Flow Networks

A **flow network** is a directed graph $G=(V,E)$ in which each edge $(u,v) \in E$ has a nonnegative **capacity** $c(u,v) \geq 0$. If $(u,v) \notin E$, we assume $c(u,v) = 0$. There are two distinguished vertices **source** s, **sink** t.

We also assume G is connected. Therefore $|E| \geq |V| - 1$.

Flow: $f : V \times V \to \mathbb{R}$ is a real-valued function.

1. for $\forall\ u, v \in V$, $f(u,v) \leq c(u,v)$.
2. " " " , $f(u,v) = -f(v,u)$
3. " " $u \in V - \{s,t\}$, $\sum_{v \in V} f(u,v) = 0$

The value of a flow is $|f| = \sum_{v \in V} f(s,v)$.

Maxflow problem is to find $\max |f|$.

Some properties of flow

$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$ for two sets

$X, Y \subseteq V$.

1. $f(X, X) = 0$
2. $f(X, Y) = -f(Y, X)$
3. $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$
4. $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.

Ford-Fulkerson Algorithm

Uses the concept of Residual capacity:

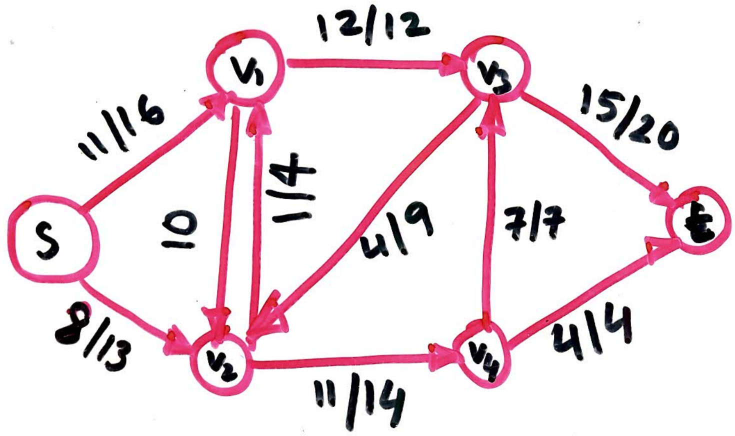
$c_f(u, v) = c(u, v) - f(u, v)$.

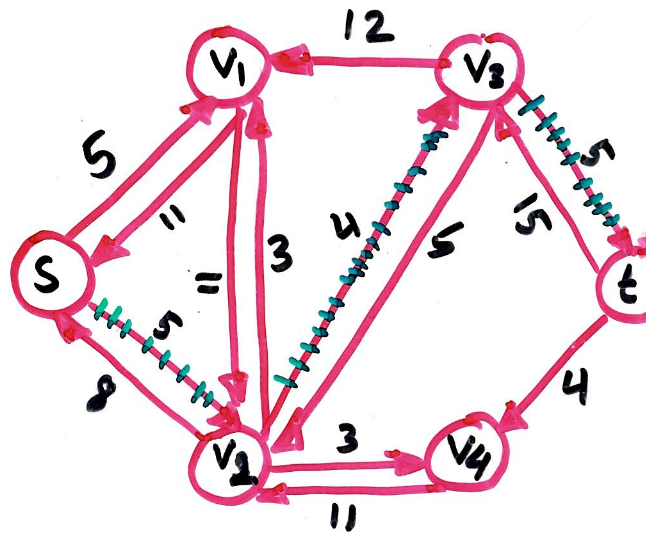
Given a flow network $G = (V, E)$ and a flow f, the residual network is

$G_f = (V, E_f)$ where

$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$.

