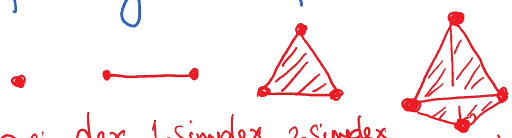
Simplicial Complex

Saturday, January 2, 2021 7:25 PM

Def (Simplex, geometric): K-Simplex σ is the convex hull of $(K+1)$ affinely independent points in $\mathbb{R}^{\geq K}$.

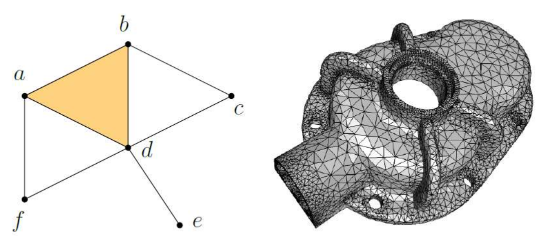
A K'-face of a K-Simplex is a K'-subsimplex.



0-Simplex vertex | 1-Simplex edge | 2-Simplex triangle | 3-Simplex tetrahedron

Def (Geometric simplicial complex): K a collection of geometric simplices:

- every face of $\sigma \in K$ is in K
- $\sigma, \sigma' \in K$ either do not intersect or intersect in a common face.



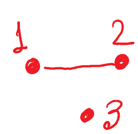
A simplicial complex with 6 vertices, 8 edges, 1 triangle (left); A complex triangulating a shape in 3D.

Def (Abstract simplex): A ground set $V(K)$ called vertex set; K is a collection of subsets of $V(K)$ called simplices in K satisfying:

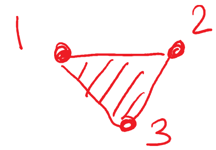
- $\sigma \in K \Rightarrow \sigma' \subseteq \sigma \in K$

Example: $V(K) = \{1, 2, 3\}$

$K = \{\{1\}, \{1,2\}, \{2\}, \{3\}\}$



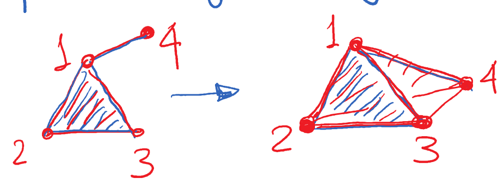
$K = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$



- σ is a k-simplex in K if $|\sigma| = k+1$
- every $\sigma' \subseteq \sigma$ is a face of σ
- $\sigma' \subseteq \sigma$ is facet of σ if $|\sigma'| = k$ and $|\sigma| = k+1$
- σ is a coface of σ' if σ' is a face of σ
- σ has dimension $|\sigma| - 1$
- K is a simplicial m-complex if highest dimension of a simplex in it is m.

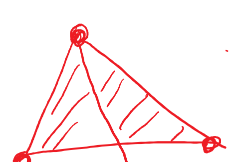
* An abstract simplicial complex with m vertices can always be embedded in $\mathbb{R}^{m-1}$ called its geometric realization.

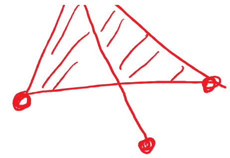
* Just consider a geometric $(m-1)$-simplex in $\mathbb{R}^{m-1}$!
Then, map K bijectively to faces of the simplex



* For every abstract simplicial complex with m vertices there is a canonical geometric realization in $\mathbb{R}^m$

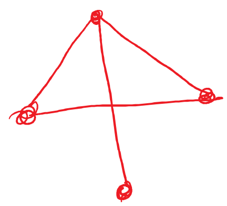
Def (Underlying space) $|K|$ denotes the underlying space of K. It is pointwise union of its canonical geometric realization.





Abstract complex not embedded in $\mathbb{R}^2$. Still we can talk about its underlying space.

Def (K-skeleton): K-skeleton of K, denoted K^k is the subcomplex of $K^k \subseteq K$ consisting of all simplices of dimension at most k.

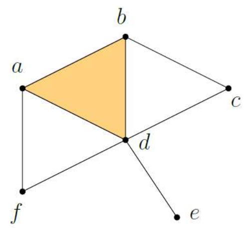


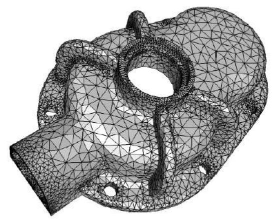
1-skeleton of the complex drawn above

Stars, Links, Simplicial map

Tuesday, January 5, 2021 2:23 PM

Def (star, link): Given $\tau \in K$, $St(\tau) = \{\sigma \mid \tau \subseteq \sigma\}$: set of all simplices containing τ constitute its star $st(\tau)$.





