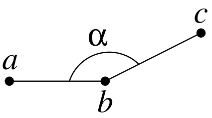
Predicates

Elisha Sacks

Planar Vector Geometry

- Vectors represent positions and directions.
- Vector u has Cartesian coordinates $u = (u_x, u_y)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector: $u/||u||$.
- Cross product: $u \times v = u_x v_y - u_y v_x$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos \alpha$.
- $u \times v = ||u|| \cdot ||v|| \cdot \sin \alpha$.

Predicates

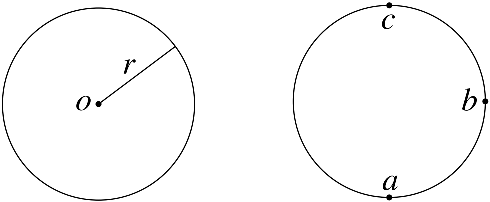


- A predicate is a polynomial in the parameters of objects.
- Our parameters are the Cartesian coordinates of points.
- We have already seen the left turn predicate for 2D points $\mathrm{LT}(a, b, c) = (c - b) \times (a - b)$.
- It has the same sign as $\sin\alpha$ with $\alpha = \angle(c - b, a - b)$.
- It can also be expressed as the determinant

$$\mathrm{LT}(a, b, c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix}$$

- Another simple predicate is the order of points a and b in direction u: $(b - a) \cdot u$ is positive if b comes after a.

Circles



- A circle can be represented by a center o and a radius r.
- A circle can also be represented by points a, b, and c.
- The first representation has three independent parameters.
- The second representation has six dependent parameters.
- Circle predicates depend on the choice of representation.
- A point p is outside an o, r circle if $||p - o|| - r$ is positive.
- The predicate can be rewritten without a square root as $(p - o) \cdot (p - o) - r^2$.

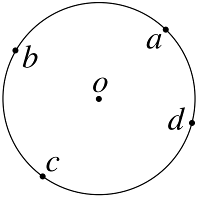
Point in Circle

- The predicate for a point p and an a, b, c circle is

$$\begin{vmatrix} a_x & a_y & a \cdot a & 1 \\ b_x & b_y & b \cdot b & 1 \\ c_x & c_y & c \cdot c & 1 \\ p_x & p_y & p \cdot p & 1 \end{vmatrix}$$

- The predicate is positive when p is outside the circle if a, b, c are in counterclockwise order around the circle.
- Replacing p with (x, y) and expanding along the last row yields $\mathrm{LT}(a, b, c)(x^2 + y^2) + ux + vy + w$.
- This is the equation of a circle after dividing by $\mathrm{LT}(a, b, c)$.
- It is the circle through a, b, c because the determinant is zero when p equals a, b, or c, since two rows are equal.
- It is positive for sufficiently large p because the LT is positive.

Angle Order



- Task: sort points counterclockwise around a point o.
- Need to define the order of points a and b around o.
- If $a_y > o_y$ and $b_y < o_y$, a is first.
- If $a_y < o_y$ and $b_y > o_y$, b is first.
- Otherwise, a is first if $\mathrm{LT}(a, o, b) < 0$.
- What are the degenerate cases?

Spatial Vector Geometry

- Vectors represent positions and directions.
- Vector u has coordinates $u = (u_x, u_y, u_z)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y + u_z v_z$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector: $u/||u||$.
- Cross product:
  $u \times v = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos \alpha$.
- $u \times v = (||u|| \cdot ||v|| \cdot \sin \alpha)\, n$ with n a unit-vector perpendicular to u and v.

Predicates

- Point d is on the counterclockwise side of triangle abc if

$$\mathrm{LT}(a,b,c,d) = \begin{vmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ d_x & d_y & d_z & 1 \end{vmatrix} > 0.$$

- Point p is outside the sphere through points a, b, c, d with $\mathrm{LT}(a,b,c,d) > 0$ if

$$\begin{vmatrix} a_x & a_y & a_z & a \cdot a & 1 \\ b_x & b_y & b_z & b \cdot b & 1 \\ c_x & c_y & c_z & c \cdot c & 1 \\ d_x & d_y & d_z & d \cdot d & 1 \\ p_x & p_y & p_z & p \cdot p & 1 \end{vmatrix} > 0.$$

