# General Persistence

Sunday, January 24, 2021 11:04 AM

## Towers

For a discrete set $A = a_0 \le a_1 \cdots \le a_n$, we have seen filtration:

$$F : X_{a_0} \hookrightarrow X_{a_1} \hookrightarrow \cdots \hookrightarrow X_{a_n}$$

Now instead of inclusions, consider continuous maps for space filtration and simplicial maps for simplicial filtration

$$x_{ij} : X_{a_i} \rightarrow X_{a_j}$$

$$\mathcal{X} : X_{a_0} \xrightarrow{x_{01}} X_{a_1} \xrightarrow{x_{12}} \cdots \xrightarrow{x_{n-1,n}} X_{a_n}$$

Space or simplicial tower

- Now consider homology groups, we get

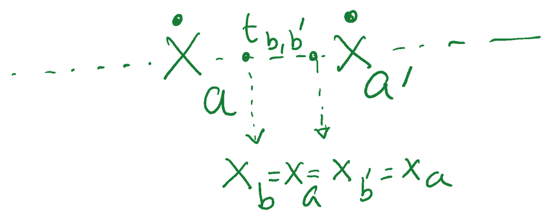
$$H_p\mathcal{X} : H_p(X_{a_0}) \xrightarrow{(x_{01})_*} H_p(X_{a_1}) \xrightarrow{(x_{12})_*} \cdots \xrightarrow{(x_{n-1,n})_*} H_p(X_{a_n})$$

Vector space tower

* Algorithms for 'inclusion' induced filtrations do not work here. Also, we need to define the 'stability' aspect again.

**Definition 1** (Tower). A *tower* with resolution $r \geq 0$ and indexed in $A = \{a \in \mathbb{R} \mid a \geq r\}$ is any collection $\mathsf{T} = \{T_a\}_{a \geq r}$ of objects $T_a$, $a \in A$, together with maps $t_{a,a'} : T_a \to T_{a'}$ so that $t_{a,a} = \mathrm{i}d$ and $t_{a',a''} \circ t_{a,a'} = t_{a,a''}$ for all $r \leq a \leq a' \leq a''$. Sometimes we write $\mathsf{T} = \{T_a \xrightarrow{t_{a,a'}} T_{a'}\}_{r \leq a \leq a'}$ to denote the collection with the maps. If the resolution $r$ is zero, we drop mentioning the resolution of the tower.

* A discrete index or in general a subposet $A \subseteq \mathbb{R}$ can be 'embedded' in $\mathbb{R}$:
  - for any $a < a' \in A$, $(a, a') \notin A$, any $a \leq b < b' < a'$, $t_{b,b'}$ is isomorphism.



**Definition 2** (Interleaving of simplicial (space) towers). Let $\mathcal{X} = \{X_a \xrightarrow{x_{a,a'}} X_{a'}\}_{a \le a'}$ and $\mathcal{Y} = \{Y_a \xrightarrow{y_{a,a'}} Y_{a'}\}_{a \le a'}$ be two towers of simplicial complexes (spaces resp.) with resolution $r$. For any real $\varepsilon \ge 0$, we say that they are $\varepsilon$-interleaved if for every $a \ge r$ one can find simplicial maps (continuous maps resp.) $\varphi_a : X_a \to Y_{a+\varepsilon}$ and $\psi_a : Y_a \to X_{a+\varepsilon}$ so that:

(i) $\psi_{a+\varepsilon} \circ \varphi_a$ and $x_{a,a+2\varepsilon}$ are contiguous (homotopic resp.),

(ii) $\varphi_{a+\varepsilon} \circ \psi_a$ and $y_{a,a+2\varepsilon}$ are contiguous (homotopic resp.).

(iii) $\varphi_{a'} \circ x_{a,a'}$ and $y_{a+\varepsilon,a'+\varepsilon} \circ \varphi_a$ are contiguous (homotopic resp.),

(iv) $x_{a+\varepsilon,a'+\varepsilon} \circ \psi_a$ and $\psi_{a'} \circ y_{a,a'}$ are contiguous (homotopic resp.).

If no such finite $\varepsilon$ exists, we say the two towers are $\infty$-interleaved.

