"How to draw a graph nicely is an art rather than a science because too little is known about it."

4 Triangulating polygons and drawing graphs

Polygons and triangulations are fundamental objects in geometry and in geometric algorithms, where they often occur in a larger or more general context. For example, we will use results on polygons and polygon triangulations to prove that every planar graph has a plane straight-line embedding.

Polygons and triangulations. A *simple closed curve* is (the image of) a continuous and injective map $\mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1$ is the circle of points in $\mathbf{R}^2$ at distance 1 from the origin. It decomposes $\mathbf{R}^2$ into two regions, the bounded *inside* and the unbounded *outside*. A *polygon* is a piecewise linear simple closed curve; it is an *n-gon* if it has n edges (linear pieces). We can think of a polygon as a plane and straight-line embedding of a cycle without repeated vertices. A *diagonal* is a line segment inside the polygon that connects two of its vertices. To *triangulate* a polygon means to draw pairwise non-crossing diagonals that decompose the inside into triangles.

THM. 4.1 Every n-gon has a triangulation, and every triangulation of the n-gon consists of $n-2$ triangles and $n-3$ diagonals.

PROOF. For $n = 3$, the polygon is already triangulated with $n - 2 = 1$ triangles and $n - 3 = 0$ diagonals. So assume $n > 3$. It suffices to prove we can draw a single diagonal inside the n-gon. It cuts the polygon into an n_1-gon and an n_2-gon, with $n_1 + n_2 = n + 2$ and $n_1, n_2 \leq n - 1$. By induction, the n_i-gon can be triangulated with $n_i - 2$ triangles and $n_i - 3$ diagonals. The two triangulations together form a triangulation of the n-gon using $(n_1 - 2) + (n_2 - 2) = n - 2$ triangles and $(n_1 - 3) + (n_2 - 3) + 1 = n - 3$ diagonals.

To prove there is a diagonal, let b be the leftmost vertex of the n-gon, and let a and c be its two neighbors. If ac is a diagonal we are done, see figure 4.1 left. Otherwise, let x be the leftmost vertex of the n-gon that lies inside the triangle abc; bx is a diagonal, see figure 4.1 right. □

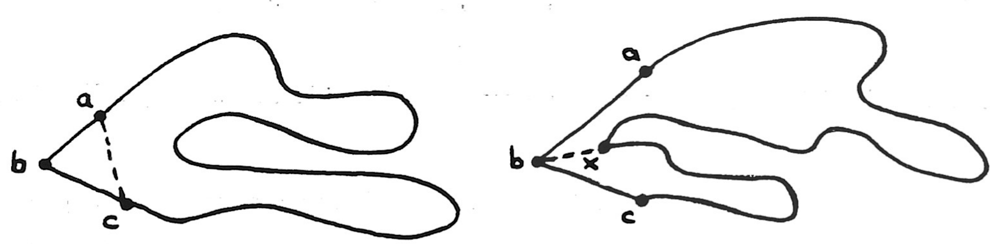

Figure 4.1: Either the neighbors of b define a diagonal (left), or there is a vertex x that defines a diagonal with b (right).

REMARK. The $n-3$ for the number of diagonals is important. Recall the claim in section 3 that a face with $k > 3$ edges in the plane embedding of a graph can be decomposed into smaller faces by adding a diagonal. A particular diagonal cannot be added if it already exists outside the face. If there is a tangling tree in the face then there is no problem because any diagonal connecting a leaf of this tree to another vertex can be added. Otherwise, there are $\frac{k(k-3)}{2}$ possible diagonals. The diagonals that already exist form a plane decomposition of the outside, so there can be at most $k - 3$ such diagonals. Since $\frac{k(k-3)}{2} > k - 3$ for all $k \geq 4$, there is always a diagonal available.

We skip the discussion of algorithms that triangulate polygons although this is an interesting topic. The plane-sweep method of section 2 can be modified so it finds the $n-3$ diagonals that triangulate an n-gon in time $O(n \log n)$. For a many years it was an open problem whether $O(n)$ time suffices, and this has recently been proven to be the case by Chazelle [1].

Dual tree. We can think of the triangles decomposing an n-gon as the nodes of a tree: two nodes are joined by an arc if they share a diagonal, see figure 4.2. Since a triangle has only three sides, each node has degree 3 or less. The number of nodes is $n-2$ and the number of arcs is $n-3$. A *leaf* is a node with only one neighbor; it corresponds to a triangle bounded by one diagonal and two edges of the n-gon. Such a triangle is called an *ear*.

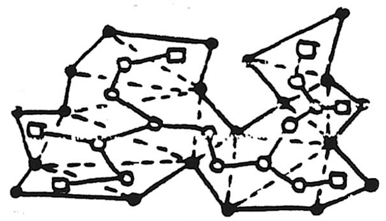

Figure 4.2: The triangles and diagonals of a triangulated polygon form a tree. The ears correspond to the leaves.

LEMMA A. Every triangulation of an n-gon, $n \geq 4$, has at least two ears.

There are at least two easy proofs of this fact. The first uses the dual tree and the observation that every tree with $n-2 \geq 2$ nodes has at least two leaves. The second is a counting argument. A triangle is bounded by 0, 1, or 2 polygon edges, and since there are n edges for $n-2$ triangles, at least two triangles receive two such edges.

