# Topological space(quotient space)

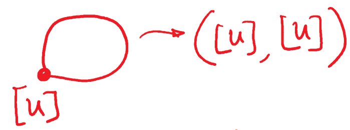
Sunday, January 3, 2021 7:39 PM

Def. (quotient space): '$\sim$' an equivalence relation on $\mathbb{T}$.
$x \in \mathbb{T}$, $[x]$ equivalence class.

- $\mathbb{S} = \mathbb{T}/\sim$ quotient set
- $(\mathbb{S}, S)$ quotient space where

$$S = \{ U \subseteq \mathbb{S} \mid \{x : [x] \in U\} \in T \}$$

Example:


$\mathbb{T} = \{u, v, (u,v)\}$

$T = \{\{u\}, \{v\}, \{(u,v)\}, \{u, (u,v)\}, \{v, (u,v)\}, \{u, v, (u,v)\}\}$

'$\sim$': $u \sim v$, $(u,v) = (u,u)$

$\mathbb{S} = \{[u], ([u],[u])\}$ $\quad S = \{\{[u]\}, \{([u],[u])\}, \{[u], ([u],[u])\}\}$

# Topological space (Metric space)

Sunday, January 3, 2021 7:57 PM

**Def. (Metric):** $(\mathbb{T}, d)$, $d: \mathbb{T} \times \mathbb{T} \to \mathbb{R}$

- (identity) $d(p,q) = 0$ iff $p = q$ $\forall p \in \mathbb{T}$
- (Symmetry) $d(p,q) = d(q,p)$ $\forall p,q \in \mathbb{T}$
- (Triangular inequality) $d(p,q) \leq d(p,r) + d(r,q)$ $\forall p,q,r \in \mathbb{T}$

- Given the three axioms, $d(x,y) \geq 0$ $\forall x,y \in \mathbb{T}$. (prove)
- Examples: in $\mathbb{R}^2$:

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$
$$d_1(x,y) = |x_1-y_1| + |x_2-y_2|$$
$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$$
$$d_\infty(f,g) = \max_x \{|f(x)-g(x)|\}$$

**Def. (Metric space):** $(M,d)$ - the open sets are metric balls
$B_o(c,r) = \{p \in M \mid d(p,c) < r\}$ for $r > 0$

**Examples:** Euclidean space $\mathbb{R}^k$

$B_o^d = \{x \in \mathbb{R}^d \mid \|x\|_2 < 1\}$ d-ball (open)
$S^d = \{x \in \mathbb{R}^d \mid \|x\|_2 = 1\}$ d-sphere
$B^d = \{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$ d-ball (closed)
$H^d = \{x \in \mathbb{R}^d \mid \|x\|_2 < 1,\ x_d \geq 0\}$ Half d-ball

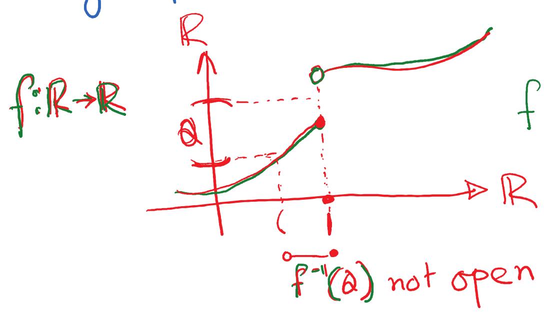
$(P,d)$: discrete points $P$ with interpoint distances


discrete topology.
What are metric balls?

# Maps

Monday, January 4, 2021 9:48 AM

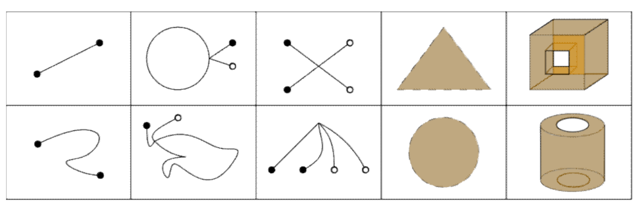
Def (Continuous): $f : \mathbb{T} \to \mathbb{U}$ is continuous if for every open $Q \subseteq \mathbb{U}$, $f^{-1}(Q)$ is open.