These four conditions are summarized by requiring that the four diagrams below commute up to contiguity (homotopy resp.):

$$\begin{tikzcd} X_a \arrow[rr, "x_{a,a+2\varepsilon}"] \arrow[dr, "\varphi_a"'] & & X_{a+2\varepsilon} \\ & Y_{a+\varepsilon} \arrow[ur, "\psi_{a+\varepsilon}"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_{a+\varepsilon} \arrow[dr, "\varphi_{a+\varepsilon}"] & \\ Y_a \arrow[ur, "\psi_a"] \arrow[rr, "y_{a,a+2\varepsilon}"] & & Y_{a+2\varepsilon} \end{tikzcd} \tag{6.2}$$

$$\begin{tikzcd} X_a \arrow[r, "x_{a,a'}"] \arrow[d, "\varphi_a"'] & X_{a'} \arrow[d, "\varphi_{a'}"] \\ Y_{a+\varepsilon} \arrow[r, "y_{a+\varepsilon,a'+\varepsilon}"'] & Y_{a'+\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} X_{a+\varepsilon} \arrow[r, "x_{a+\varepsilon,a'+\varepsilon}"] & X_{a'+\varepsilon} \\ Y_a \arrow[u, "\psi_a"] \arrow[r, "y_{a,a'}"'] & Y_{a'} \arrow[u, "\psi_{a'}"'] \end{tikzcd}$$

- If we replace '+' with multiplication '·', we get $\mathcal{X}$ & $\mathcal{Y}$ are multiplicatively $\epsilon$-interleaved.

Def (Interleaving distance):

$$d_I(\mathcal{X}, \mathcal{Y}) = \inf_{\epsilon} \{ \mathcal{X} \,\&\, \mathcal{Y} \ \epsilon\text{-interleaved} \}$$

**Definition 4** (Interleaving of vector space towers)**.** Let $\mathbb{U} = \{\mathrm{U}_a \xrightarrow{u_{a,a'}} \mathrm{U}_{a'}\}_{a\le a'}$ and $\mathbb{V} = \{\mathrm{V}_a \xrightarrow{v_{a,a'}} \mathrm{V}_{a'}\}_{a\le a'}$ be two vector space towers with resolution $r \ge 0$. For any real $\varepsilon \ge r$, we say that they are $\varepsilon$-interleaved if for each $a \ge r$ one can find linear maps $\varphi_a : \mathrm{U}_a \to \mathrm{V}_{a+\varepsilon}$ and $\psi_a : \mathrm{V}_a \to \mathrm{U}_{a+\varepsilon}$ so that:

(i) for all $a \ge r$, $\psi_{a+\varepsilon} \circ \varphi_a = u_{a,a+2\varepsilon}$,

(ii) for all $a \ge r$, $\varphi_{a+\varepsilon} \circ \psi_a = v_{a,a+2\varepsilon}$.

(iii) for all $a' \ge a \ge r$, $\varphi_{a'} \circ u_{a,a'} = v_{a+\varepsilon,a'+\varepsilon} \circ \varphi_a$,

(iv) for all $a' \ge a \ge r$, $u_{a+\varepsilon,a'+\varepsilon} \circ \psi_a = \psi_{a'} \circ v_{a,a'}$.

If no such finite $\varepsilon$ exists, we say the two towers are $\infty$-interleaved.

Analogous to the simplicial (space) towers, if we replace the operator '+' by the multiplication '·' in the above definition, then we say that $\mathbb{U}$ and $\mathbb{V}$ are *multiplicatively $\varepsilon$-interleaved*.

**Definition 5** (Interleaving distance between vector space towers)**.** The interleaving distance between two towers of vector spaces $\mathbb{U}$ and $\mathbb{V}$ is:

$$\mathsf{d}_I(\mathbb{U},\mathbb{V}) = \inf_{\varepsilon}\{\mathbb{U} \text{ and } \mathbb{V} \text{ are } \varepsilon\text{−}\textit{interleaved}\}.$$

* Suppose $\mathcal{X} : \{X_a \xrightarrow{x_{a,a'}} X_{a'}\}$

$\mathcal{Y} : \{Y_a \xrightarrow{y_{a,a'}} Y_{a'}\}$

two towers giving vector space towers.

