The Union-Find Problem

We look at the problem maintaining a system of sets that are pairwise disjoint. It should support two operations: (1) Find (2) Union.

C: collection of subsets of $\{1, 2, \dots, n\}$

s.t. $\bigcup_{I \in C} I = \{1, 2, \dots, n\}$ and $I \cap J = \emptyset$ if $I, J \in C$.

Find(i): determines the set $I \in C$ with $i \in I$.

Union(I, J): Joins sets I and J in C.

Often in applications we need the above two operations in the following way:

$I :=$ Find(i); $J :=$ Find(j);

If $I \neq J$ then Union(I, J) endif.

Here it does not really matter what I and J really are, except that they need to be different iff they represent different sets.

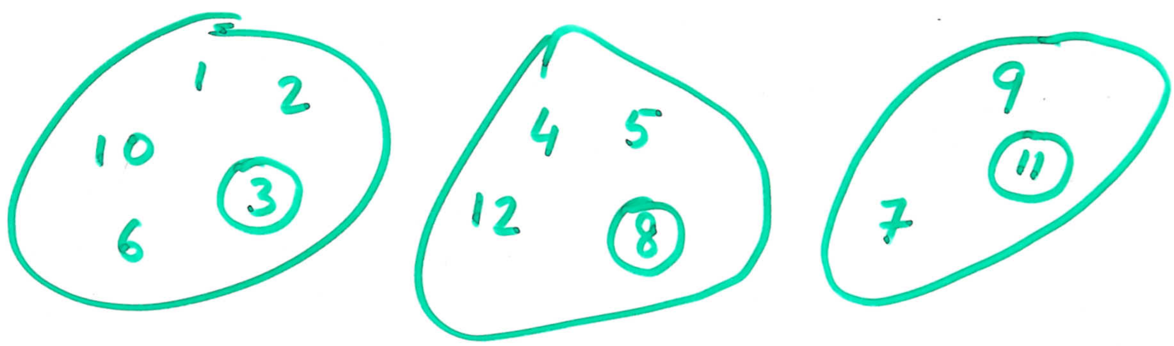
A simple solution.

C: array [1...n] of integers

Each set is represented by one of its elements, and C[i] stores the name (the index of the representative) of the set containing i.

C:

Set	3	3	3	8	8	3	11	8	11	3	11	8
Size			5					4			3	
next	1	2	3	4	5	6	7	8	9	10	11	12



Finding a set takes $O(1)$, but union takes $\Theta(n)$ since the entire array needs to be scanned in the worst-case.

The previous solution can be improved by storing (3)

(i) the elements of a set in a linked list (next pointer)

(ii) the size of a set at its representative

```
function Find(i)
  return C[i].set

procedure Union(I, J)
  if C[I].size < C[J].size then I <--> J endif
  C[I].size := C[I].size + C[J].size;
  Second := C[I].next; C[I].next := J;
  t := J;   loop
              C[t].set := I;
              if C[t].next := 0 then
                    C[t].next := second
                    exit loop
              endif
              t := t.next
         endloop
```

The worst-case of a single union operation is still $\Theta(n)$, as before, but now we can show a logarithmic amortized bound.

Claim n-1 union operations take time $O(n \log n)$

proof We consider the size of the set that contains the element i. So define

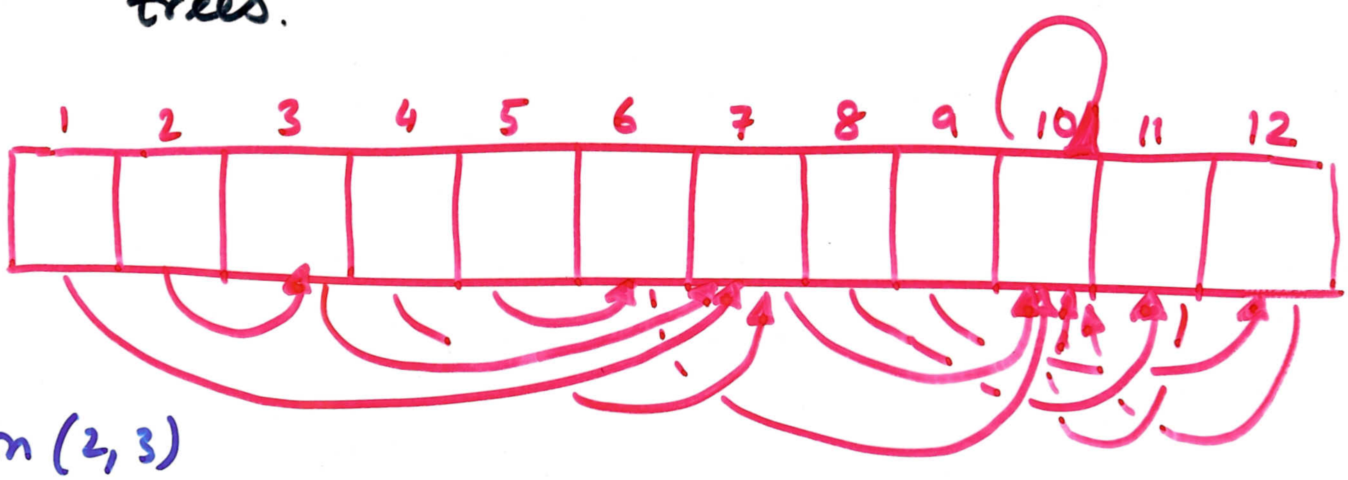
$$\sigma(i) = C[\mathrm{Find}(i)].\mathrm{size}.$$

$\sigma(i)$ changes whenever i is touched in the union operation; in this case the new $\sigma(i)$ is at least twice as large as the old one. This is because i is touched only if it belongs to the smaller of the two sets joined. Define k as the number of times element i is touched. Then $\sigma(i) \geq 2^k \Rightarrow k \leq \log n$.

