_i \sigma_i) = \sum \alpha_i \partial_p \sigma_i$ for p-chain $c \in C_p$.

$\partial_1 (v_1 v_2 + v_0 v_2 + v_0 v_1)$



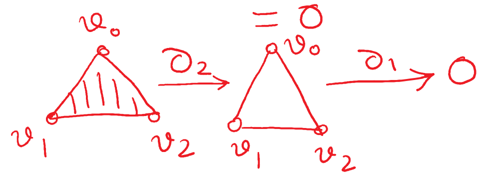
$v_1 \quad v_2$

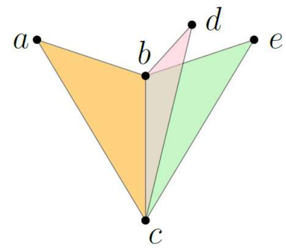
$= (v_1+v_2) + (v_0+v_2) + (v_0+v_1)$

$= 0$

$\partial_1 \partial_2 (v_0 v_1 v_2) = \partial_1 (v_1v_2 + v_0v_2 + v_0v_1)$

$= 0$





$\partial_2(abc + bcd) = ab + ac + cd + bd$ (bc gets cancelled)

$\partial_2(abc + bcd + bce) = ab + bd + be + dc + ec + ac + bc$
(bc is not cancelled because it appears odd # of times in the boundary)

Proposition 1: $\partial_{p-1} \circ \partial_p (c) = 0$

Proof: Sufficient to show $\partial_{p-1} \circ \partial_p(\sigma) = 0$ for every p-simplex σ

$\partial_p(\sigma)$ gives the set of $(p-1)$-faces of σ

∂_{p-1} on these $(p-1)$-faces gives $(p-2)$-faces of σ

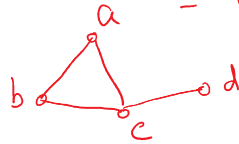
Every $(p-2)$-face is in exactly in 2 $(p-1)$-faces and thus gets cancelled.

Fact
- Group C_p is freely generated by p-simplices. It is an abelian free group.

Any p-chain can be written as a finite sum of p-simplices uniquely. So, they form a basis.

Def (cycle): A p-cycle c_p is a p-chain whose boundary is zero, $\partial_p c_p = 0$.

$$\partial_1(ab+bc+ca) = (a+b)+(b+c)+(c+a) = 0$$



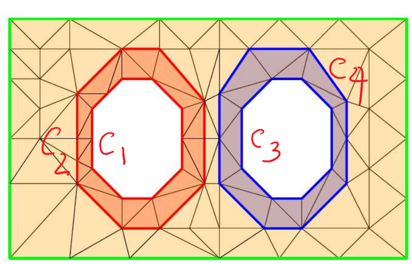
Def (cycle and boundary groups):
A p-cycles together form the p-cycle group Z_p under the addition of chain groups.

- Because of Proposition 1, boundary of a p-chain is a $(p-1)$-cycle
$$\partial_{p-1}(\partial_p c_p) = 0$$

$(p-1)$-chains obtained by applying ∂_p on p-chains are $(p-1)$-boundaries forming the boundary group B_{p-1}.

$$B_{p-1} = \partial_p(C_p) \subseteq C_{p-1}$$

- Since $\partial_{p-1}(B_{p-1}) = \partial_{p-1}(\partial_p(C_p)) = 0$,
$$B_{p-1} \subseteq Z_{p-1} \subseteq C_{p-1}$$



$c_1 + c_2 \in C_1$
$c_1 + c_2$ is a boundary in B_1
$c_3 + c_4 \in B_1$, $c_3 + c_4 \in C_1$
$c_1 \notin B_1$, $c_1 \in Z_1$

Boundary of the complex above: green cycle + inner red and blue cycles.

Boundary of the reddish triangles are the

two 'red' cycles (so two together form boundary)
Boundary of the bluish triangles are the two blue cycles

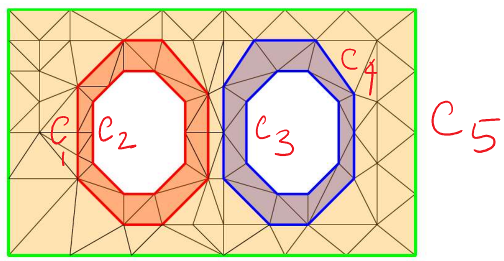
Fact $B_p \subseteq Z_p \subseteq C_p$ and all are free abelian (with $\mathbb{Z}_2$ coefficients)

Homology Groups (cont.)

Wednesday, January 6, 2021 7:08 PM

Def (homology group): $H_p = Z_p / B_p$ is the quotient group.