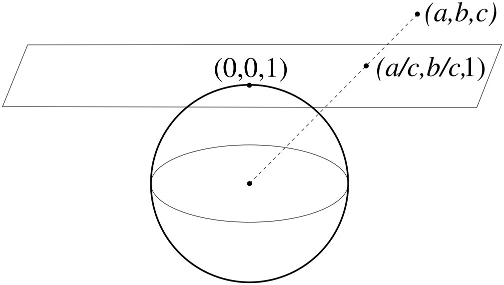
Projective Geometry

Elisha Sacks

Motivation

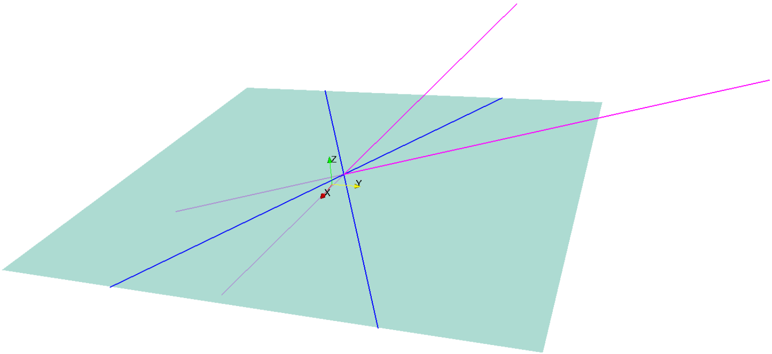
- The projective plane adds points at infinity to the affine plane.
- Two parallel lines intersect at a point at infinity.
- Asymptotes of algebraic curves are points at infinity.
- These concepts remove special cases from affine geometry.
- Any two projective lines intersect at a unique point.
- Every projective algebraic curve consists of closed loops.

Projective Points



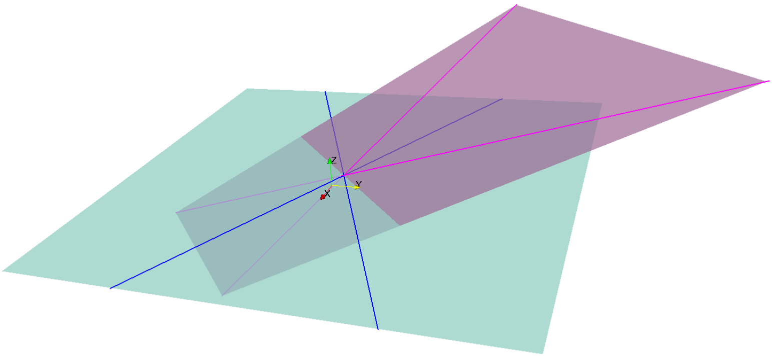
- A projective point is a line through the origin of $\Re^3$.
- Its homogenous coordinates are any point (a, b, c) on the line.
- If $c \neq 0$, it intersects the $z = 1$ plane at $(a/b, b/c, 1)$ and represents the affine point $(a/c, b/c)$.
- If $c = 0$, it is at infinity.

Projective Points



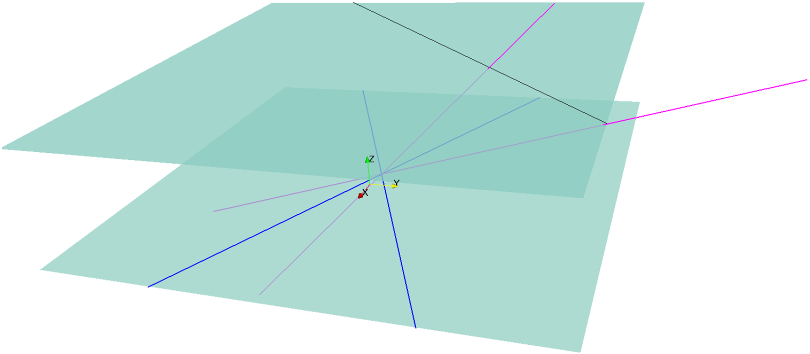
- The pink lines are affine points.
- The blue lines are points at infinity.

