> "Everything has many sides and angles, and nothing can be fully comprehended without understanding everything else, no?"

12 Lifting and polarity

As explained in section 5, the Delaunay triangulation of a finite set $S \subseteq \mathbf{R}^2$ can be obtained by projecting the lower faces of a convex polytope in $\mathbf{R}^3$ [1, 3]. In a similar fashion, the Voronoi diagram can be obtained by projecting the faces of a three-dimensional convex polyhedron. These relations can be extended to points with weights, and they are useful in explaining alpha shapes from a three-dimensional perspective.

Lifting the Voronoi diagram. For a point $p = (\phi_1, \phi_2) \in \mathbf{R}^2$, let η_p be the plane $x_3 = 2\phi_1 x_1 + 2\phi_2 x_2 - (\phi_1^2 + \phi_2^2)$. It is easy to see that η_p is tangent to the paraboloid $\varpi : x_3 = x_1^2 + x_2^2$ and touches ϖ in $\lambda(p) = (\phi_1, \phi_2, \phi_1^2 + \phi_2^2)$, see figure 12.1. Using the same names, we define maps $\varpi, \eta_p : \mathbf{R}^2 \to \mathbf{R}$ whose graphs are the paraboloid ϖ and the plane η_p. A most important property is the following straightforward result.

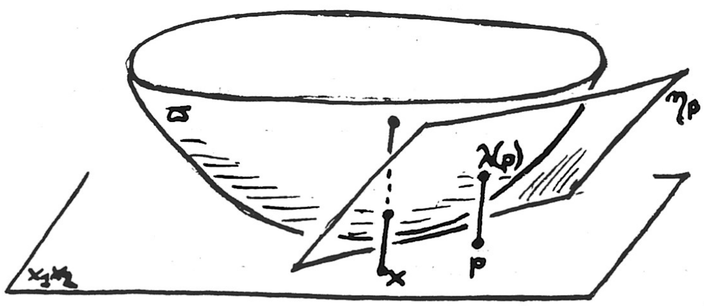

Figure 12.1: The plane η_p is tangent to the paraboloid ϖ.

LEMMA A. For $x, p \in \mathbf{R}^2$, $|xp|^2 = \varpi(x) - \eta_p(x)$.

PROOF. Take points $x = (\xi_1, \xi_2)$ and $p = (\phi_1, \phi_2)$ and evaluate the right side of the relation:

$$\begin{aligned} \varpi(x) - \eta_p(x) &= \xi_1^2 - 2\phi_1\xi_1 + \phi_1^2 + \xi_2^2 - 2\phi_2\xi_2 + \phi_2^2 \\ &= (\xi_1 - \phi_1)^2 + (\xi_2 - \phi_2)^2; \end{aligned}$$

this is the square of the distance between x and p. □

For a finite set $S \subseteq \mathbf{R}^2$, consider the collection of planes or maps $\eta_S = \{\eta_p \mid p \in S\}$. Since $\varpi(x)$ is fixed for any fixed $x \in \mathbf{R}^2$, lemma A implies that x belongs to the Voronoi cell V_p of p iff $\eta_p(x) \geq \eta_q(x)$ for all $q \in S$. This suggests we consider the half-spaces bounded from below by the planes η_p, and in particular, the common intersection of these half-spaces. Define

$$\eta_p^+ = \{y = (v_1, v_2, v_3) \in \mathbf{R}^3 \mid v_3 \geq \eta_p(y'), y' = (v_1, v_2)\},$$

$\eta_S^+ = \{\eta_p^+ \mid p \in S\}$, and $Z_S = \bigcap \eta_S^+$, see figure 12.2. The *facets* of Z_S are the sets $Z_S \cap \eta_p$, $p \in S$. Each such set is non-empty since all planes are tangent to ϖ, which is a convex surface. It follows that each plane η_p carries a facet, and the projection to $\mathbf{R}^2$ of $Z_S \cap \eta_p$ is the Voronoi cell V_p of p.