- Space of cycles up to boundary
- Element of H_p is a coset $c + B_p = [c]$ called ~~equivalent~~ homology class of c
- Equivalent class of c is cosists of all p-cycles that can be obtained by adding c to any boundary p-cycle



- $c_2 \in [c_1]$ because $c_2 = c_1 + \underbrace{(c_1 + c_2)}_{\text{in } B_1}$

  So, $[c_1] = [c_2]$

- $[c_3] = [c_4]$, but $[c_3] \neq [c_1]$
- $c_5 \in [c_2 + c_3]$ because

  $c_5 + c_2 + c_3 \in B_1$, $c_5 = (c_2 + c_3) + \underbrace{(c_2 + c_3 + c_5)}_{\text{in } B_1}$

- c_1 are and c_2 are <u>homologous</u> if $[c_1] = [c_2]$

* The group operation for H_p is given by

$$[c] + [c'] = [c + c']$$

* H_p is free and abelian under Z_2 coefficient.

* It has a basis. It is a vector space.

A set of classes $B \subseteq H_p$ s.t. all classes in H_p can be uniquely expressed as a linear combination of classes in B.

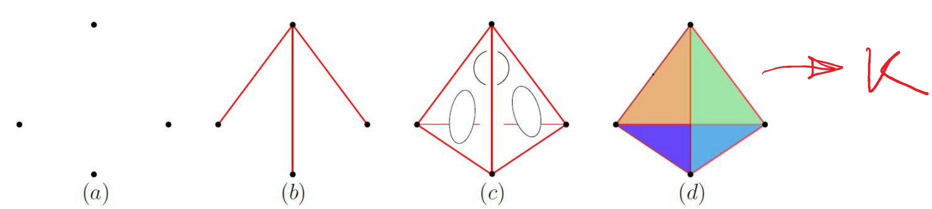
rank $H_p = |B|$ for any basis B.

$\beta_p = \text{rank } H_p = \dim \underbrace{H_p}_{\text{treated as vector space}}$

Betti number

- In the example above, $[c_2], [c_3]$ forms a basis, $[c_2], [c_2+c_3]$ also form a basis, $[c_2], [c_5]$ another basis.

Example :



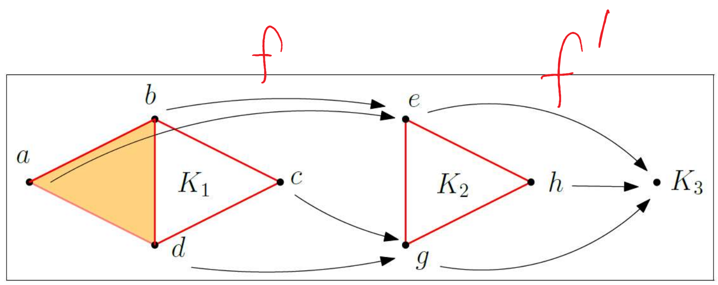
(a) H_0 generated by four vertices, $\beta_0 = 4$

(b) H_0 " " any one of the four vertices because they all are in same class, $\beta_0 = 1$

(c) H_1 has thee independent cycles, fourth one is the linear combination of the others, $\beta_1 = 3, \beta_0 = 1$

(d) H_2 has one 2-cycle, $\beta_2 = 1$, $\beta_1 = 0$, $\beta_0 = 1$

Def (Chain map): $f: K_1 \to K_2$ a simplicial map.
$f_\#: C_p(K_1) \to C_p(K_2)$ induced chain map defined as:
$c = \sum \alpha_i \sigma_i$ a p-chain
$f_\#(c) = \sum \alpha_i \tau_i$ where
$$\tau_i = \begin{cases} f(\sigma_i) & \text{if } f(\sigma_i) \text{ is } p\text{-simplex in } K_2 \\ 0 & \text{o.w.} \end{cases}$$



- $f_\#(ab) = 0$, $f_\#(bd) = eg$, $f_\#(ad) = eg$
- $f'_\#(eg) = 0$, $f'_\#(eh) = 0$
- $f_\#(bc + cd + bd) = eg + eg = 0$

Proposition: $f: K_1 \to K_2$ a simplicial map; $\partial_p^{K_1}$, $\partial_p^{K_2}$ are boundary homomorphisms for dimension p. The following diagram commutes.

$$\begin{array}{ccc} C_p(K_1) & \xrightarrow{f_\#} & C_p(K_2) \\ \downarrow{\scriptstyle \partial_p^{K_1}} & & \downarrow{\scriptstyle \partial_p^{K_2}} \\ C_{p-1}(K_1) & \xrightarrow{f_\#} & C_{p-1}(K_2) \end{array}$$

This means $\partial_p^{K_2}(f_\#(c)) = f_\#(\partial_p^{K_1}(c)) \ \forall c \in C_p(K_1)$

