Rounding in LP

- We have seen that in LP, there were no constraints that the variables take only integer values. But, in some cases we need such constraints.
  Then, the LP with integrality constraints become what we call **Integer Programs**

- It turns out that some NP-hard problems can be reduced to integer programming (IP). In fact, we will reduce the vertex cover (VC) problem to IP. Since VC $\in$ NPC, IP $\in$ NPC.

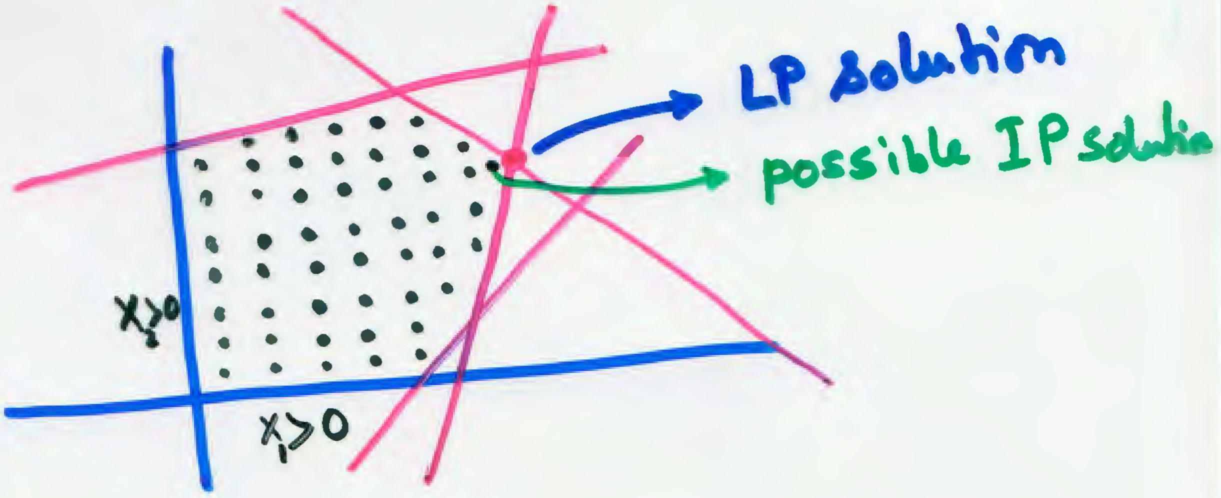
- Since IP is NP-hard, we relax it to LP and solve LP. The possible fractional solution to LP is rounded to integral one.

Integer program: (0-1) optimization:

$$\text{Minimize } \sum_i w_i x_i$$
$$A\cancel{x_i} \le b$$
$$x_i \geq 0$$
$$x_i \in \{0, 1\} \Rightarrow \text{this is integral constraint.}$$

Linear Program formulation:

$$\text{Minimize } \sum_i w_i x_i$$
$$Ax \le b$$
$$x_i \geq 0$$
$$x_i \in [0, 1] \Rightarrow \text{this is LP relaxation.}$$



Solving NP-hard problems with Approximations

- Reduce an NP-hard optimization problem to IP
- Relax IP to LP by dropping the integrality constraints
- Find optimal solution to LP
- Round the optimal solution to LP to an integral solution.

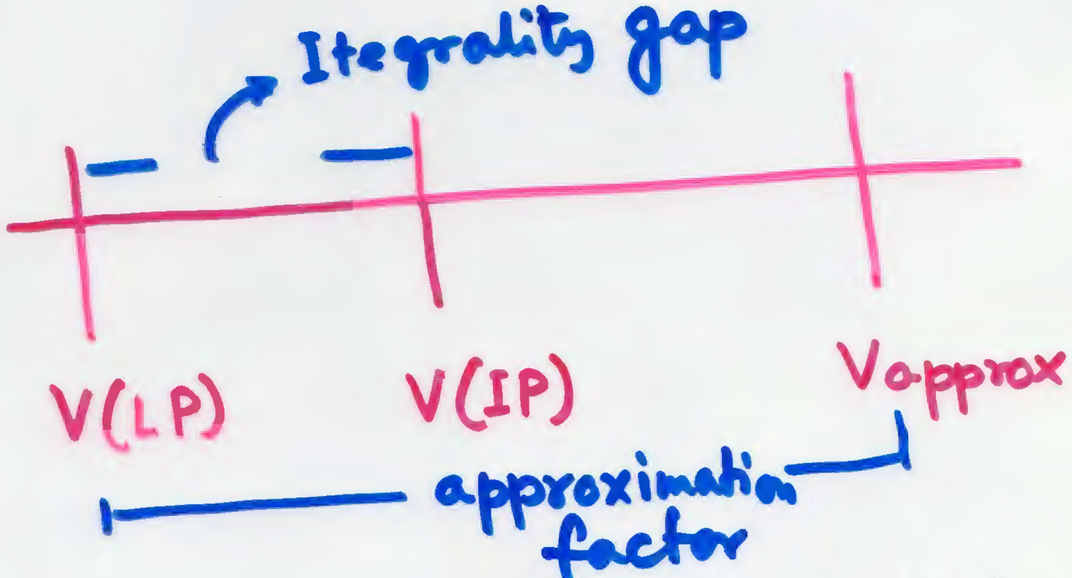
Let $V(IP)$ denote the value of the IP solution. Let $V(LP)$ be the same for LP relaxation and V_{approx} be the actual solution obtained after rounding

Claim. $V(LP) \le V(IP) \le V_{approx}$.

assuming we are solving a minimization

Proof. The first inequality $V(LP) \leq V(IP)$ is obvious because any integral solution is also a feasible solution for LP.