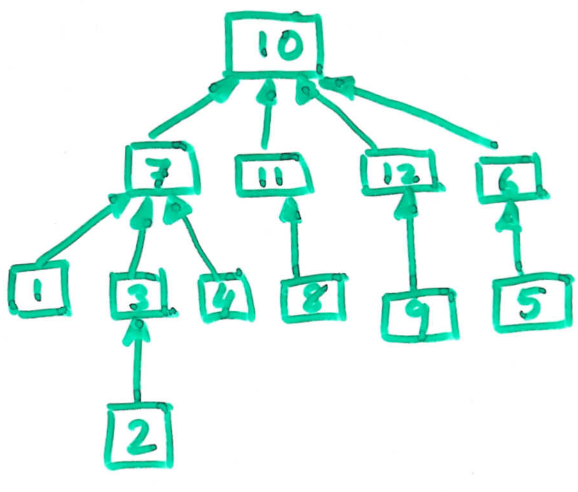
Tree Representation

We consider representing each set as a tree.

idea – each set is represented by
- Find(i) traverses the path from i up to the root.
- Union(I, J) links the two trees.



Ex.
Union (2, 3)
" (4, 7)
" (2, 4)
" (1, 2)
" (4, 10)
" (9, 12)
" (12, 2)
" (8, 11)
" (8, 2)
" (5, 6)
" (6, 1)



Union takes O(1) time,
Find takes time proportional to the depth of the ~~tree~~. node.

Weighted Merging. The same idea as before improves time: instead of joining arbitrarily, join the smaller to the larger tree.

Assume: C has fields

p.. index of parent,
index to itself if root

h... height of the tree

```
function Find(i)
  if C[i].p = i then return i
     else return Find(C[i].p)
  endif

procedure Union(I, J)
 if C[I].h < C[J].h then
                    C[I].p := J
    else
            C[J].p := I;
            if C[I].h = C[J].h then
                    C[I].h := C[I].h + 1
            endif
 endif
```

Claim. The height of a tree with n nodes is at most $\log n$.

So, Find takes $O(\log n)$ time.
Union takes $O(1)$ time.

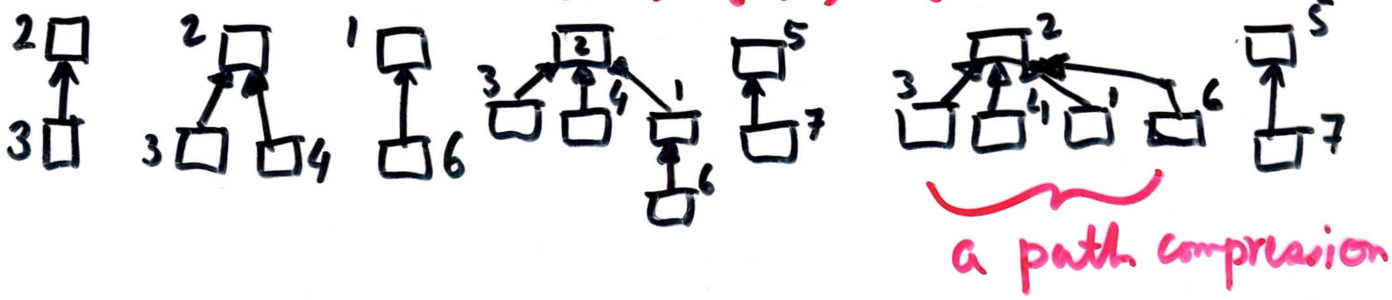
Path Compression.

The idea is to connect all nodes visited during a Find operation directly to the root.

function Find(i)
  if C[i].p $\neq$ ~~∅~~ i then C[i].p := Find(C[i].p) endif
  return C[i].p

Example (i, j) stands for
  I := Find(i); J := Find(j);
  If I $\neq$ J then Union(I, J)

(2,3) (2,4) (1,6) (2,6) (5,7) (4,6)



Ackermann's Function

It can be shown that m find operations take $O(m\,\alpha(m))$ time where

$\alpha(m)$ is the slowly growing inverse Ackermann's function.

Def.

$$\left.\begin{aligned} A_k(1) &= 2 \quad \text{for } k \geq 1 \\ A_1(n) &= 2n \quad \text{for } n \geq 1 \\ A_k(n) &= A_{k-1}(A_k(n-1)) \quad \text{for } k, n \geq 2 \end{aligned}\right\} \text{Ackermann's function}$$

	$n=1$	2	3	4	5	6
$k=1$	2	4	6	8	10	12
2	2	4	8	16	32	64
3	2	2^2	2^{2^2}	$2^{2^{2^2}}$	...	
4	2	2^2	$2^{2^{2^2}}$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}$ tower of height $2^{2^{2^2}}$		
5	2	2^2	$2^{\cdot^{\cdot^{\cdot}}}$			

$$\alpha(m) = \min\{n \mid A_n(n) \geq m\}$$

For all practical purposes $\alpha(m) \leq 4$, but $\alpha(m)$ goes to infinity as m goes to ∞.