Projective Lines



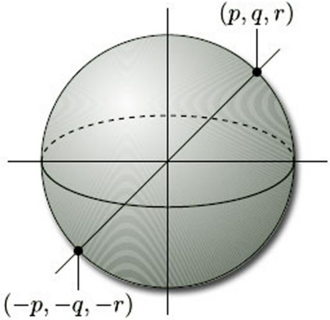
- A projective line is a plane through the origin of $\Re^3$.
- The line $ux + vy + wz = 0$ is written as $\langle u, v, w \rangle$.
- It consists of the affine points $(a, b, 1)$ with (a, b) on the affine line $ux + vy + w = 0$, plus $(-v, u, 0)$ at infinity.
- The line at infinity $z = 0$ consists of all the points at infinity.

Plane Model



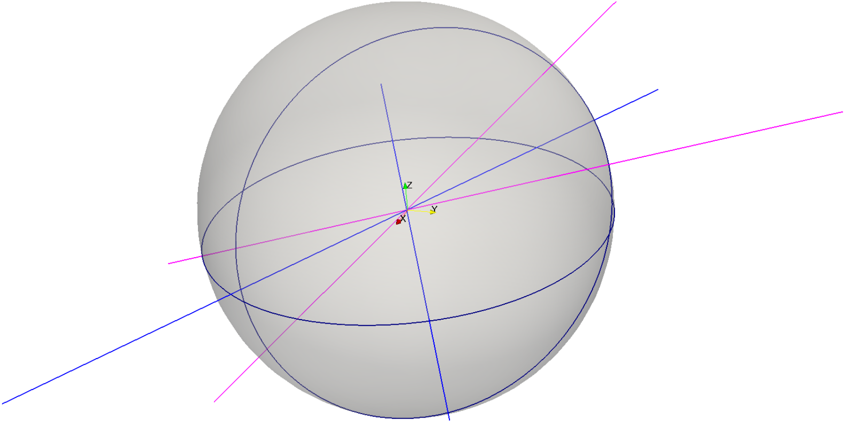
- Map an affine point to its intersection with the $z = 1$ plane.
- Map the line at infinity to the $z = 0$ plane.
- Affine lines are on the $z = 1$ plane.
- Their points at infinity are on the $z = 0$ plane.

Sphere Model



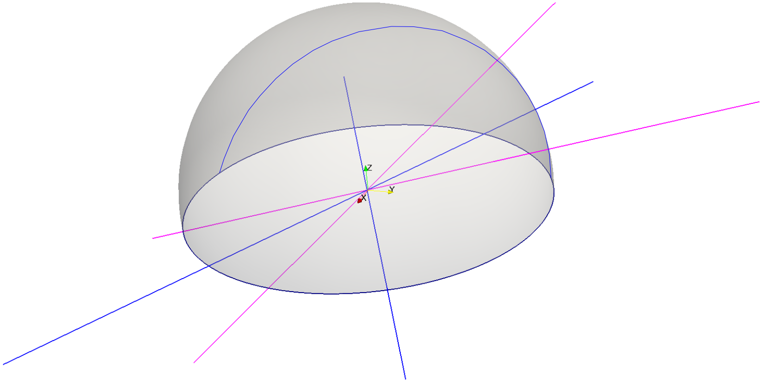
- Map a point to its two intersections with the unit sphere.
- Lines map to great circles.
- The line at infinity maps to the equator.

Sphere Model



- The pink lines (affine points) lie on the great circle.
- The blue lines (points at infinity) lie on the equator.

Hemisphere Model



- Map a point to its intersection with the northern hemisphere.
- Affine lines map to great semicircles.
- The line at infinity maps to the equator.

Points and Lines

- The line through points p and q has normal $p \times q$.
- Lines m and n intersect at the point $p = m \times n$.
- If m and n are affine and non parallel, p is affine.
- If m and n are parallel, p is at infinity.
  $(u, v, w) \times (u, v, w') = (-v(w - w'), u(w - w'), 0) = (-v, u, 0)$
- If m is the line at infinity, p is n's point at infinity.
  $(0, 0, 1) \times (u, v, w) = (-v, u, 0)$
- The line at infinity is parallel to every affine line.

Examples

The affine lines $x + y - 1 = 0$ and $x - y - 1 = 0$ intersect at $(1, 0)$. The projective lines $x + y - z = 0$ and $x - y - z = 0$ intersect at $(1, 1, -1) \times (1, -1, -1) = (1, 0, 1)$.