$\mathbb{V}_X : \{H_p(X_a) \xrightarrow{(x_{a,a'})_*} H_p(X_{a'})\}$

$\mathbb{V}_Y : \{H_p(Y_a) \xrightarrow{(y_{a,a'})_*} H_p(Y_{a'})\}$

<u>Proposition:</u> $d_I(\mathbb{V}_X, \mathbb{V}_Y) \le d_I(\mathcal{X}, \mathcal{Y})$.

* From a vector space tower, we can get Interval module decomposition as we have seen before.

* So, we can define $Dgm(\mathbb{V})$: persistence diagram

with the endpoints of the intervals.

<u>Theorem</u> $d_b(\mathrm{Dgm}(\mathbb{U}), \mathrm{Dgm}(\mathbb{V})) = d_I(\mathbb{U}, \mathbb{V})$.

$$\Downarrow$$

$$d_b(\mathrm{Dgm}(\mathbb{U}), \mathrm{Dgm}(\mathbb{V})) \le d_I(\mathcal{X}, \mathcal{Y})$$

# Towers continued

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## log-scale diagrams

- We want to apply previous results for multiplicatively interleaved towers.
- So, we define $\mathrm{Dgm}_{\log}(\cdot)$ as the persistence diagram drawn with log-scale $\{\log x, \log y \mid (x,y) \in \mathrm{Dgm}\}$.

<u>Fact</u> If $\mathcal{X}$ & $\mathcal{Y}$ are multiplicatively $c$-interleaved, then

$$d_b\left(\mathrm{Dgm}_{\log}(\mathbb{V}_{\mathcal{X}}), \mathrm{Dgm}_{\log}(\mathbb{V}_{\mathcal{Y}})\right) \le \log c.$$

## Interleaving between Čech & Rips filtrations

Let $P \subseteq (M, d)$ be a finite subset (points).

$$\mathcal{R} : \left\{ \mathbb{VR}(P)^{\epsilon} \hookrightarrow \mathbb{VR}(P)^{\epsilon'} \right\}_{\epsilon \leq \epsilon'} \quad \text{Rips filtration}$$

$$\mathcal{C} : \left\{ \mathbb{C}(P)^{\epsilon} \hookrightarrow \mathbb{C}^{\epsilon'}(P) \right\}_{\epsilon \leq \epsilon'} \quad \text{Čech filtration}$$

We know

$$\underline{\mathbb{C}^{\epsilon}(P) \subseteq \mathbb{VR}(P)^{\epsilon} \subseteq \mathbb{C}^{2\epsilon}(P)}$$

We get a multiplicatively <u>2-interleaved</u> simplicial towers (filtrations)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{C}^{\varepsilon} \arrow[r] \arrow[d] \arrow[dr] & \mathbb{C}^{2\varepsilon} \arrow[r] \arrow[d] \arrow[dr] & \mathbb{C}^{4\varepsilon} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{VR}^{\varepsilon} \arrow[r] \arrow[ur] & \mathbb{VR}^{2\varepsilon} \arrow[r] \arrow[ur] & \mathbb{VR}^{4\varepsilon} \arrow[r] & \cdots
\end{tikzcd}$$

- So, we get

$$d_b\left(\mathrm{Dgm}_{\log}(H_p\mathcal{R}), \mathrm{Dgm}_{\log}(H_p\mathcal{C})\right) \leq \log 2 = 1$$

# Algorithm for tower

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## Computing persistence for simplicial tower

- $\mathcal{K}: K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} \cdots \rightarrow K_n = K$
- $\mathcal{H}_p K: H_p(K_1) \xrightarrow{f_{1*}} H_p(K_2) \rightarrow \cdots H_p(K_n)$
- Maintain a consistent basis for homology mapped by $f_{i*}$'s linearly.
- Instead we maintain a cohomology basis for

  $\mathcal{H}^p K: H^p(K_1) \xleftarrow{f_1^*} H^p(K_2) \leftarrow \cdots \xleftarrow{f_{n-1}^*} H^p(K_n)$

  (The sequence is reversed for cohomology)

- As we proceed, if we have a basis $B_i$ for $H^p(K_i)$, then using its preimage $f_i^{*-1}(B_i)$ we compute a basis $B_{i+1}$ for $H^p(K_{i+1})$.
- This is done by Annotations.