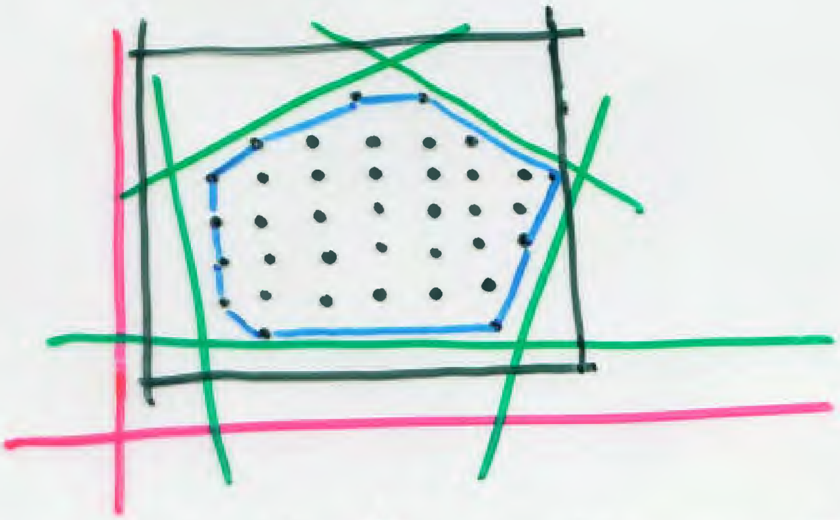
The second inequality is also obvious since an optimal solution to IP is better than any integral solution.



- $V(LP)$ is a lower bound to the optimal solution which $V(IP)$
- The difference $V(IP) - V(LP)$ is the integrality gap.

- The approximation factor to the actual optimal solution cannot be smaller than the integrality gap.

- So, the goal is to write an IP that has low integrality gap, and do the rounding efficiently so that approximation factor remains low.



The convex hull of all integer points inside the constraint polytope is the tightest (blue). But, it may involve too many constraints. Therefore, for efficiency one may choose more relaxed but smaller set. (black in fig.)

Approximate Weighted Vertex Cover

Given an undirected graph $G=(V,E)$ and a weight $\omega: V \to \mathbb{R}$ function for the vertex set, the weight of a vertex cover $V' \subseteq V$ is $\sum_{v \in V'} \omega(v) = \omega(V')$.

The problem is to find a vertex cover of minimum weight.

We cast the problem as an integer programming.

Let $x(v)$ be a variable for vertex v.

minimize $\sum_{v \in V} \omega(v) x(v)$

s.t.

$$x(u) + x(v) \geq 1 \quad \text{for } \forall (u,v) \in E$$

$$x(v) \in \{0,1\} \quad \text{for } \forall v \in V.$$

For each edge $(u,v) \in E$ at least one of $x(u)$ or $x(v)$ has to be 1.

We know that integer programming is NP-hard. So, we convert the integer programming to LP by relaxing that $x(v)$ can take any values between 0 and 1.

$$\text{Minimize } \sum_{v \in V} x(v)\, w(v)$$

s.t.

$$x(u) + x(v) \geq 1 \quad \text{for } \forall (u,v) \in E$$

$$x(v) \leq 1 \quad \text{for } \forall v \in V$$

$$x(v) \geq 0 \quad \text{for } \forall v \in V.$$

Any feasible solution to IP is also feasible for LP. Therefore, an optimal solution to LP is a lower bound for the optimal solution to IP and hence the minimum-weight VC problem.

Approx-Min-Weight-VC (G, ω)

```
C := ϕ;
compute x̄, the optimal solution to LP
for each v ∈ V do
    if x̄(v) ≥ 1/2 then
        C := C ∪ {v}
    endif
endfor
return C.
```

Theorem. Approx-Min-Weight-VC is a polynomial-time 2-approximation algorithm.

Proof. Since LP is polynomial-time solvable, the algorithm is ~~?~~ in P.

Let C^* be an optimal solution to the minimum weight vertex cover problem.

Let z^* be ~~an~~ the value of an optimal solution to LP.

Since an optimal cover is a feasible solution to LP, we have

(1) .... $z^* \le w(C^*)$

rounding the fractional values of the variables $\bar{x}(v)$, we produce a set C. We show C is a vertex cover.

- $x(u) + x(v) \geqslant 1$ for any $(u, v) \in E$

  $\Downarrow$

  at least one of $x(u)$ or $x(v) \geqslant \frac{1}{2}$.

  $\Downarrow$

  at least one of u or v will be in C.

  $\Downarrow$

  all edges $(u, v) \in E$ are covered by C.

- To see approximation consider

$$
\begin{aligned}
z^* &= \sum_{v \in V} w(v)\, \bar{x}(v) \\
&\geqslant \sum_{v \in V \mid \bar{x}(v) \geqslant \frac{1}{2}} w(v)\, \bar{x}(v) \\
&\geqslant \sum_{v \in V \mid \bar{x}(v) \geqslant \frac{1}{2}} w(v) \cdot \frac{1}{2} \\
&= \frac{1}{2} \sum_{v \in C} w(v). \\
&= \frac{1}{2} w(C).
\end{aligned}
$$

Combining with (1)

$\Rightarrow w(C) \le 2z^* \le 2w(C^*)$