* These stars define the Alexandrov topology

- $St(f) = \{f, (f,d), (f,a)\}$
- $St(a) = \{a, (a,b), (a,d), (a,f), (a,b,d)\}$
- $St(ad) = \{ad, (abd)\}$

Closed star $\overline{St}(\tau)$ is the closure of $st(\tau)$ wrt face relations, ie., $\overline{St}(\tau) = st(\tau) \cup \{\sigma \mid \sigma \subseteq \sigma' \in st(\tau)\}$

- $\overline{St(f)} = st(f) \cup \{a, d\}$
- $\overline{St(ad)} = st(ad) \cup \{ab, bd, a, b, d\}$

__Link__: $Lk(\tau) = \{\sigma \in \overline{St}(\tau) \mid \sigma \cap \tau = \phi\}$

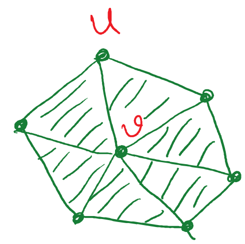
These are simplices in the closed star of τ that are disjoint from τ.

- $Lk(f) = \{a, d\}$
- $Lk(a) = \{b, d, f, bd\}$
- $Lk(ad) = \{b\}$

Def (Triangulation of a manifold): A simplicial k-complex K is a triangulation of a k-manifold if

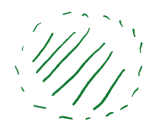
$$|K| \simeq M$$

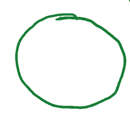
- $|St(v)| \simeq \mathbb{B}_o^k$ (open k-ball) if v maps to interior of M
- $|St(v)| \simeq \mathbb{H}^k$ (open k-half ball) if v maps to $Bd(M)$.
- $|Lk(v)| \simeq \mathbb{S}^{k-1}$ if v maps to interior
- $|Lk(v)| \simeq \mathbb{B}^{k-1}$ (close $(k-1)$-ball) if v maps to $Bd(M)$.



A triangulated 2-ball

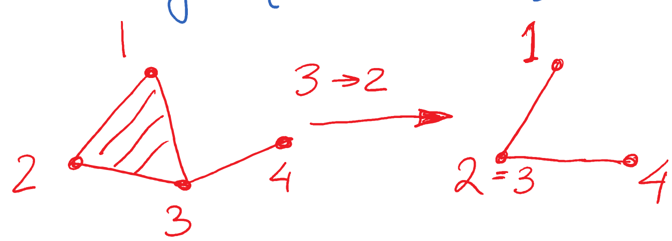
$|St(u)|$ is a open half ball 

$|St(v)|$ is a open 2-ball 

$|Lk(v)|$ is a 1-sphere 

$|Lk(u)|$ is a closed 1-ball 

Def (Simplicial map): $f: K_1 \to K_2$ is simplicial if for every $\{v_0, v_1, \cdots v_k\} \in K_1$, $\{f(v_0) \cdots f(v_k)\} \in K_2$.



$f(123) = \{12\}$

$f(23) = \{2\}$

Def (vertex map) : $f: V(K_1) \to V(K_2)$

* not every vertex map induces a simplicial map

* Simplicial maps are discrete version of continuous maps.

Fact: Every continuous map $f: |K_1| \to |K_2|$ can be arbitrarily approximated by a simplicial map on appropriate subdivisions of K_1 & K_2.

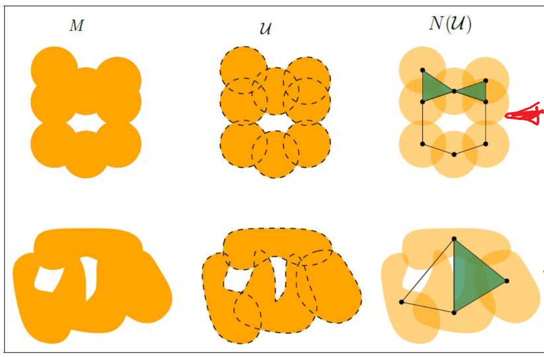
Nerves, Cech , Rips Complex

Tuesday, January 5, 2021 8:35 PM

Def (Nerve): $\mathcal{U} = \{U_i\}_{i \in A}$ finite collection of sets.