Power diagrams. In the case of a Voronoi diagram, all planes η_p are constrained to be tangent to ϖ. By moving planes up or down, into ϖ or away from ϖ, it is possible to capture the effect of weights and the definition of power diagrams. Extend the definition of η to points $p = (\phi_1, \phi_2)$ with weights $w_p \in \mathbf{R}$:

$$\eta_p : x_3 = 2\phi_1 x_1 + 2\phi_2 x_2 - (\phi_1^2 + \phi_2^2 - w_p),$$

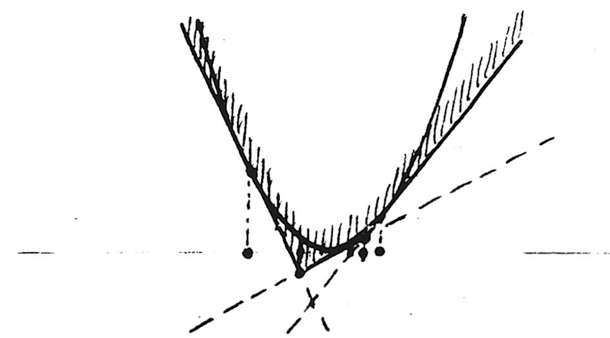

Figure 12.2: For points p in one dimension, the Voronoi cells V_p are projections of the edges of a two-dimensional convex polygonal region. The region is the common intersection of the half-planes bounded from below by the lines η_p.

and define $\eta_p : \mathbf{R}^2 \to \mathbf{R}$ so its graph is the plane η_p. Similar to lemma A, we have

LEMMA B. For $x \in \mathbf{R}^2$ and $p \in \mathbf{R}^2 \times \mathbf{R}$, $\pi_p(x) = \varpi(x) - \eta_p(x)$.

The proof is the same as for lemma A. As before, let η_p^+ be the closed half-space bounded from below by η_p, and for a set $S \subseteq \mathbf{R}^2 \times \mathbf{R}$, define $\eta_S^+ = \{\eta_p^+ \mid p \in S\}$ and $Z_S = \bigcap \eta_S^+$. Because of the weights, the sets $Z_S \cap \eta_p$ are possibly empty, and if general position is not required, they can be line segments or points. The facets of Z_S are the non-empty two-dimensional sets of the form $Z_S \cap \eta_p$. By Lemma B, the power cell of $p \in S$ is the projection to $\mathbf{R}^2$ of the facet $Z_S \cap \eta_p$.

REMARK. For any set of non-vertical planes, H, there is a set of weighted points, $S \subseteq \mathbf{R}^2 \times \mathbf{R}$, so that $H = \eta_S$. This is not true for unweighted points. Although power diagrams are possibly less intuitive than Voronoi diagrams, this correspondence with arbitrary plane collections indicates that power diagrams are rather natural objects.

Polarity. Duality is an important concept in the theory of convex polytopes, and in general in geometry. For example, the cube is dual to the octahedron. Indeed, the cube has 8 vertices, each endpoint of 3 edges, and the octahedron has 8 facets, each bounded by 3 edges. Furthermore, the cube has 6 facets, each bounded by 4 edges, and the octahedron has 6 vertices, each endpoint of 4 edges. The notion of duality is unspecific about metric properties, such as the length of edges and the area of facets. A related and more concrete geometric notion is polarity discussed below.

The *polar* plane of a point $a = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3$ is $a^* : x_3 = 2\alpha_1 x_1 + 2\alpha_2 x_2 - \alpha_3$. Symmetrically, we call a the *polar point* of a^* and write this as $a = a^{**}$. Observe that a^* is non-vertical, so the closed half-space a^{*+} bounded from below by a^* is unambiguous, and so is the closed half-space a^{*-} bounded from above by a^*. An important property of polarity is that it preserves incidences and reverses vertical order.

LEMMA C. Let $x, a \in \mathbf{R}^3$.

(i) $x \in a^*$ iff $a \in x^*$.
(ii) $x \in a^{*+}$ iff $a \in x^{*+}$.
(iii) $x \in a^{*-}$ iff $a \in x^{*-}$.