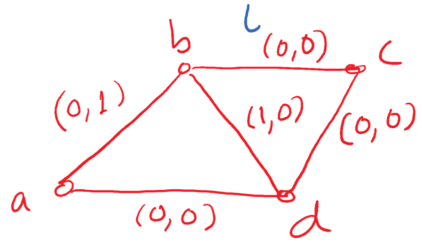
• This is done by <u>Annotations</u>.

Def (Annotation): Let $K(p)$ be the set of $p$-simplices in $K$. An annotation $a: K(p) \to \mathbb{Z}_2^g$ assigns a binary vector of length $g$ to every $p$-simplex, where

$$g = \operatorname{rank} H_p(K)$$

$a_c$: annotation for $p$-chains $c = \sum \alpha_i \sigma_i$

$$= \sum_i \alpha_i a(\sigma_i)$$




$$a_{(ab+bc+cd+ad)} = (0,1)+(0,0)+(0,0)+(0,0) = (0,1)$$

<u>Def</u> An annotation $a: K(p) \to \mathbb{Z}_2^g$ is valid if

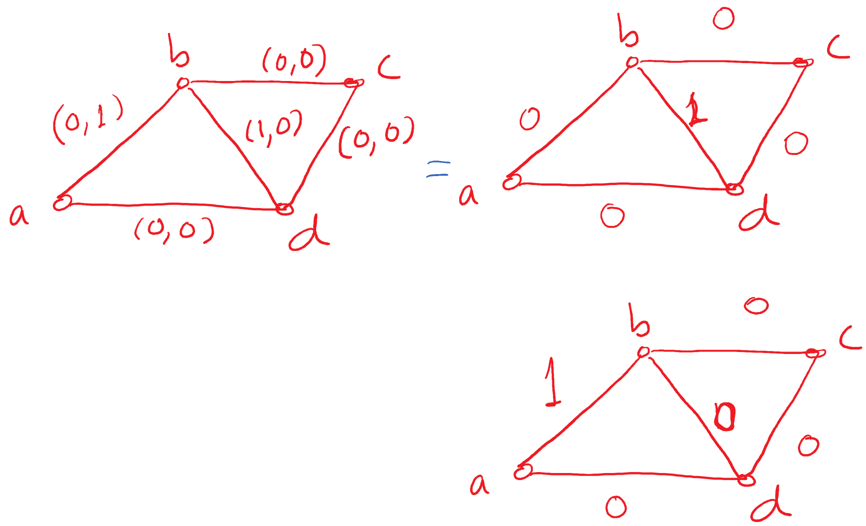
(i) $g = \operatorname{rank} H_p(K)$

(ii) For any two $p$-cycles $z_1, z_2$
$$a_{z_1} = a_{z_2} \text{ iff } [z_1] = [z_2]$$

<u>Fact</u> $a: K(p) \to \mathbb{Z}_2^g$ is valid <u>iff</u>

Cochains $\{\phi_i\}_{i=1 \ldots g}$ given by $\phi_i(\sigma) = a_\sigma[i]$

Cochains $\{\phi_i\}_{i=1,..,g}$ given by $\phi_i(\sigma) = a_\sigma[i]$ produces cohomology classes $[\phi_i]_{i=1,\dots,g}$ that form a basis of $H^p(K)$.




- A cochain $c^p$ assigns either '0' or '1' to a p-simplex. Then $c^p : C_p \to \mathbb{Z}_2$ is a homomorphism.

$$c^p(c_p) = \sum \alpha_i \, c^p(\sigma_i) \quad \text{where } c_p = \sum_i \alpha_i \sigma_i$$

Then, one can define cocycles, coboundaries and cohomology groups (Details are in the book)

- We will not go into the details of cohomology groups, but notice the similarity between annotations and cochains.