Nerve $N(\mathcal{U})$ = simplicial complex with

- $V(N(\mathcal{U})) = A$
- $\sigma = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \in N(\mathcal{U})$ iff $U_0 \cap U_1 \cdots \cap U_k \neq \phi$.



homotopy equivalent to M

not homotopy equivalent

Theorem (Nerve): $\mathcal{U}$ open (or closed) cover of a metric space M. The nerve $|N(\mathcal{U})|$ is homotopy equivalent to M if every non-empty intersection

$$\bigcap_{\alpha \in A} U_\alpha \text{ is homotopy equivalent to a point.}$$

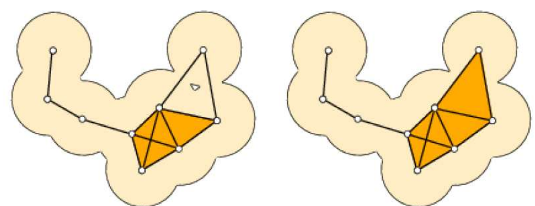
Def (Čech complex): (M, d): metric space. $P \subseteq M$ a point sample. Čech complex $\check{C}^r(P)$ for $r > 0$

- $B(p_i, r) = \{x \in M \mid d(x, p_i) \leq r\}$
- $\mathcal{U} = \{B(p_i, r) \mid p_i \in P\}$
- $\check{C}^r(P) = N(\mathcal{U})$

* If M is Euclidean, $B(p_i, r)$ are convex.
* intersections of convex sets are convex.
* Intersections are contractible
* $\check{C}^r(P)$ is homotopy equivalent to $\bigcup_{p \in P} B(p, r)$

Def (Vietoris-Rips Complex): (P, d): finite metric space.
$VR^r(P)$: σ is a simplex if every edge $pq \subset \sigma$ satisfies $d(p,q) \le 2r$

* Edges determine simplices



Čech Complex   Vietoris-Rips complex

Fact: 1-skeleton of $\check{C}^r(P)$ and $VR^r(P)$ coincide.

Proposition: P a finite subset of (M, d).
interleaving: $\check{C}^r(P) \subseteq VR^r(P) \subseteq \check{C}^{2r}(P)$

* if M is Euclidean
$\check{C}^r(P) \subseteq VR^r(P) \subseteq \check{C}^{\sqrt{2}r}(P)$.

Sparse Complexes

Wednesday, January 6, 2021 7:02 AM

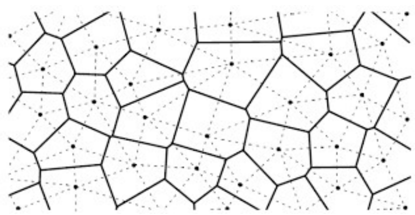
Def (Delaunay Complex): Given $P \subseteq \mathbb{R}^k$. For $p \in P$, let

$$V_p = \{x \mid d(x,p) \leq d(x,q) \ \forall q \in P\}$$

$$\mathrm{Del}\, P = N(\{V_p\})$$

* Collection of V_p constitute Voronoi diagrams

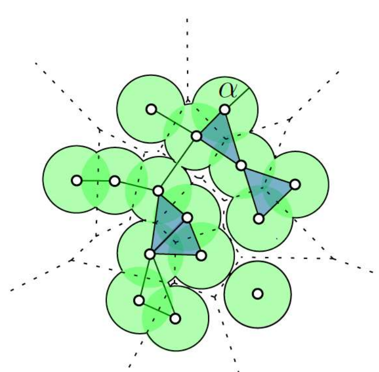
* Delaunay triangulation of P is the nerve of the Voronoi diagram.