PROOF. We only consider (ii) and use ξ and α to denote the coordinates of x and a, as usual. The algebraic condition for $x \in a^{*+}$ is

$$\xi_3 \geq 2\alpha_1\xi_1 + 2\alpha_2\xi_2 - \alpha_3.$$

By trivial reordering of terms we see this is equivalent to $a \in x^{*+}$. □

Consider a finite set $A \subseteq \mathbf{R}^3$, and apply polarity to get $A^* = \{a^* \mid a \in A\}$. A point x lies above all planes in A^* iff each point of A lies above x^*, see figure 12.3. This implies a correspondence between the convex bodies defined

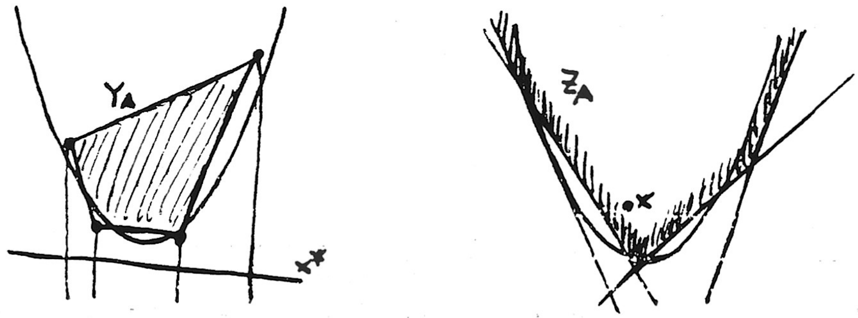

Figure 12.3: The points in A correspond to planes in A^*, and the planes below all points correspond to points above all planes.

by A and A^*. We need definitions to be specific. $Z_A = \bigcap_{a \in A} a^{*+}$ is the convex polyhedron above all planes in A^*. Its boundary consists of faces of dimension 0 (vertices), 1 (edges), and 2 (facets). $Y_A = \operatorname{conv} A$ is also a convex polyhedron. A *lower* face of Y_A is contained in a non-vertical plane so that all other points of Y_A lie vertically above this plane. There is a bijection between the faces of Z_A and the lower faces of Y_A. To see this, take a face f of Z_A and consider a point $x \in \operatorname{int} f$. The polar plane, x^*, contains all points $p \in A$ whose polar planes contain x and therefore f. The bijection maps vertices, edges, facets of Z_A to lower facets, lower edges, and lower vertices of Y_A, and it reverses the direction of containment.

Weighted Delaunay triangulations. For a finite set $S \subseteq \mathbf{R}^2$, the relation between Voronoi diagram, intersection of half-spaces, convex hull, and Delaunay triangulation can be expressed by a commuting diagram:

$$\begin{array}{ccc} \mathcal{V}_S & \overset{\text{nerve}}{\longrightarrow} & \mathcal{D}_S \\ \downarrow\eta & & \uparrow' \\ \bigcap \eta_S^+ & \overset{\text{polarity}}{\longrightarrow} & \operatorname{conv} \lambda(S), \end{array}$$

where the prime next to the arrow going up to $\mathcal{D}_S$ means projection of lower faces. A similar diagram exists for a finite set $S \subseteq \mathbf{R}^2 \times \mathbf{R}$ of points with weights. The only missing piece is the *weighted Delaunay triangulation* $\mathcal{R}_S$ of S, also known as the *regular triangulation* of S:

$$\begin{array}{ccc} \mathcal{P}_S & \overset{\text{nerve}}{\longrightarrow} & \mathcal{R}_S \\ \downarrow\eta & & \uparrow' \\ \bigcap \eta_S^+ & \overset{\text{polarity}}{\longrightarrow} & \operatorname{conv} \eta_S^*. \end{array}$$

$\mathcal{R}_S$ can be defined as the nerve of the collection of power cells, geometrically realized by mapping P_p to p. Alternatively, we can obtain $\mathcal{R}_S$ by projecting the lower faces of $\operatorname{conv} \eta_S^*$ to $\mathbf{R}^2$. Note that we implicitly stretch the notion of a geometric triangulation by allowing simplicial complexes $\mathcal{K}$ with $|\mathcal{K}| = \operatorname{conv} S$ and $\mathcal{K}^{(0)} \subseteq S$; compare with the definition in section 8 where $\mathcal{K}^{(0)} = S$ is required.

