# An 1.5-approximation for optimal TSP



We improve upon the 2-approximation TSP algorithm that we saw earlier. The new algorithm is based on the observation that one can obtain an Eulerian Tour with a better approximation factor than doubling the MST. The algorithm is due to Christofides [19].

Plan:

- Augment the MST so that it has an Eulerian Tour
  (A cycle of edges that traverse every edge)
- Now modify this Eulerian tour to be a TSP by deleting repeated vertices.
- We already saw that the above operation does not increase cost under triangle prop.

**Fact 1.** Any connected graph with all vertex degree equal to an even number has an Eulerian tour which can be found in linear time $O(V+E)$.

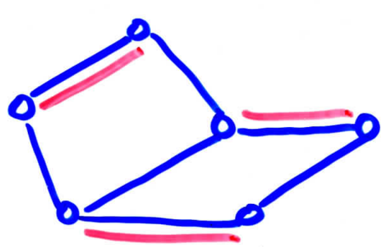
**Proof.**
- Start with any vertex and continue traversing edges not visited so far. Because of even parity of degrees, one is guaranteed to leave a vertex if entered except the first one.
- When we reach the first one we discover a tour.
- We may not finish covering all edges. Start a new tour with a vertex that has an unvisited edge and continue.
- At the end, the final tour is the concatenation of all tours found.

- Our goal is to find edges that can augment the MST to have an Eulerain tour.

- From Fact 1., we set for adding edges to the odd degree vertices in MST.

- The edges we add come from a perfect matching.

Perfect Matching: Given a graph $G=(V,E)$, a perfect matching $E' \subseteq E$ is a set of edges so that every vertex in $V$ is incident to exactly one edge in $E'$.



- $E' \subset E$

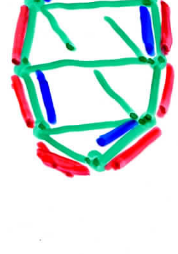
$E'$ is a Matching.

Prove: Every ~~connected~~ complete graph has a perfect matching iff it has even number of vertices.

**Lemma!** A minimum weight perfect matching in a graph $G=(V,E)$ has weight at most half the cost (weight) of minimum TSP of $G$.

$$\text{Cost (min. Perfect match)} \leq \frac{1}{2} \text{ min. TSP.}$$

**Proof.**
- Since $G$ has even number of vertices (otherwise no perfect match. could exist), TSP has even vertices.
- Split TSP into two sequence of edges of alternating edges in the tour.
- The sequence with the smaller wt. is $\leq \frac{1}{2}$ cost(TSP).
- Alternating sequence of edges constitutes a perfect matching.
- min. wt. perfect matching is even smaller.



– TSP
– perfect matching

# Algorithm (G is a complete graph)

Step 1. Construct MST T of $G=(V,E)$

Step 2. Let $D \subseteq V$ be the set of odd degree vertices in T. Let $G'=(D,E')$ be the subgraph induced by E on D.

# odd vertices is even (why?)

Step 3. Compute a perfect min. weight matching $E'' \subseteq E'$ in $G'$.

Since |D| is even, perfect match exists

Step 4. Augment T by adding edges $E''$ to T.

T has now only even degrees

Step 5. Compute an Eulerian tour in augmented T.

Eulerian tour exists in T now

Step 6. Delete repeated vertices in the Eulerian tour to get an approximate TSP.

Triangle inequality lets deleting repeats without increasing cost.

**Theorem.** The algorithm computes a 1.5-approximation of min. TSP.

Proof.
- The Eulerian tour has edges from MST T and the edges from perfect min. wt. matching computed in step 3.
- wt.(T) $\leq$ wt.(TSP) because deleting an edge from TSP creates a spanning tree.
- wt.(Matching) $\leq \frac{1}{2}$(TSP) by Lemma 1.
- Therefore wt.(Eulerian tour) $\leq$ 1.5 (TSP).
- Step 6 makes the computed TSP even smaller than the Eulerian tour.