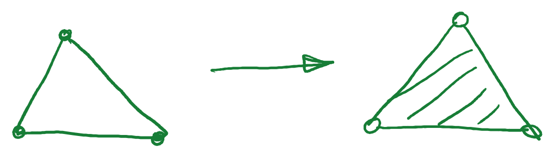
Elementary simplicial map

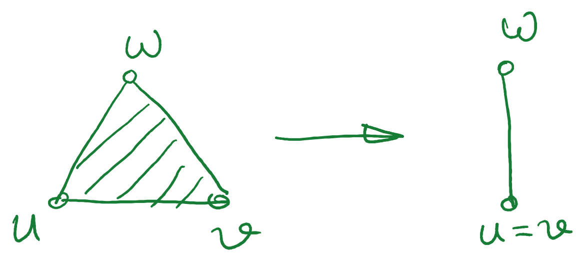
# Elementary simplicial map

$f: K \to K'$ is elementary if

- $f$ is injective and $K' \setminus K$ is at most one simplex [elementary inclusion]

- $f$ is surjective; $f_V$ is injective everywhere except on one pair $\{u, v\} \in V(K)$ [elementary collapse]


elementary inclusion



elementary collapse

- Every simplicial map $f$ can be decomposed into $f = f_1 \circ f_2 \circ \cdots \circ f_n$ where each $f_i$ is elementary.

- So, we assume elementary simplicial maps for the tower.

# Elementary inclusions

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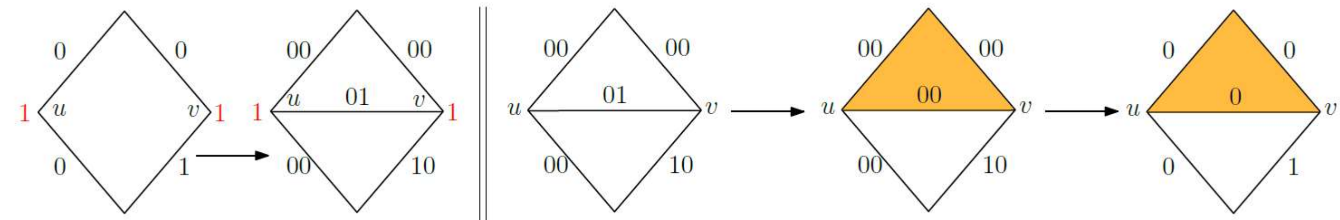
## Elementary inclusions

$K_i \xhookrightarrow{\sigma} K_{i+1}$ , $\sigma$ a $p$-simplex

We have two cases:

Case(i): $a_{\partial\sigma} = 0$, addition of $\sigma$ creates a $p$-cycle, or dually a $p$-cocycle is killed going from $K_{i+1} \to K_i$.

- For every $p$-simplex $\tau$, update $a_\tau = [b_1, b_2, \dots b_g]$ to $[b_1, b_2, \dots b_g, \underline{b_{g+1}}]$ with time stamp of $b_{g+1}$ to be $\underline{i+1}$
- Set $b_{g+1} = 0$ for every $\tau \neq \sigma$ and $b_{g+1} = 1$ for $\sigma$




(a) Case(i) (b) Case(ii)

Figure 6.2: Case(i) of inclusion: the boundary $\partial uv = u + v$ of the edge $uv$ has annotation $1 + 1 = 0$. After its addition, every edge gains an element in its annotation which is 0 for all except the edge $uv$. Case (ii) of inclusion: the boundary of the top triangle has annotation 01. It is added to the annotation of $uv$ which is the only edge having the second element 1. Consequently the second element is zeroed out for every edge, and is then deleted.

Case ii: $a_{\partial\sigma} \neq 0$: the class $[\partial\sigma]$ is non-trivial in $H_{p-1}(K_i)$. So, $\sigma$ kills this cycle, or dually a co-cycle is born in the reverse direction.

