Subset-Sum Approximation

Given a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers, and a, $t \geq 1$, positive integer, find a subset s.t. its sum is largest possible not exceeding t.

We design a polynomial time approximation scheme (PTAS). {It is fully polynomial since polynomial in n & $\frac{1}{\epsilon}$.}.

An exact solution.

1. compute all subsets $S' \subseteq S$
2. return the subset with largest sum $\leq t$.

Compute the above sums iteratively.

$L_i = \{$ sums of all subsets of $\{x_1, x_2, \ldots x_i\}$ not exceeding t $\}$.

Then return the maximum in L_n.

L: List of positive integers.

L+x: list of positive integers derived from L by adding x to each element in L.

$L = \{1, 2, 4, 7\}$, $L+2 = \{3, 4, 6, 9\}$.

Similarly,

$S+x : \{ s+x \mid s \in S \}$.

Exact-Subset-Sum(S, t)

$n := |S|$;

$L_0 := \{0\}$;

for $i := 1$ to n do

$L_i :=$ Merge-Lists(L_{i-1}, $L_{i-1}+x_i$)

remove from L_i each element $> t$

endfor

return the largest in L_n.

Since length of L_i can be as much as 2^i, the algorithm above runs in exponential time and space.

FPTAS: Trim the list L_i. Let $0<\delta<1$ be a parameter.

Let L' be the resulting list from a list L. For every element y removed from L, there is an element $z \in L'$ s.t.

$$\frac{y}{1+\delta} \leq z \leq y.$$

z "represents" y in L'.

Ex. $L = \{10, 11, 12, 15, 20, 21, 22, 23, 24, 29\}$

$L' = \{10, 12, 15, 20, 23, 29\}$ with $\delta = 0.1$.

Trim (L, δ) : $L = \{y_1, \cdots y_m\}$ sorted increasing order.

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m := |L|
L' := {y_1}
last := y_1
for i := 2 to m do
    if y_i > last·(1+δ) then
        append y_i to L'
        last := y_i
    endif
endfor
return L'
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With $0 < \epsilon < 1$, we write the approximation scheme

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Approx-Subset-Sum (S, t, ε)
 n := |S|;
 L_0 := {0};
 for i := 1 to n do
     L_i := Merge-List (L_{i-1}, L_{i-1} + x_i)
     L_i := Trim (L_i, ε/2n)
     remove from L_i each element > t
 endfor
 return z*, the maximum in L_n.
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Ex. $S = \{104, 102, 201, 101\}$

$t = 308$, $\epsilon = 0.4$

$\delta = \epsilon/8 = 0.05$

$L_0 = \{0\};$

$L_1 = \{0, 104\}$

$L_2 = \{0, 102, 104, 206\}$

$L_2 = \{0, 102, 206\}$

$L_2 = \{0, 102, 206\}.$

$L_3 = \{0, 102, 201, 206, 303, 407\}$

$L_3 = \{0, 102, 201, 303, 407\}$

$L_3 = \{0, 102, 201, 303\}$

$L_4 = \{0, 101, 102, 201, 203, 302, 303, 404\}$

$L_4 = \{0, 101, 201, 302, 404\}$

$L_4 = \{0, 101, 201, 302\}$.

$z^* = 302$ is within $\epsilon = 40\%$ of the optimal $307 = 104 + 102 + 101$.

Theorem Approx-Subset-Sum is a FPTAS for subset-sum problem.

Proof. Let P_i be the set of values that can be obtained by selecting subsets of $\{x_1, x_2, \dots x_i\}$ and summing its members.

Let $y^* \in P_n$ denote the optimal value.

$z^* \leq y^*$ (follows from definition).

need to show

$$\frac{y^*}{z^*} \leqslant 1+\epsilon .$$

By induction on i, it can be shown for each $y \in P_i$, $y \leq t$, there is a $z \in L_i$ s.t.

1 $$\frac{y}{(1+\epsilon/2n)^i} \leq z \leq y .$$

In particular, $\dfrac{y^*}{(1+\epsilon/2n)^n} \leq z \leq y^*$

$$\Downarrow$$

$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n$$

$$\Downarrow$$

2 $$\frac{y^*}{z^*} \leq \left(1+\frac{\epsilon}{2n}\right)^n$$

Since $\frac{d}{dn}\left(1+\frac{\epsilon}{2n}\right)^n > 0$, the function $\left(1+\frac{\epsilon}{2n}\right)^n$ increases with n as it its limit $\lim_{n\to\infty}\left(1+\frac{\epsilon}{2n}\right)^n = e^{\epsilon/2}$.

we have

$$\left(1+\frac{\epsilon}{2n}\right)^n \le e^{\epsilon/2}$$

$$\le 1+\frac{\epsilon}{2}+\left(\frac{\epsilon}{2}\right)^2$$

3 $$\le 1+\epsilon$$

from 2 & 3

$$y^* \le (1+\epsilon)\, z^*.$$

To show fully polynomial behavior, we derive a bound on $|L_i|$.

Due to trimming two successive elements z and z' of L_i should satisfy

$$z'/z > 1+\epsilon/2n.$$

Thus, L_i contains 0, possibly 1, and upto $\lfloor \log_{1+\epsilon/2n} t \rfloor$ additional values.

Thus, $$|L_i| \le \log_{1+\epsilon/2n} t + 2$$

$$= \frac{\ln t}{\ln\left(1+\frac{\epsilon}{2n}\right)} + 2$$

$$\le \frac{2n\left(1+\frac{\epsilon}{2n}\right)\ln t}{\epsilon} + 2 \le \frac{4n \ln t}{\epsilon} + 2$$

(polynomial in n & $\frac{1}{\epsilon}$, $\lg t$)