Weighted alpha shapes. How does all this relate to alpha shapes? Take a finite set $S \subseteq \mathbf{R}^2 \times \mathbf{R}$. The facets of $Z_S = \bigcap \eta_S^+$ project to the power cells of the points in S. Imagine Z_S moving vertically upwards. Each plane $\eta_p \in \eta_S$ intersects the paraboloid, ϖ, in a possibly empty ellipse whose projection in $\mathbf{R}^2$ is a possibly empty circle growing around p. Initially, the radius of this circle is $\sqrt{w_p}$, and after moving the plane up along a vertical vector of length β, the radius is $\sqrt{w_p + \beta}$. Downward motion is indicated by negative values of β. For $\alpha = \sqrt{\beta}$, let $D_p(\alpha)$ be the disk bounded by this circle; it is empty if $w_p + \beta < 0$, but not necessarily if $\beta < 0$. Since β can take on any real value, positive or negative, α ranges over all non-negative reals and all non-negative multiples of the imaginary unit. The *(weighted) α-complex* of S, $\mathcal{K}(\alpha)$, is the nerve of the collection of cells $E_p(\alpha) = P_p \cap D_p(\alpha)$, geometrically realized as usual [2], see figure 12.4. for $\alpha_1^2 \leq \alpha_2^2$, the cells $E_p(\alpha_1)$ defining $\mathcal{K}(\alpha_1)$ are subsets of the cells $E_p(\alpha_2)$, simply because $D_p(\alpha_1) \subseteq D_p(\alpha_2)$. In the limit, when α reaches $+\infty$, $D_p(\alpha)$ covers all of P_p, so $E_p(\alpha)$ becomes equal to P_p. Therefore,

(i) $\mathcal{K}(\alpha_1) \subseteq \mathcal{K}(\alpha_2) \subseteq \mathcal{R}_S$ whenever $\alpha_1^2 \leq \alpha_2^2$.

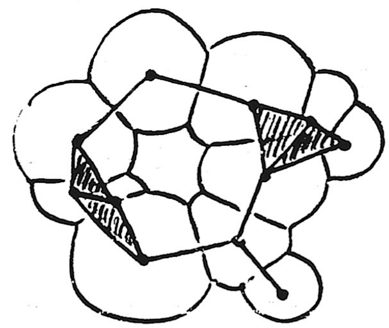

Figure 12.4: The weighted alpha complex is a subcomplex of the weighted Delaunay triangulation.

The *(weighted) α-shape* of S is $|\mathcal{K}(\alpha)|$. Because of property (i), the α-shape grows as α^2 grows, albeit not continuously. alpha complexes and alpha shapes for unweighted points are just a special case of their weighted counterparts. In the unweighted case, we can restrict β to non-negative real numbers, since negative values generate nerves and complexes that consist of the empty set only. It follows that α ranges only over the set of non-negative reals.

Weighted alpha complexes naturally define filtrations, in the same way unweighted alpha complexes do, see section 11. Using the map $P_S \to Z = \bigcap \eta_S^+$, we can be specific about when exactly a simplex $\sigma \in \mathcal{R}_S$ belongs to $\mathcal{K}(\alpha)$. Consider the corresponding face, f, of Z. Move Z upward along a vertical vector of length $\beta = \alpha^2$ (downward if $\beta < 0$). Consider f after the vertical translation. If f lies below ϖ the corresponding cells $E_p(\alpha)$ have no common points, so $\sigma \notin \mathcal{K}(\alpha)$. Otherwise, f intersects ϖ or lies above ϖ, in which case the cells $E_p(\alpha)$ have a non-empty common intersection, and $\sigma \in \mathcal{K}(\alpha)$.

Homework exercises

12.1 Let $S \subseteq \mathbb{R}^2$ be a finite set of points in general position, and for each $p \in S$ call

$$F_p = \{x \in \mathbb{R}^2 \mid |xp| \geq |xq|, q \in S\}$$

the *furthest-point Voronoi cell* of p. Show there is a set $T \subseteq \mathbb{R}^2 \times \mathbb{R}$ and a bijection $\beta : S \to T$ so that $F_p = P_{\beta(p)}$ for each $p \in S$.

12.2 Let S be as in exercise 12.1. Define the *furthest-point Delaunay triangulation* as the nerve of $F_S = \{F_p \mid p \in S\}$, geometrically realized by mapping F_p to p. Prove that abc is a triangle in this triangulation iff all points of $S - \{a, b, c\}$ lie inside the circumcircle of abc.

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