"The question whether or not a graph can be drawn without edges crossing each other is topological. Once it is drawn, questions become geometric."

3 Planar graphs

In the work of Turán and others, graphs were predominantly considered geometric or topological objects drawn in the plane. The historical development has soon lead to a more abstract view, in which a graph is just a set of elements (vertices or nodes) and a set of pairs of elements (edges or arcs). One of the strongholds of the connection between the abstract and the concrete is the area of planar graphs. Graphs that can be drawn crossing free in the plane enjoy many properties that do not hold for general graphs.

Graphs. We begin by recollecting common definitions needed in this section. A *graph* is an ordered pair of sets, $G = (V, E)$. The elements in V are called *vertices* and the elements in E are vertex pairs and called *edges*. We consider only finite undirected simple graphs, that is, V is finite and $E \subseteq \binom{V}{2}$. So each edge is an unordered pair. The *degree* of a vertex, u, is the number of edges that contain u. A *path* is a sequence of vertices, $t_0, t_1, \ldots, t_j$ so that $\{t_{i-1}, t_i\} \in E$ for all $1 \leq i \leq j$; it *starts* at t_0 and *ends* at t_j. Two vertices $s, u \in V$ are *adjacent* if they form an edge in E, and they are *connected* if there is a path starting at s and ending at u. G is *connected* if every pair of vertices is connected. A graph $F = (U, D)$ is a *subgraph* of G if $U \subseteq V$ and $D \subseteq E$. The subgraph of G *induced* by a subset $U \subseteq V$ is $(U, E \cap \binom{U}{2})$. A *(connected) component* of G is an induced subgraph that is connected and maximal. A *cycle* is a path that starts and ends at the same vertex. A *(free) tree* is a connected graph without cycle, and a *spanning tree* of a graph $G = (V, E)$ is a tree whose vertex set is V. We will use the fact that every connected graph has a spanning tree, and that a tree with n vertices has precisely $n - 1$ edges.

Embedding in the plane. Whenever we draw a graph in the plane, $\mathbb{R}^2$, we draw a vertex as a point or little circle and an edge as a curve connecting two such points. We make this more formal. A *simple curve* is (the image of) a continuous and injective map $[0, 1] \to \mathbb{R}^2$. The images of 0 and 1 are the *endpoints* of the curve. An *embedding*, ε, of $G = (V, E)$ in the plane maps a vertex $u \in V$ to a point $\varepsilon(u) \in \mathbb{R}^2$ an edge $uv = \{u, v\} \in E$ to a simple curve $\varepsilon(uv)$ with endpoints $\varepsilon(u)$ and $\varepsilon(v)$. To simplify the discussion, an embedding is often implicitly assumed and no distinction is made between a vertex and its point or an edge and its simple curve. By definition, a single edge has no self-intersections, but it is certainly possible that two edges cross. An embedding ε is *plane* if

(i) $\varepsilon(u) \neq \varepsilon(v)$ for different vertices u and v, and

(ii) $\varepsilon(st) \cap \varepsilon(uv) = \begin{cases} \emptyset & \text{if } \{s,t\} \cap \{u,v\} = \emptyset \\ \varepsilon(s) & \text{if } s = u \text{ and } t \neq v. \end{cases}$

An embedding is *straight-line* if every edge is a (straight) line segment. Finally, we call $G = (V, E)$ *planar* if it has a plane embedding. These concepts are illustrated in figure 3.1.

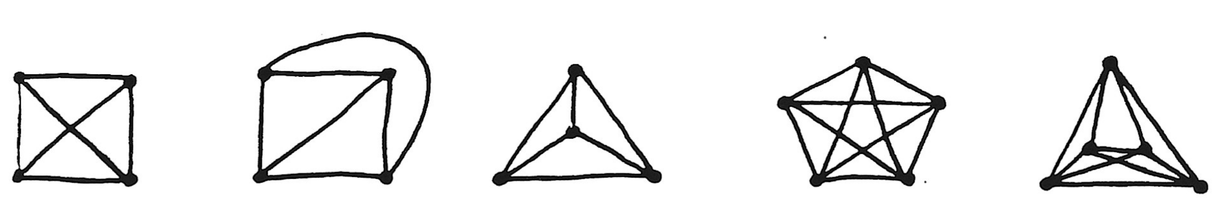

Figure 3.1: K_4 is the complete graph on 4 vertices. The first three embeddings from left to right are non-plane and straight-line, plane and not straight-line, and plane and straight-line. On the right, the two embeddings of K_5 are both non-plane, but this does not yet imply that K_5 is not planar; it might be we just missed a plane embedding.

Euler's relation. A plane embedding of a graph decomposes $\mathbf{R}^2$ into disconnected regions, called *faces*. Informally, the faces are the pieces of $\mathbf{R}^2$ formed by cutting along the edges of the embedding. As an example, consider the embedding of the "cube" in figure 3.2, which has $n = \text{card}\, V = 8$ vertices, $e = \text{card}\, E = 12$ edges, and $f = 6$ faces. Note that the outside region is also counted as a face.

Figure 3.2: A spanning tree of the "cube" has 7 edges, and the cube can be completed by adding the other 5 edges.

THM. 3.1 (Euler's relation). Let G be a planar graph with c components. Then $n - e + f = 1 + c$.

PROOF. Consider an arbitrary but fixed plane embedding of G. Assume first that G is connected, and let $T = (V, A)$ be a spanning tree, see figure 3.2. Then $\text{card}\, V = n$, $\text{card}\, A = n - 1$, and there is only one face. Observe that $n - (n-1) + 1 = 2$, so the assertion holds for T. Any additional edge leaves n unchanged and it increases e by 1. Because it forms a cycle in T, the edge decomposes a face into two, so it also increases f by 1. So adding an edge does not change $n - e + f$, which completes the proof assuming G is connected.

