Linear Programming

A linear constraint is a linear inequality or equality.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \begin{smallmatrix}\leq\\=\\\geq\end{smallmatrix} b$$

A linear objective function is

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

A linear program is to minimize or maximize a linear objective function under linear constraints.

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

minimize

subject to

$$a_{11}x_1 + \cdots + a_{1n}x_n \begin{smallmatrix}\geq\\\leq\\=\end{smallmatrix} b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n \begin{smallmatrix}\geq\\\leq\\=\end{smallmatrix} b_2$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$$

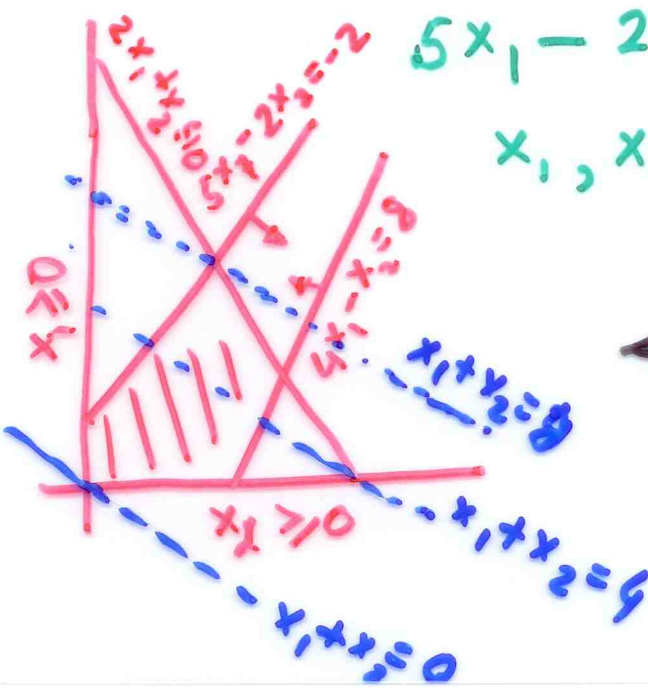
- The x_i's are called <u>variables</u>.
- The values of variables that satisfy all constraints constitute <u>feasible region</u>
- A solution to the constraints is a <u>feasible solution</u>
- An <u>optimal</u> <u>feasible</u> <u>solution</u> is the goal of linear of programming.

<u>Ex.</u>

Maximize $x_1 + x_2$

S.t.

$$4x_1 - x_2 \le 8$$
$$2x_1 + x_2 \le 10$$
$$5x_1 - 2x_2 \ge -2$$
$$x_1, x_2 \ge 0$$



← <u>Geometric interpretation</u>

- The feasible region is <u>convex</u>.

- The optimality happens (if at all) at a boundary points of the convex feasible region

- It is either a vertex or a line segment (in 2D), or a k-flat (in general dimension)

- Even if there are multiple optimal solutions, there is one vertex where optimality happens

- The convex region can be unbounded

- If unbounded, the optimal solution could be infinity

- Even if the feasible region is unbounded, the solution may not be unbounded.

Standard Form

Given: n real numbers $c_1, \cdots c_n$

m real numbers $b_1, \cdots b_m$

mn " " a_{ij}, $i = 1 \ldots m$, $j = 1 \ldots n$

To find: n real numbers $x_1, x_2, \cdots x_n$ that

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \cdots, m$$

$$x_j \ge 0 \quad \text{for } j = 1, \cdots, n.$$

The last set of constraints are called <u>nonnegativity constraints</u>.

Matrix form:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

A : $m \times n$ matrix

b : m-dimensional vector

c : n-dimensional vector.

Converting to standard form

1. Objective function may be minimization
2. Variables without nonnegativity constraints
3. may be equality constraints
4. greater-than-equal-to inequality constraints

We convert a LP L to a LP L' s.t. L' is in standard form but L' is **equivalent** to L: each feasible solution $\bar{x}$ to L with objective value z, there is a $\bar{x}'$ with objective value z for L' and vice versa.

(−z if min-max interchanged)

For 1: Simply negate the objective function. i.e. if minimize $c^T x$ then do: maximize $[-c]^T x$.

