Topological Sort

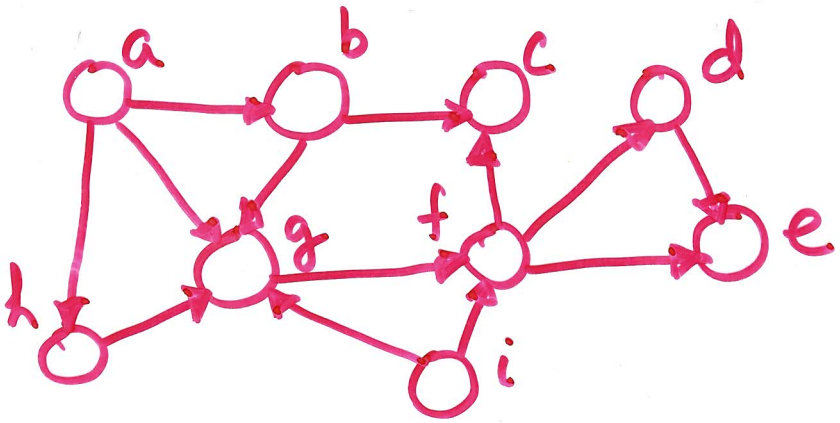
(V,E) is a directed graph. A cycle is a sequence of vertices $v_0, v_1, \cdots, v_m$, $v_i \in V$ and $m \geq 1$ so that $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq m-1$, and $(v_m, v_0) \in E$.

(V,E) is a directed acyclic graph (DAG), if it has no cycle.

To produce a topological sort of (V,E) we need to order vertices that is compatible with E. Formally, a sequence of vertices $V_1, V_2, \cdots V_n$ is a topological sort so that $i < j$ if $(v_i, v_j) \in E$.

In other words $i > j$ only if $(v_i, v_j] \notin E$.

Ex.



a, b, h, i, g, f, c, d, e is a topological sort

A vertex with no incoming edge is a source, with no outgoing edge is a sink.

Claim (V,E) is acyclic if and only if DFS search yields no back edge.

The idea is to use DFS to produce a topological sort. We report a vertex when it is finished. It produces a topological sort back to front. It is justified by the following.

Claim If $(u,v) \in E$ then $V[u].f > V[v].f$.

proof Consider the time (u,v) is explored by DFS.

Case 1. v is an ancestor of u (contradiction to no back edge claim)

Case 2. v is yet unexplored.
Then v becomes descendant of u.
So, $V[u].f > V[v].f$

Case 3. v is already finished.
Then $V[u].f > V[v].f$ automatically.