If G is not connected we get $n_i - e_i + f_i = 2$ for the ith component. Clearly, $n = \sum_i n_i$ and $e = \sum_i e_i$. To count the faces, consider adding a component at a time. Initially, there is one face, $\mathbf{R}^2$, and when we add the ith component we decompose one face into f_i faces. Thus, $f = 1 + \sum_i (f_i - 1) = 1 - c + \sum_i f_i$. We finally have

$$n - e + f = 1 - c + \sum_i (n_i - e_i + f_i) = 1 + c.$$

□

Note that the embedding does not play a role in Euler's relation, as long as it is plane so that faces are defined. It is thus a theorem for planar graphs, and not only for plane embeddings. Generalizations of the Euler relation are abundant in the geometry and topology literature, see e.g. [5]. Examples are graphs embedded on surface other than the plane and complexes embedded in 3 or higher dimensions.

Maximally connected planar graphs. A planar graph $G = (V, E)$ is *maximally connected* if adding any more edge to E would result in a violation of planarity. In a maximally connected planar graph with 3 or more vertices, every face (also the outside) must be bounded by precisely three edges. For if a face is bounded by only one or two edges, G is not a simple graph, and if a face is bounded by more than three edges then it can be further subdivided.[1]

LEMMA A. A maximally connected graph with $n \geq 3$ vertices has $e = 3n - 6$ edges and $f = 2n - 4$ faces.

PROOF. Every face has 3 edges and every edge belongs to 2 faces. So if we count the 3 edges of each face we count each edge twice, that is $3f = 2e$. Therefore, $n - e + \frac{2e}{3} = 2$ or $e = 3n - 6$ and $n - \frac{3f}{2} + f = 2$ or $f = 2n - 4$. □

This implies $e \leq 3n - 6$ and $f \leq 2n - 4$ for general planar graphs of $n \geq 3$ vertices. Another useful consequence of the above lemma is the following.

[1] This claim is not entirely trivial. Consider a face bounded by a cycle of $k > 3$ edges. A *diagonal* is an edge that connects two non-contiguous vertices and goes through the face. Not every diagonal can be used to subdivide the face, because the vertex pair it connects might already be connected by an edge outside the face. It is, however, sufficient to show that there is at least one diagonal that can be used for subdividing the face, see also section 4.

LEMMA B. Every planar graph has a vertex of degree at most 5.

PROOF. The average degree is $\frac{2e}{n} \leq \frac{6n-12}{n} < 6$. Not all vertices can have degree higher than the average. □

Testing for planarity. A popular topic in planar graph theory is testing whether or not a given graph is planar. This can be done in time $O(n)$ by fairly involved algorithms, see e.g. [1, 2]. A related topic is to structurally characterize when a graph is planar. Possibly the best known characterization is due to Kuratowski [3] who showed that a graph is planar as long as it does not have a subgraph homeomorphic to K_5 and $K_{3,3}$. We begin by arguing that K_5 and $K_{3,3}$ are not planar.

LEMMA C. K_5 and $K_{3,3}$ are not planar.

PROOF. Recall that K_5 has 5 vertices and $\binom{5}{2} = 10$ edges. Assume K_5 has a plane embedding. As a consequence of lemma A, we get $f \leq 2n - 4 = 6$, and thus $n - e + f \leq 5 - 10 + 6 = 1 < 2$, which contradicts Euler's relation.

$K_{3,3}$ consists of $3 + 3$ vertices and all 9 edges connecting the first 3 with the second 3 vertices. Note that every cycle in $K_{3,3}$ has even length, so every face in an assumed plane embedding of $K_{3,3}$ is bounded by 4 or more edges. Hence, $4f \leq 2e = 18$, and therefore $n + e - f \leq 6 - 9 + 4 = 1 < 2$, again a contradiction to Euler's relation. □

It is possibly surprising that these two graphs, K_5 and $K_{3,3}$, are the quint-essential non-planar graphs, as observed by Kuratowski. To make this formal, we need the notion of a homeomorphism between graphs. Two graphs are *homeomorphic* if one can be obtained from the other by a sequence of the following two operations: (a) if x is adjacent to exactly two other vertices, u and v, then substitute uv for ux and xv and delete x, and (b) add a vertex x and substitute ux and xv for the edge uv. We omit the elementary but somewhat lengthy proof of Kuratowski's characterization.

THM. 3.2 (Kuratowski). A graph G is planar iff no subgraph of G is homeomorphic to K_5 or to $K_{3,3}$.

Homework exercises

3.1 Consider the following algorithm for testing whether or not a given graph is planar. It is based on Kuratowski's characterization of planar graphs.

`step 1.` Remove every degree-1 vertex and the edge that contains it.

`step 2.` Remove every degree-2 vertex and replace its two edges by an edge connecting its two neighbors.

`step 3.` Check whether the resulting graph has a K_5 or a $K_{3,3}$ as a subgraph; if not proclaim the graph is planar.

Why is this algorithm not correct?

3.2 Let S be a set of n points in the plane so that the distance between any two points is at least 1. Prove that the number of point pairs at distance exactly 1 is less than $3n$.
(Hint. Prove first that the shortest side of a convex quadrilateral is strictly shorter than the longest diagonal.)

3.3 A graph $G = (V, E)$ is *bipartite* if there is a partition $V = A \dot{\cup} B$ so that every edge in E connects a vertex in A with a vertex in B. For example, every tree is bipartite. Show that every graph with m edges has a bipartite subgraph with at least $\frac{m}{2}$ edges. How fast can such a subgraph be constructed?
(Hint. Try an iteration that moves vertices back and forth between the two vertex sets.)

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