The affine lines $x - y - 1 = 0$ and $x - y - 2 = 0$ are parallel. The projective lines $x - y - z = 0$ and $x - y - 2z = 0$ intersect at $(1, -1, -1) \times (1, -1, -2) = (1, 1, 0)$.

The affine points $(1, 1)$ and $(2, 3)$ define the line $-2x + y + 1 = 0$. The projective points $(1, 1, 1)$ and $(2, 3, 1)$ define the line $-2x + y + z = 0$, since $(1, 1, 1) \times (2, 3, 1) = (-2, 1, 1)$.

The affine line through (a, b) in direction (c, d) is the projective line $(a, b, 1) \times (c, d, 0)$.

Duality

There is a natural duality between the point $p = (a, b, c)$ and the line $\hat{p} = \langle a, b, c \rangle$.

Unlike the affine case, every line has a dual.

If a point p is on a line l, $\hat{l}$ is on $\hat{p}$, since the original equation is $p \cdot l = 0$ and the dual equation is $\hat{l} \cdot \hat{p} = 0$.

If a line l passes through points p and q, $\hat{p}$ and $\hat{q}$ intersect at $\hat{l}$, since $l = p \times q$ implies $l \cdot p = 0$ and $l \cdot q = 0$, so $\hat{l} \cdot \hat{p} = 0$ and $\hat{l} \cdot \hat{q} = 0$.

Projective Varieties

A projective variety is the zero set of a homogeneous polynomial $p(x, y, z)$; every term of the polynomial has the same degree d.

Examples: a projective line is homogeneous with $d = 1$ and $xy - z^2$ is homogeneous with $d = 2$.

A homogeneous polynomial is zero or nonzero for all the homogeneous coordinates of a projective point.

The projective variety $p(x, y, z) = 0$ consists of the affine variety $p(x, y, 1) = 0$, which is its intersection with the plane $z = 1$, plus the points at infinity $p(x, y, 0) = 0$, which are its intersection with the plane $z = 0$.

Example: $xy - z^2 = 0$ consists of the hyperbola $xy = 1$ plus the points at infinity $(1, 0, 0)$ and $(0, 1, 0)$.

Homogenization

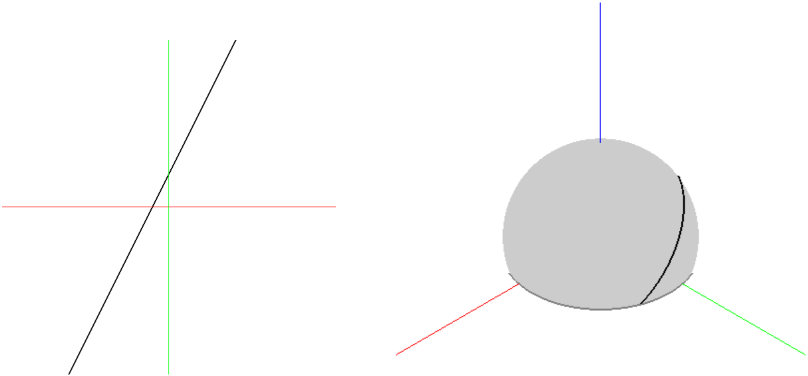
Homogenization: Convert an affine polynomial $p(x, y) = 0$ to a homogeneous polynomial in x, y, z by substituting x/z for x and y/z for y then clearing the denominator.

Example: the hyperbola $xy - 1 = 0$ homogenizes to $xy - z^2 = 0$.

Dehomogenization: Convert a homogeneous polynomial to an affine polynomial by substituting $z = 1$.

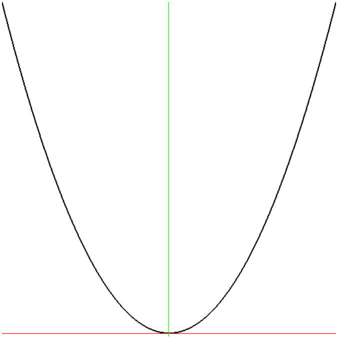
Let $q(x, y, z)$ be the homogenization of $p(x, y)$. The affine variety of p equals the affine part of the projective variety of q, that is the points with $z = 1$. The points at infinity of q are the zeroes of the leading (highest degree) terms of p, since the other terms of q are zero for $z = 0$.