Convexity and star-convexity. Intuitively, a polygon or its inside is convex if all pairs of contiguous edges turn the same way. We present a different definition that generalizes to other objects and higher dimensions. A subset $A \subseteq \mathbf{R}^2$ is *convex* if $a, b \in A$ implies the line segment $ab = \{x = \lambda a + (1-\lambda)b \mid 0 \leq \lambda \leq 1\}$ is contained in A, see figure 4.3. A polygon is *convex* if its inside is convex. The *kernel* of A is the set of points $a \in A$ so that $ab \subseteq A$ for all $b \in A$, see figure 4.3. For example, if A is convex then A is its own kernel. It is easy to construct polygons with empty kernel, and there are examples already for $n = 6$ edges. A is *star-convex* if its kernel is non-empty. The kernel of a polygon can be constructed by taking an open half-plane for each edge and forming the common

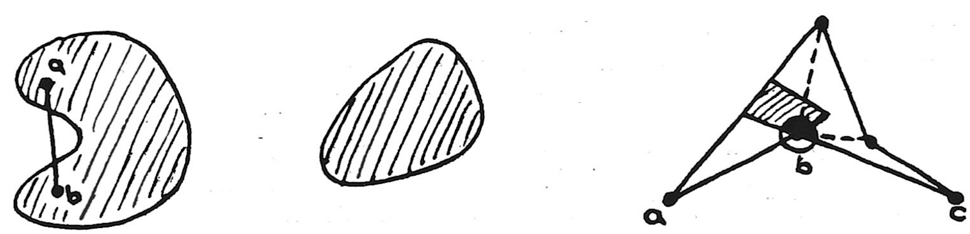

Figure 4.3: The set on the left is not convex, the set in the middle is convex, and the right shows the kernel of a non-convex pentagon.

intersection of these half-planes. More specifically, the half-plane defined by an edge xy is bounded by the line through x and y and consists of all points on the same side of the line as the points inside the polygon sufficiently close to the midpoint, $\frac{x+y}{2}$. If non-empty, the kernel is the inside of a convex polygon.

LEMMA B. Every 5-gon is star-convex.

PROOF. Triangulate the 5-gon. The two diagonals used share a common vertex, b. Let a and c be the neighboring vertices of b. Take a sufficiently small open disk around b and intersect it with the open half-planes defined by ab and bc. The result is a non-empty open sector, see figure 4.3 right. The other three half-planes contain the disk and therefore the sector. ◻

Drawing graphs with straight edges. Plane embeddings of graphs are not restricted to straight edges, but it turns out that straight edges suffice. Note however, that once the locations of the vertices are fixed, it might be necessary to use non-straight edges in order to avoid crossings, see figure 3.1.

THM. 4.2 (Fáry). Every planar graph has a straight-line embedding.

PROOF. Let $G = (V, E)$ be the planar graph and add edges until it is maximally connected. Find an arbitrary plane embedding, and let a, b, c be the vertices of the outside face. If a, b, c are the only vertices in V we may assume the embedding is straight-line. Otherwise, modify the embedding using the following recursive algorithm.

step 1. Find a vertex $v \in V - \{a, b, c\}$ with degree $k \leq 5$ and remove v together with all edges that contain it. Add $k - 3$ edges to decompose the thus generated face into triangles.

step 2. Recursively, find a straight-line embedding of the new graph.

step 3. Remove the edges added in **step 1** and choose a point p in the kernel of the thus created k-gon, $k \leq 5$. Use straight edges to join p to the vertices of the k-gon.

Note that a vertex v as required in **step 1** can be found unless G has only 3 vertices. This is because a, b, c have degree 3 or more each, so the total degree of the other $n - 3$ vertices is at most $(6n - 12) - 9$. The average degree is at most $\frac{6n-21}{n-3} < 6$, hence there exists a vertex v different from a, b, c whose degree is 5 or less. □

REMARK. The proof of theorem 4.2 is formulated as a recursive algorithm. Equivalently, we can think of **step 2** as the induction hypothesis applied to the smaller graph. It is easy to translate the proof into a concrete algorithm. A drawback of the algorithm is that it tends to cluster vertices in small regions. Methods that avoid tight clusters can be found in [2, 5].

Homework exercises

4.1 Denote $\{1, 2, \ldots, k\}$ by $[k]$. A *k-coloring* of a graph $G = (V, E)$ is an assignment $\chi : V \to [k]$ so that $\chi(\nu) \neq \chi(\mu)$ if $\nu\mu \in E$. Prove that every planar graph has a 6-coloring.
Remark. In fact, every planar graph has a 4-coloring [4] (which is a 6-coloring with two unused colors), but no short proof of this fact is known. To prove the existence of a 5-coloring is still relatively straightforward and only slightly more involved than for a 6-coloring.)

4.2 Let $T = (V, E)$ be a degree-3 tree, that is, each vertex has degree 3 or less. The removal of an edge, e, decomposes T into two trees, $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$, with $V_1 \cup V_2 = V$ and $E_1 \cup E_2 = E - \{e\}$. Define $n = \text{card } V$ and $\beta(e) = \min\{\text{card } V_1, \text{card } V_2\}$. Show that T has an edge e_0 with $\beta(e_0) \geq \frac{n-1}{3}$.
(Remark. Note that a polygon triangulation corresponds to a degree-3 tree, and if the polygon has $n + 2$ edges the tree has n vertices. The tree edge e_0 with $\beta(e_0) \geq \frac{n-1}{3}$ corresponds to a diagonal that cuts the $(n + 2)$-gon into an $(n_1 + 2)$-gon and an $(n_2 + 2)$-gon so that $\min\{n_1, n_2\} \geq \frac{n+1}{3}$. Equivalently, $\min\{n_1 + 2, n_2 + 2\} \geq \frac{n+5}{3} = \frac{n+2}{3} + 1$, that is, each side contains at least one third of the original polygon edges.)

References

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