- We force $a_{\partial\sigma} = 0$ in $K_{i+1}$
- Let $i_1 < i_2 < \cdots < i_k$ be the indices of non-zero elements in $a_{\partial\sigma} = [b_1, b_2, \ldots b_{i_k}, \ldots b_g]$
- Cocycle $\phi = \phi_{i_1} + \phi_{i_2} + \cdots + \phi_{i_k}$ needs to become coboundary, i.e.,

$$\phi = 0 \Rightarrow \phi_{i_k} = \phi_{i_1} + \cdots + \phi_{i_{k-1}}$$

we choose to kill $\phi_{i_k}$ (the youngest one)

- So a bar $[b_{i_k}, i+1]$ is produced
- To kill $\phi_{i_k}$ and adjusting basis, we

add $a_{\partial\sigma}$ to all $(p-1)$-simplices $\sigma'$'s annotations where $a_{\sigma'}[b_{i_k}] = 1$.

- This makes $a_{\sigma'}[b_{i_k}] = 0$ for every $(p-1)$-simplex $\sigma'$ thus allowing to delete this bit.

Below we give an algorithm to compute an annotation for all simplices in a complex $K$.

**Algorithm 1** ANNOT($K$)

**Input:**
$K$: input complex
**Output:**
Annotation for every simplex in $K$

Let $m := |K^0|$;
For every vertex $v_i \in K^0$, assign an $m$-vector $\mathsf{a}(v_i)$ where $\mathsf{a}(v_i)[j] = 1$ iff $j = i$.
**for** $p = 1 \to d$ **do**
  **for all** simplex $\sigma \in K^p$ **do**
    Let annotation of every $p$-simplex be an $s$-vector so far.
    **if** $\mathsf{a}(\partial\sigma) \neq 0$ **then**
      assign $\mathsf{a}(\sigma)$ to be a 0 vector of size $s$
      pick any non-zero entry $b_u$ in $\mathsf{a}(\partial\sigma)$
      add $\mathsf{a}(\partial\sigma)$ to every $(p-1)$-simplex $\sigma'$ s.t. $\mathsf{a}(\sigma')[u] = 1$
      delete $u$th entry from annotation of every $(p-1)$-simplex
    **else**
      extend $\mathsf{a}(\tau)$ for every $p$-simplex $\tau$ so far added by appending a 0 bit;
      create $(s+1)$-vector $\mathsf{a}(\sigma)$ with the last bit being 1
    **end if**
  **end for**
**end for**

## Elementary collapse

- Insert simplices to satisfy Link condition.
$$Lk(uv) = Lk(u) \cap Lk(v)$$

• $Lk(\sigma) = \overline{St\,\sigma} \setminus St\,\sigma$ (see the notes)

• Annotation transfer : prepares for collapse

• collapse

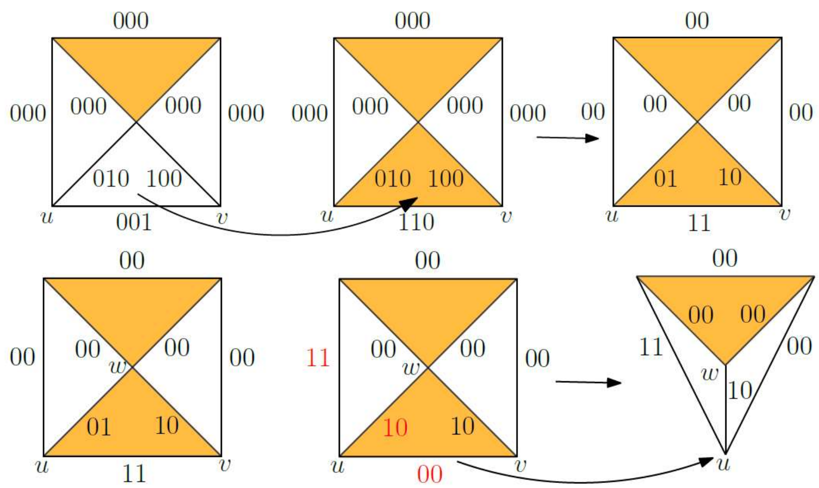



Figure 6.3: Annotation updates for elementary collapse: inclusion of a triangle so as to satisfy the link condition (upper row), annotation transfer and actual collapse (lower row); annotation 11 of the vanishing edge *uv* is added to all edges (cofaces) adjoining *u*.