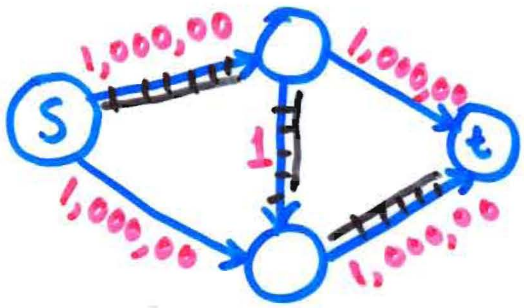
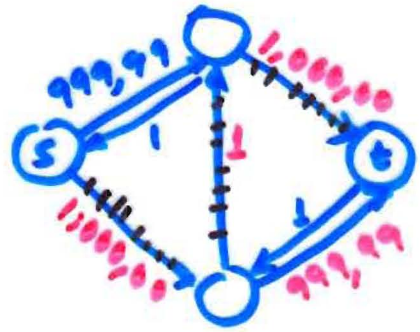
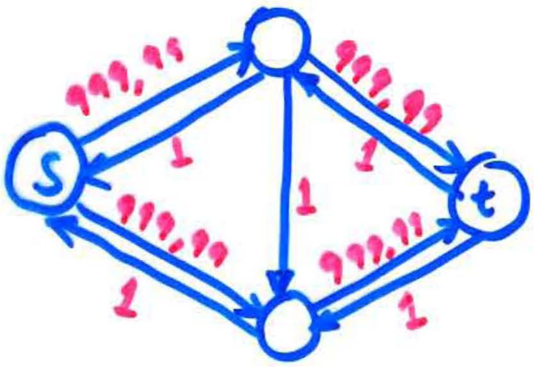
① TKD

Max-flow (Edmond-Karp algorith.

- Ford-Fulkerson's algorithm takes $O(|f^*|E)$ time which could be costly.







- The choice of paths in the residual network makes a difference.

- Edmond-karp algorithm chooses the augmenting path by a breadth-first search. This means the augmenting $p: s \leadsto t$ is the shortest path in G_f w.r.t. # of edges (property of BFS).

We analyze that E-K algorithm takes $O(VE^2)$ time. ② TKD

Lemma 1 For all vertices $v \in V - \{s, t\}$, the shortest distance $\delta_f(s,v)$ in G_f increases monotonically with flow augmentation.

Proof. • Suppose $\delta_f(s,v)$ decreases, i.e.,

* $\delta_{f'}(s,v) < \delta_f(s,v)$ where f' is the flow just after f.

• Assume v be the vertex with min. $\delta_{f'}(s,v)$ for which the decrease occurs.

• Let $p = s \rightsquigarrow u \rightarrow v$ be the s.p. in $G_{f'}$

① $\delta_{f'}(s,u) = \delta_{f'}(s,v) - 1$ and $(u,v) \in E_{f'}$.

• Because of our choice of v,

② $\delta_{f'}(s,u) \geq \delta_f(s,u)$.

Claim: $(u,v) \notin E_f$.

If (u,v) were in E_f, we would have

$$\delta_f(s,v) \leq \delta_f(s,u) + 1 \underset{②}{\leq} \delta_{f'}(s,u) + 1 \underset{①}{=} \delta_{f'}(s,v),$$ contradicting *

- We have $(u,v) \notin E_f$ and $(u,v) \in E_{f'}$!
  $\Rightarrow$ augmentation increased flow ~~along~~ from v to u. E-K algorithm augments only along shortest path
  $\Rightarrow$ s.p. in G_f from s to u has (v,u) as the last edge

- $$\begin{aligned} \delta_f(s,v) &= \delta_f(s,u) - 1 \\ &\underset{2}{\leq} \delta_{f'}(s,u) - 1 \\ &\underset{1}{=} \delta_{f'}(s,v) - 2 \end{aligned}$$
  Contradicts again *.

- Conclusion: vertex such as v does not exist.

Theorem: Total # of flow augmentations in E-K algorithm is $O(VE)$.

Proof. (u,v) is critical in G_f if $(u,v) \in P$ s.t. $c_f(P) = c_f(u,v)$.

④, TKD

- prove that each edge can become critical only at most $|V|/2 - 1$ times which also bounds # augmentations

- consider (u,v) being critical.

3 $\delta_f(s,v) = \delta_f(s,u) + 1$

- Edge (u,v) disappears from G_f

- (u,v) can reappear only if (v,u) appears on augmenting path later. for a flow f'

- $$\delta_{f'}(s,u) = \delta_{f'}(s,v) + 1$$
$$\geq \delta_f(s,v) + 1 \quad \text{(Lemma 1)}$$
$$\overset{3}{=} \delta_f(s,u) + 2$$

- distance of u from s increases by at least 2 for two consecutive times when (u,v) becomes critical.

- Since distance cannot be more than $O(|V|)$, (u,v) can be critical only $O(|V|)$ time.

- SO, $|E|$ edges can become $O(|VE|)$ times critical bounding # augmentations.