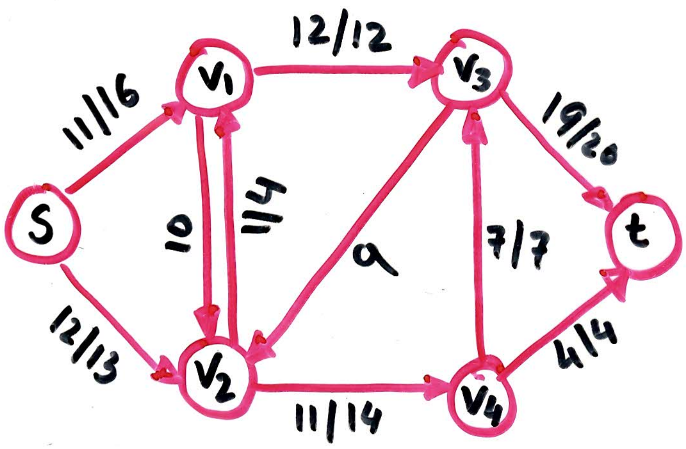
Example.

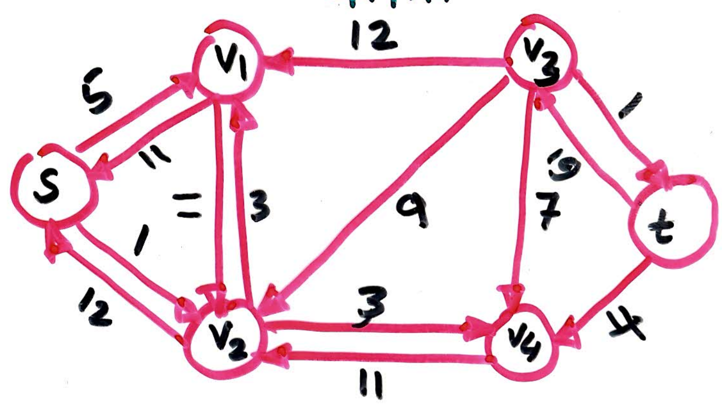




Residual Network

Augmenting path





No augmenting path

Augmenting path. A path p from s to t in the residual network G_f. The residual capacity of p is given by

$$c_f(p) = \min \{ c_f(u,v) \mid (u,v) \in p \}$$

Algorithm

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Ford-Fulkerson(G, s, t)
  for each edge (u,v) ∈ E
      do f[u,v] := 0
         f[v,u] := 0
  endfor
  while there exists a path p from s to t
        in G_f do
      c_f(p) := min{ c_f(u,v) | (u,v) ∈ p }
      for each edge (u,v) ∈ p
          f[u,v] := f[u,v] + c_f(p)
          f[v,u] := -f[u,v]
      endfor
  endwhile
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If capacities are integers, the above algorithm runs in $O(|E||f^*|)$ time where $|f^*|$ is the max flow. Each iteration in the while loop can be implemented in $O(|E|)$ time increasing the flow at least by one unit.

Max flow - Min Cut

Lemma 1

Let G_f be the residual network induced by flow f. Let f' be a flow in G_f. Then $f+f'$ is a flow in G where

$$|f+f'| = |f| + |f'|.$$

Proof.

Skew symmetry:

$$\begin{aligned}
(f+f')(u,v) &= f(u,v) + f'(u,v) \\
&= -(f(v,u) + f'(v,u)) \\
&= -(f+f')(v,u)
\end{aligned}$$

Capacity constraint:

$$\begin{aligned}
(f+f')(u,v) &= f(u,v) + f'(u,v) \\
&\le f(u,v) + c_f(u,v) \\
&= f(u,v) + c(u,v) - f(u,v) \\
&= c(u,v)
\end{aligned}$$

Flow conservation: for $\forall u \in V - \{s,t\}$

$$\begin{aligned}
\sum_{v \in V} (f+f')(u,v) &= \sum_{v \in V} (f(u,v) + f'(u,v)) \\
&= \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) \\
&= 0 + 0 = 0
\end{aligned}$$

Also,
$$|f+f'| = \sum_{v\in V} f(s,v) + \sum_{v\in V} f'(s,v)$$
$$= |f| + |f'|$$

Lemma 2. Let p be an augmenting path in G_f. Define $f_p : V\times V \to \mathbb{R}$ as:
$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is in } p \\ -c_f(p) & \text{if } (v,u) \text{ is in } p \\ 0 & \text{otherwise.} \end{cases}$$
Then f_p is a flow in G_f with $|f_p| = c_f(p) > 0$.