Def (Alpha Complex): $\mathrm{Del}^\alpha(P)$: For $\alpha \geq 0$, Let

$$D_p^\alpha = \{x \in B(p,\alpha) \mid d(x,p) \leq d(x,q) \ \forall q \in P\}$$

$$\mathrm{Del}^\alpha(P) = N(\{D_p^\alpha\})$$



* $\mathrm{Del}^\alpha(P) \subseteq \mathrm{Del}(P)$, $\mathrm{Del}^\alpha(P) = \mathrm{Del}(P)$ for $\alpha = \infty$

* $P \subseteq \mathbb{R}^d$, $Del(P)$ can be computed in

$\Theta(n \log n)$ time if $d=2$

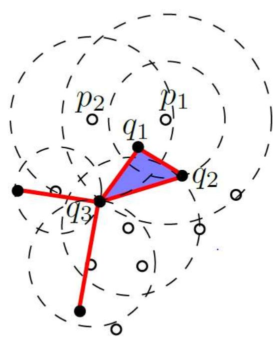
$\Theta(n^2)$ time if $d=3$

$\Theta(n^{\lceil d/2 \rceil})$ time if $d>2$

Witness Complex

Def (weak witness): Given finite metric space (P, d). $Q \subseteq P$: landmarks. A simplex $\{q_1, \cdots q_k\}$, $q_i \in Q$, is weakly witnessed by $x \in P \setminus Q$ if

$\forall q_i$, $d(x, q_i) \leq d(x, p)$ $\forall p \in P \setminus Q$



$q_1 q_2 q_3$ is weakly witnessed by p_1

- Faces of a weakly witnessed simplex may not be weakly witnessed

Def (Witness Complex): $W(Q, P)$ collection of all simplices whose faces are also weakly witnessed.

Let $Q \subseteq \mathbb{R}^d$.

Def (strong witness): $\sigma = \{q_1, \cdots q_k\}$ is strongly witnessed

by $x \in \mathbb{R}^d$ if σ is weakly witnessed and additionally
$$d(q_1, x) = d(q_2, x) = \cdots = d(q_k, x).$$

Witness Complex (cont.)

Wednesday, January 6, 2021 7:41 AM

When $Q \subset \mathbb{R}^d$

- A simplex σ is strongly witnessed iff all of its faces are weakly witnessed.
- $\sigma \in \mathrm{Del}\, Q$ iff σ is strongly witnessed by points in $\mathbb{R}^d$!

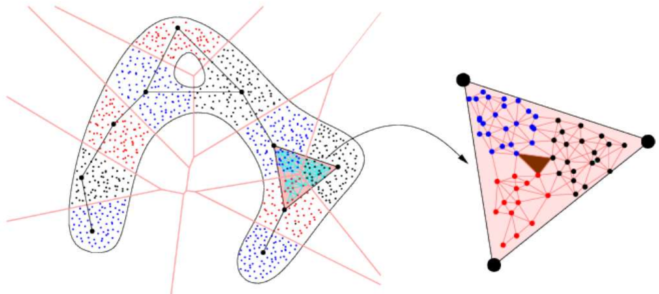
<u>Proposition:</u> If $Q \subseteq P \subset \mathbb{R}^d$, then $W(Q,P) \subseteq \mathrm{Del}\, Q$.

Proposition: (i) $W(Q, \mathbb{R}^d) = \mathrm{Del}\, Q$

(ii) $W(Q,M) = \mathrm{Del}|_M Q$ if $M \subseteq \mathbb{R}^d$ is a smooth 1- or 2-manifold.

Graph Induced Complex (GIC)

Wednesday, January 6, 2021 7:49 AM



Def (GIC): Given • (P,d) finite metric space
• $G(P)$ a graph with vertices in P
• $Q \subseteq P$

Let $\gamma : P \to 2^Q$ nearest point map
$\gamma(p) = \operatorname{argmin} d(p, Q)$

$\mathcal{G}(G(P), Q, d)$ is the complex where $\sigma = \{q_1, \cdots q_k\}$, $q_i \in Q$, is in $\mathcal{G}$ iff $\exists$ clique $\{p_1, \cdots p_k\}$ in $G(P)$ s.t.
$\gamma(p_i) = q_i \ \forall i \in [1, k]$.

* Input Graph can be k-nearest neighbor graph for a point cloud

* Q can be subsampled from P.
  - take q_0 arbitrarily, $Q \leftarrow \{q_0\}$
  - Choose $p \in P \setminus Q$ and delete all $p' \in P$ s.t. $d(p, p') \le \delta$
  - $Q \leftarrow Q \cup \{p'\}$
  - Continue till P is exhausted.

* The above procedure produces a

δ-sparse, δ-sample of P.

$\forall p \in P, \exists q \in Q, d(p,q) \leq \delta$

$\forall q, q' \in Q, d(q,q') \geq \delta$.

* Metric d: It can be the shortest path metric in $G(P)$.