For 2: if x_j appears without nonnegativity constraits, replace x_j with $x_j' - x_j''$ and add $x_j' \geq 0$, $x_j'' \geq 0$. Thus, replace $c_j x_j \rightarrow c_j x_j' - c_j x_j''$ and $a_{ij} x_j \rightarrow a_{ij} x_j' - a_{ij} x_j''$

2 continued: a feasible solution $\overline{x}$ to the new LP corresponds to a feasible solution to the original LP with

$$\overline{x}_j = \overline{x}_j' - \overline{x}_j''.$$

For 3: Convert an equality constraint as:

$$f(x) = b \Rightarrow f(x) \le b, \text{ and } f(x) \ge b$$

Then, by (4) convert the greater-than-equal-to to less-than-equal-to.

For 4:

Convert

$$\sum_{j=1}^{n} a_{ij} x_i \ge b_i \Rightarrow \sum_{j=1}^{n} -a_{ij} x_j \le -b_i$$

Example.

Minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

Apply 1:

Maximize $2x_1 - 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

Apply 2:

Maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$
$$x_1 - 2x_2' + 2x_2'' \leq 4$$
$$x_1, x_2', x_2'' \geq 0$$

Apply 3:

Maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' \leq 7$$
$$x_1 + x_2' - x_2'' \geq 7$$
$$x_1 - 2x_2' + 2x_2'' \leq 4$$
$$x_1, x_2', x_2'' \geq 0$$

Apply 4:

$$\left.\begin{array}{l} \text{Maximize } 2x_1 - 3x_2 + 3x_3 \\ \text{s.t.} \\ x_1 + x_2 - x_3 \leq 7 \\ -x_1 - x_2 + x_3 \leq -7 \\ x_1 - 2x_2 + 2x_3 \leq 4 \\ x_1, x_2, x_3 \geq 0 \end{array}\right\} \text{Standard form}$$

Converting LP to slack form

In simplex algorithm that we will study, the constraints except the nonnegativity ones are all equalities. This is called slack form.

Let $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ be an inequality constraint.

Introduce a new slack variable s.

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$s \geq 0$$

Instead of s, we write

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$x_{n+i} \geq 0.$$

Applying slack variables to the example we took, we get:

$$\left.\begin{aligned} &\text{maximize} \quad 2x_1 - 3x_2 + 3x_3 \\ &\text{s.t.} \\ &\qquad x_4 = 7 - x_1 - x_2 + x_3 \\ &\qquad x_5 = -7 + x_1 + x_2 - x_3 \\ &\qquad x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ &\qquad x_1, x_2, \ldots, x_6 \geq 0 \end{aligned}\right\} \text{Slack form}$$

The variables on the left side of equality constraints are called <u>basic variables</u> and the rest <u>non-basic</u> <u>variables</u>.

Drop "maximize", "s.t.", and write objective function with a linear equation, drop non-negative constraints (implicit)

$$\begin{aligned} z &= 2x_1 - 3x_2 + 3x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -7 + x_1 + x_2 - x_3 \\ x_6 &= 4 - x_1 + 2x_2 - 2x_3 \end{aligned}$$

N: index set non-basic variables, $|N| = n$

B: index set of basic variables, $|B| = m$

C: objective function vector

b: constraint vector

A: constraint matrix

Thus, we can express the slack form by a tuple (N, B, A, b, c, v): All are explained except v which represents a possible constant term in objective equation.

$$\left.\begin{aligned} z &= v + \sum_{j \in N} c_j x_j \\ x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B. \end{aligned}\right\} \text{concise slack form}$$

Example

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$

$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2},$$

We have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

$$A = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}, \quad v = 28.$$

Formulating problems with LP

Shortest Path

Given $G=(V,E)$ with weight $\omega: E \to \mathbb{R}$, a source $s \in V$ and a destination $t \in V$, determine s.p. length $d[t] = \delta(s,t)$.

Maximize
~~Minimize~~ $d[t]$

s.t.

$$d[v] \le d[u] + \omega(u,v)$$

for each $(u,v) \in E$

$$d[s] = 0$$

In this LP, there are $|V|$ variables $d[v]$.
There are $|E|+1$ constraints.

? How would you formuate single-source shortest path using LP. The above was only for single-pair shortest path.

Maximum flow

$G=(V,E)$, each $u,v \in E$ has capacity $c(u,v) \geq 0$

Source s, sink t. A flow:

$f: V \times V \to \mathbb{R}$, satisfies three constraints.

$$\text{Maximize} \sum_{v \in V} f(s,v)$$

s.t.