$f$ : not continuous

Def (homeomorphism): $h : \mathbb{T} \to \mathbb{U}$ is a homeomorphism if

- $h$ is bijective, continuous
- $h^{-1}$ is continuous

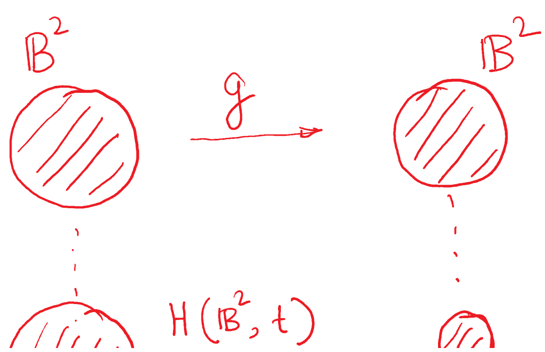


$\mathbb{T}$ & $\mathbb{U}$ are homeomorphic if $\exists$ a homeomorphism $h : \mathbb{T} \to \mathbb{U}$

- If $\mathbb{T}$ & $\mathbb{U}$ are compact metric spaces, then it is sufficient that $h$ is continuous because $h^{-1}$ in that case is necessarily continuous.

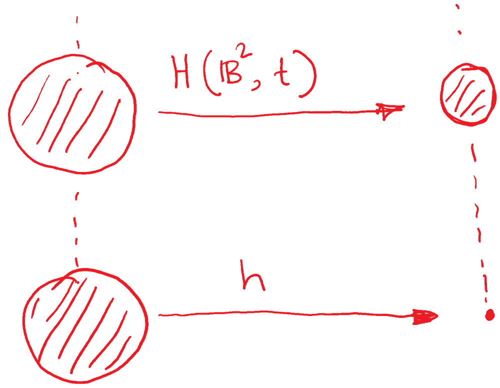
Def (homotopy): $g : \mathbb{X} \to \mathbb{U}$, $h : \mathbb{X} \to \mathbb{U}$ are homotopic if $\exists$ continuous $H : \mathbb{X} \times [0,1] \to \mathbb{U}$ called homotopy s.t.

- $H(\mathbb{X}, 0) = g$, $H(\mathbb{X}, 1) = h$



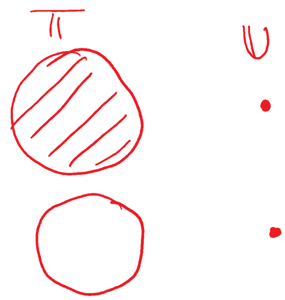

$g$ is identity map

$h$ is constant map




$\mathbb{U}$
$h$ is constant map

Def (homotopy equivalent): $\mathbb{U}$ & $\mathbb{T}$ are homotopy equivalent if $\exists\ g: \mathbb{T} \to \mathbb{U}$, $h: \mathbb{U} \to \mathbb{T}$ s.t. $h \circ g$ is identity $\mathbb{T} \to \mathbb{T}$, and $g \circ h$ is identity $\mathbb{U} \to \mathbb{U}$.

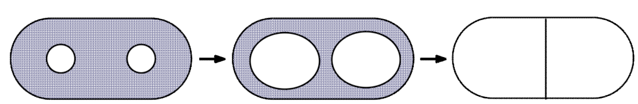



homotopy equivalent

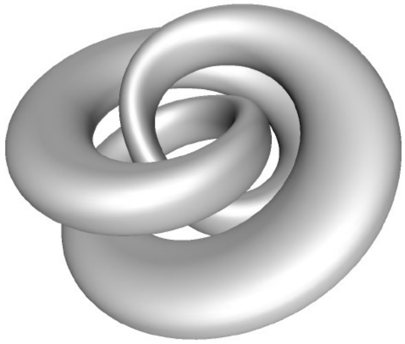
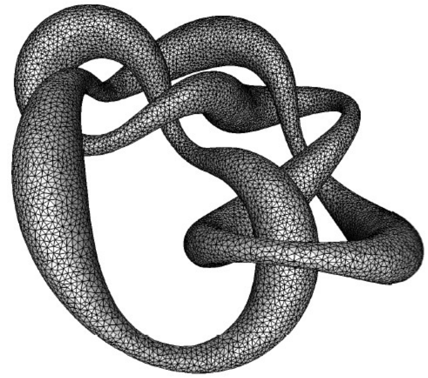
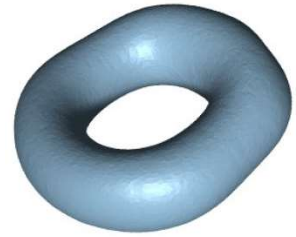
not homotopy equivalent