Corollary 1. Define $f' : V\times V \to \mathbb{R}$ by $f' = f + f_p$. Then f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

Proof. Combine ~~Lemma~~ Lemma 1 and Lemma 2.

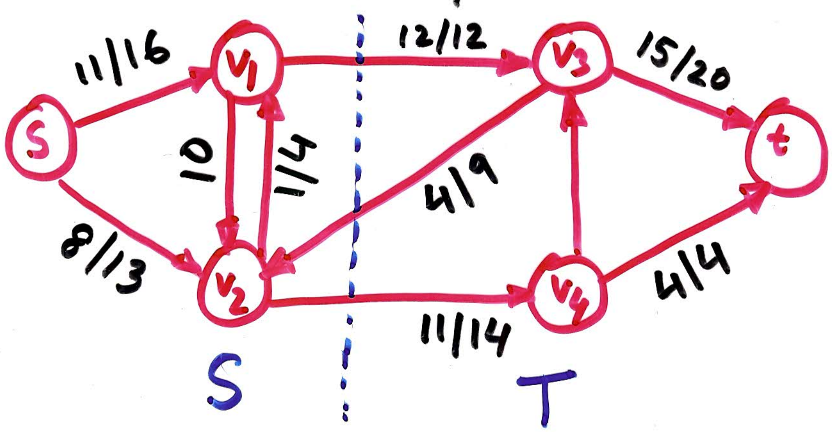
Cuts

A cut (S,T) of a flow network $G=(V,E)$ is a partition of V into S and $T=V-S$ so that $s \in S$ and $t \in V-S=T$.

<u>Net flow</u> across a cut (S,T) is $f(S,T)$.

<u>Capacity</u> of a cut (S,T) is

$$c(S,T)=\sum_{\substack{u \in S\\ v \in T}} c(u,v).$$



$$f(S,T)=f(v_1,v_3)+f(v_2,v_3)+f(v_2,v_4)$$
$$=12+(-4)+11=19.$$

$$c(S,T)=c(v_1,v_3)+c(v_2,v_4)=12+14=26.$$

Lemma 3 $f(S,T) = |f|$.

proof. $f(S,T) = f(S,V) - f(S,S)$

$$= f(S,V) = f(s,V) + f(S-s,V)$$

$$= f(s,V) = |f|.$$

Corollary 2. The value of a flow f in G is bounded from above by the capacity of any cut in G.

proof. By Lemma 3, $|f| = f(S,T)$.

By definition

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v)$$

$$\leq \sum_{u\in S}\sum_{v\in T} c(u,v)$$

$$= c(S,T).$$

We will combine all the above results to prove a main theorem in network flow.

Max flow-Min cut Theorem

Theorem. The following conditions are equivalent:

1. f is a maximum flow in G.
2. G_f contains no augmenting path.
3. $|f| = c(S,T)$ for some cut (S,T) of G.

proof. $1 \Rightarrow 2$: For contradiction, suppose f is maximum and G_f contains an augmenting path p. Then, by Corollary 1, $f + f_p$ is a valid flow in G with $|f+f_p| > |f|$ contradicting the maximality of f.

$2 \Rightarrow 3$: G_f contains no path from s to t. Define $S = \{v \in V \mid \text{there is a path from } s \text{ to } v \text{ in } G_f\}$ and $T = V - S$. The partition (S,T) is a cut: since $s \in S$ and $t \notin S$ (there is no path from s to t). We have $f(u,v) = c(u,v)$ $\forall (u,v) \in V \times V$ where $u \in S$, $v \in T$ since otherwise $(u,v) \in E_f$ and v will be in S.
Then, by Lemma 3, $|f| = f(S,T) = c(S,T)$.

$3 \Rightarrow 1$: By Corollary 2 $|f| \le c(S,T)$ for any cut (S,T). Since $|f| = c(S,T)$ we must have f to be maximum.