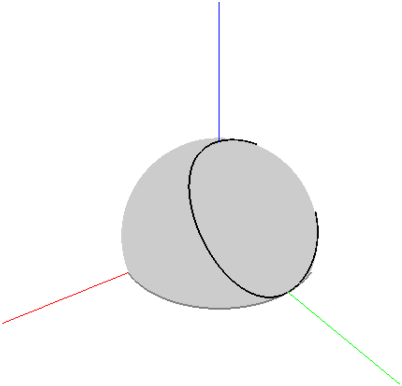
Line



The line $y = 2x + 2$ homogenizes to $2x - y + 2z = 0$ with point at infinity $(1, 2, 0)$ that equals $(0.447, 0.894, 0)$ in the hemisphere model. This point converts the affine line into a loop.

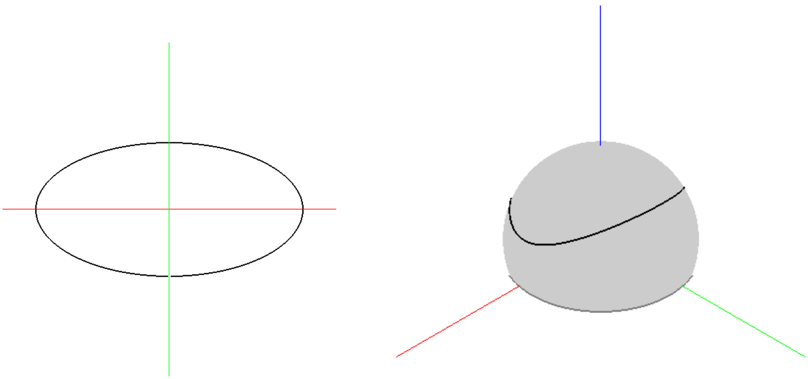
Parabola





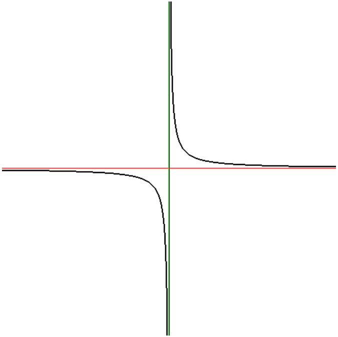
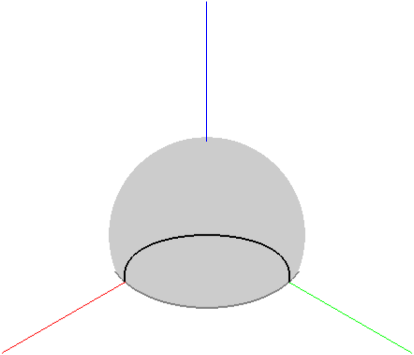
The parabola $y = x^2$ homogenizes to $yz - x^2 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine parabola into a loop.

Ellipse



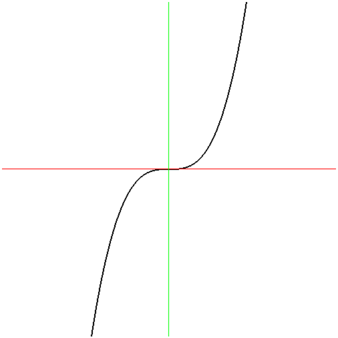
The ellipse $x^2 + 4y^2 = 4$ homogenizes to $x^2 + 4y^2 - 4z^2 = 0$ with no points at infinity, since the affine ellipse is already closed.

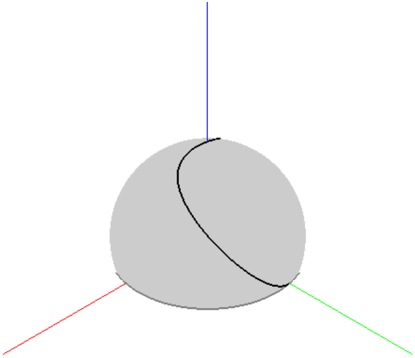
Hyperbola

The hyperbola $xy = 1$ homogenizes to $xy - z^2 = 0$ with points at infinity $(1, 0, 0)$ and $(0, 1, 0)$. These points convert the two components of the affine hyperbola into a single loop.