$f(u,v) \leq c(u,v)$ for each $u,v \in V$

$f(u,v) = -f(u,v)$ for each $u,v \in V$

$\sum_{v \in V} f(u,v) = 0$ for each $u \in V - \{s,t\}$.

This LP has $|V|^2$ variables (each pair $(u,v) \in V \times V$)

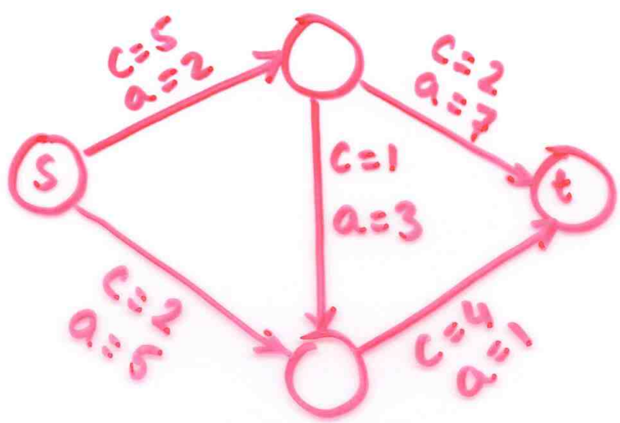
$2|V|^2 + |V| - 2$ constraints.

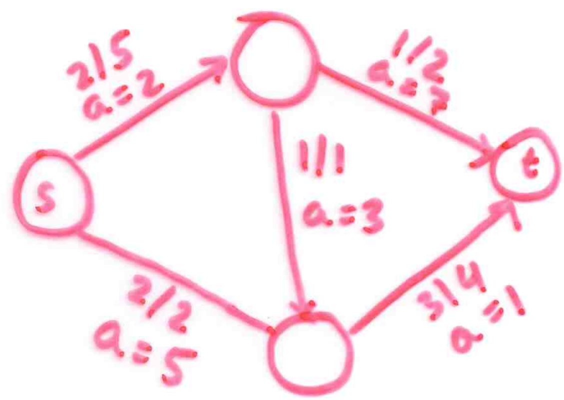
Can you rewrite the LP for the above with $O(|V| + |E|)$ constraints?

Minimum-cost flow

Each edge (u,v) has real valued cost $a(u,v)$ in addition to cost $c(u,v)$. If $f(u,v)$ is the flow over (u,v), we incur cost $a(u,v)\,f(u,v)$.

We are also given target flow d.
Goal: Send d units flow from s to t so that cost $\sum_{(u,v)\in E} a(u,v)\,f(u,v)$ is minimum.





Min-cost flow with target 4 units.

Minimize $\sum_{(u,v)\in E} a(u,v)\,f(u,v)$

s.t.

$$f(u,v) \le c(u,v) \quad \text{for } (u,v)\in \cancel{E}\,V\times V$$

$$f(u,v) = -f(v,u) \quad \text{for } (u,v)\in V\times V$$

$$\sum_{v\in V} f(u,v) = 0 \quad \text{for } u\in V-\{s,t\}$$

$$\sum_{v\in V} f(s,v) = d.$$

Multicommodity flow

Each edge $(u,v) \in E$ has $c(u,v) \geq 0$.

Given K commodities $K_1, K_2, \ldots K_k$.

$K_i = (s_i, t_i, d_i)$: specifies flow demand of d_i from source s_i to sink t_i.

- Let $f_i(u,v)$: flow over (u,v) on commodity K_i. It should satisfy the three flow constraints.

- $f(u,v) = \sum_{i=1}^{K} f_i(u,v) \leq c(u,v)$ (constraint)

Minimize 0

s.t.

$$\sum_{i=1}^{K} f_i(u,v) \leq c(u,v) \quad \text{for } u,v \in V \times V$$

$$f_i(u,v) = -f_i(u,v) \quad \text{for } i = 1, \ldots k, \text{ and } u,v \in V \times V$$

$$\sum_{v \in V} f_i(u,v) = 0 \quad \text{for } i = 1, \ldots, k \text{ and } u \in V - \{s,t\}.$$

$$\sum_{v \in V} f_i(s,v) = d_i \quad \text{for } i = 1, \ldots, k. ~~\text{and}~~$$