Def (Deformation retract): $\mathbb{U} \subseteq \mathbb{T}$ is a deformation retract of $\mathbb{T}$ if $\exists$ homotopy $H: \mathbb{T} \times [0,1] \to \mathbb{U}$ with

- $H(\mathbb{T}, 0)$ = identity on $\mathbb{T}$
- $H(\mathbb{T}, 1)$ = retract $r: \mathbb{T} \to \mathbb{U}$, $r(x) = x$ if $x \in \mathbb{U}$
- $H(\mathbb{T}, t)(x) = x$ if $x \in \mathbb{U}$.



- if $\mathbb{U}$ is a deformation retract of $\mathbb{T}$, then $\mathbb{T}$ & $\mathbb{U}$ are homotopy equivalent.

- Homeomorphic spaces, but their embeddings are different. (All are torus)

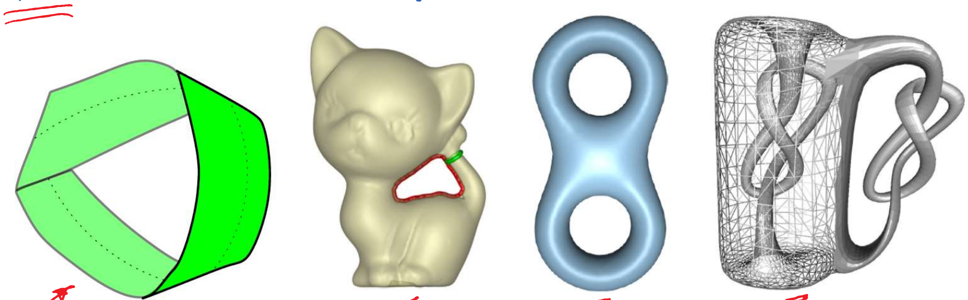
# Manifolds

Tuesday, January 5, 2021 10:10 AM

Def. (Manifold): $M$ is a manifold if every $x \in M$ has a neighborhood homeomorphic to $\mathbb{B}_0^m$ or $\mathbb{H}^m$.

open ball    Half ball

$m$ is the dimension of $M$ called $m$-manifold.



Möbius band    Torus    Double torus

Def (Interior, boundary): points in $M$ with neighborhood of open balls constitute $\operatorname{Int} M$. $\operatorname{Bd} M = M \setminus \operatorname{Int} M$. $\operatorname{Bd} M$ consists of points with neighborhood of $\mathbb{H}^m$.

$\left.\begin{array}{l}\operatorname{Bd} \mathbb{B}^2 \\ \operatorname{Int} \mathbb{B}^2\end{array}\right\}$ closed 2-ball

Orientablety: One can "orient" the manifold to define two "distinct" sides.

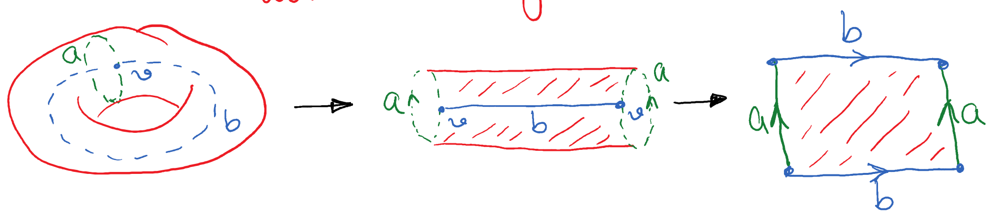
Möbius band is non-orientable.

- All compact 2-manifolds without boundary in $\mathbb{R}^3$ are orientable.
- Compact 2-manifolds without boundary that are non-orientable embed in $\mathbb{R}^4$.

Klein bottle (non-orientable 2-manifold)

Def (Genus): All 2-manifolds are also called surfaces.

- Every 2-manifold can be cut open into a disc.
- min # loops a surface need to be cut to make it a disc equals twice its genus.




- Every genus g surface (orientable without boundary) can be represented with a rectangle of sides 2g, called its polygonal schema.

- Every surface (orientable, without boundary) is a genus g-surface. For g=0, it is sphere, otherwise g-tori.

3-tori