* Since $B_p(K_1) \subseteq Z_p(K_1)$, $f_\#(B_p(K_1)) \subseteq f_\#(Z_p(K_1))$

* Define induced map on quotient space

$$f_*\left(Z_p(K_1)/B_p(K_1)\right) := f_\#(Z_p(K_1)) \,/\, f_\#(B_p(K_1))$$

* From commutative diagram

$$f_\#(Z_p(K_1)) \subseteq Z_p(K_2)$$
$$f_\#(B_p(K_1)) \subseteq B_p(K_2)$$
$$\Downarrow$$
$$f_*: Z_p(K_1)/B_p(K_1) \to Z_p(K_2)/B_p(K_2)$$
$$\Downarrow$$
$$f_*: H_p(K_1) \to H_p(K_2)$$

- A class $c + B_p(K_1)$ is mapped by f_* to

$$f_\#(c) + f_\#(B_p(K_1))$$

Example. $B_1(K_1) = \{0, ab+bd+ad\}$

$c = bd + dc + cb$

$f_*[c] = \{f_\#(c),\ f_\#(c) + f_\#(ab+bd+ad)\}$

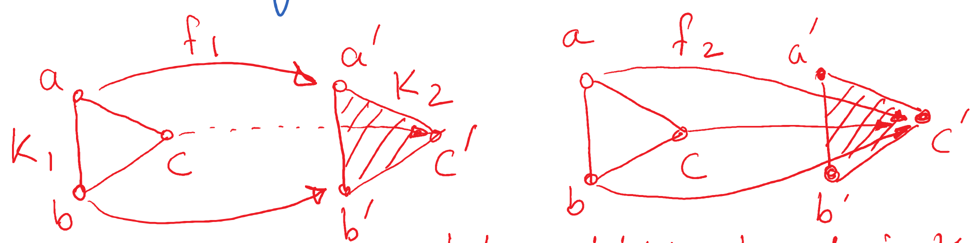
$= \{0,\ 0+0\} = \{0\}$

Def (Contiguous maps): $f_1 : K_1 \to K_2$, $f_2 : K_1 \to K_2$ are contiguous if $\forall \sigma \in K_1$, $f_1(\sigma) \cup f_2(\sigma)$ is a simplex in K_2.

Theorem: $f_1 : K_1 \to K_2$, $f_2 : K_1 \to K_2$ are contiguous,

$$f_{1*} : H_p(K_1) \to H_p(K_2)$$
$$f_{2*} : H_p(K_1) \to H_p(K_2)$$

are equal homomorphisms.



$f_1(ab) = a'b'$
$f_2(ab) = c'$ } $a'b'c'$ is a triangle in K_2

$f_{1*}(H_1(K_1)) = 0$ $\quad f_{1*}([ab + bc + ac]) = [a'b' + b'c' + a'c'] = 0$

$f_{2*}(H_1(K_1)) = 0$ $\quad f_{2*}([ab + bc + ac]) = 0$

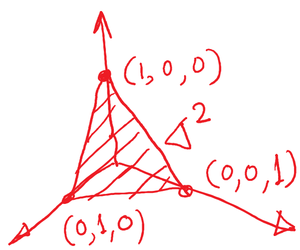
Singular homology ; Cohomology

Saturday, January 9, 2021 1:47 PM

Singular Homology

- Defined for topological space X.
- No simplex here, so simplicial homology so far defined does not make sense
- Δ^p : standard simplex in $\mathbb{R}^{p+1}$

$$\{(x_1, x_2 \cdots x_i \cdots x_{p+1}) \mid x_i = 1,\ x_j = 0 \text{ for } j \neq i\}_{i=1,\dots,p+1}$$



- $\sigma : \Delta^p \to X$ a map from the standard simplex to X.
- $\partial\sigma = \tau_1 + \tau_2 + \cdots + \tau_p$ where $\tau_i : (\partial\Delta^p)_i \to X$,

  restriction of σ on the ith facet $(\partial\Delta^p)_i$.

- Now define chains groups C_p, cycle groups Z_p and boundary groups B_p
- Still $\partial_{p-1} \circ \partial_p (c) = 0$ holds.
- Thus, $B_p \subseteq Z_p$

$H = Z_p / B$

- $H_p = Z_p / B_p$

<u>Proposition</u>: For a triangulable space X, its singular homology is isomorphic to its simplicial homology

$H_p(X) \simeq H_p(K)$ where K is a triangulation of X.

<u>Cohomology</u>

- It is a dual concept to homology group.
- It is denoted H^p
- Under Coefficient field $\mathbb{Z}_2$,

$$H^p(K) \simeq H_p(K) \text{ for all } p \geq 0.$$

- $f : K_1 \to K_2$ simplicial map induces homomorphisms

$$\left.\begin{array}{l} f_* : H_p(K_1) \to H_p(K_2) \\ f^* : H^p(K_2) \to H^p(K_1) \end{array}\right\} \text{reverses direction}$$

<u>Its definition with intuitive examples are given in the book.</u>