Cubic





The cubic $y = x^3$ homogenizes to $yz^2 - x^3 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine variety to a loop.

Complex Projective Geometry

The true setting for algebraic geometry is complex projective space.

Example: The circle $x^2 + y^2 = 1$ homogenizes to $x^2 + y^2 = z^2$ with points at infinity $(\pm 1, i)$.

Bezout's theorem If polynomials p and q of degrees m and n do not have a common component, they have mn complex projective roots counting multiplicity.

Example: The intersection of two circles consists of two real or complex affine points and the two points at infinity $(\pm 1, i)$.

Projective Geometry in n Dimensions

- Every affine space k^n has a projective space $P(k^n)$.
- The projective points are lines through the origin of k^{n+1}.
- The homogeneous coordinates are $(x_1, \ldots, x_{n+1})$.
- If $x_{n+1} \neq 0$, x maps to the affine point $(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$.
- If $x_{n+1} = 0$, x is at infinity.
- The plane, sphere, and hemisphere models generalize.
- The points at infinity are isomorphic to $P(k^{n-1})$.
- The space $P(\Re^3)$ is used in graphics.

Limitations of Projective Geometry

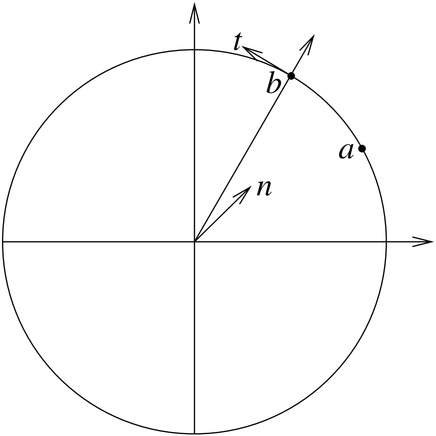
- Although the projective plane eliminates the special cases of the affine plane, it also has disadvantages.
- The projective plane is not orientable.
- Lines have one side: removing a line leaves a connected set.
- Segments are ambiguous: two points split their line into two connected parts that cannot be distinguished.
- Likewise, the direction from a to b is ambiguous, e.g. each point at infinity lies in two directions from every affine point.
- Convexity is undefined.

Oriented Projective Geometry

- Stolfi [1] defines an oriented version of projective geometry that solves these problems at the cost of increased complexity.
- Each projective point is split into two oriented points: the line ka is split into the rays ka and $-ka$ with $k > 0$.
- Each projective line is split into two oriented lines likewise.
- In the sphere model, opposite points are no longer identified and great circles are oriented.
- The convex hull of a set of points is the dual of the envelope of the dual lines.

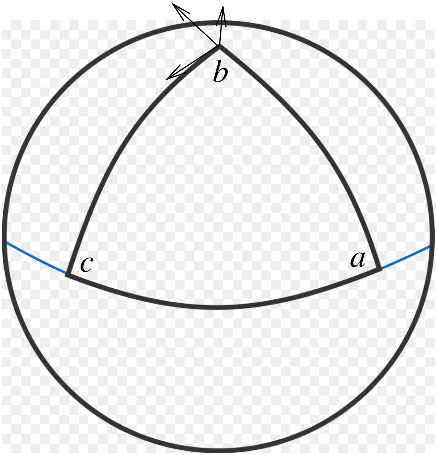
[1] J. Stolfi, Oriented Projective Geometry, Academic Press, 1991.

Spherical Computational Geometry



- A point a has normal vector a.
- A segment ab lies in the plane with normal $n = a \times b$ and is traversed counterclockwise around n.
- The tangent to ab at b is $t(ab, b) = (a \times b) \times b = n \times b$.

Spherical Computational Geometry



- The path *abc* is a left turn if $b \cdot t(ab, b) \times t(bc, b) > 0$.
- The segment intersection predicate is as before.

Spherical Computational Geometry

- Some algorithms transfer easily from the plane to the sphere.
- Some rely on properties of the plane that differ on the sphere.
- For example, the sum of the angles of a triangle is not 180°.
- Spherical geometry is an instance of Riemann geometry.