"Are the shape of the universe, the shape of a car, and the shape a person is in related concepts or a fluke of common language?"

11 Alpha shapes

The word 'shape' is commonly used to describe a class of geometric properties that come in degrees, such as the degree of non-convexity, or the degree of non-connectivity, etc. There is an inherent notion of level of detail, similar to the distinction between global and local and everything between. For a real parameter α, the family of α-shapes of a finite point set offers a concrete substitute for the fuzzy notion of shape of the set [1, 2].

Growing disks. Let S be a finite set in $\mathbb{R}^2$, and for $p \in S$ and $\alpha \geq 0$, let $D_p(\alpha) = \{x \in \mathbb{R}^2 \mid |xp| \leq \alpha\}$ be the disk of radius α around p. By gradually and continuously increasing α starting at 0, we can grow a disk D_p around each $p \in S$. The disks grow simultaneously and at uniform speed. The boundary of the union behaves like the parabolic front in section 10: the breakpoints where arcs meet move along Voronoi edges, and the arcs sweep out Voronoi cells, see figure 11.1.

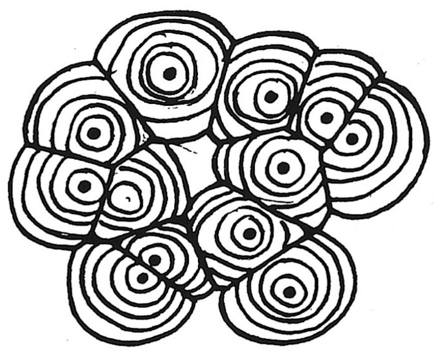

Figure 11.1: The breakpoints where arcs meet sweep out the 1-skeleton of the Voronoi diagram.

We introduce some notation. The set of disks of radius α is $D_S(\alpha) = \{D_p(\alpha) \mid p \in S\}$, the disk union is $\bigcup D_S(\alpha) = \bigcup_{p \in S} D_p(\alpha)$, and its boundary is $\operatorname{bd} \bigcup D_S(\alpha)$. A straightforward but most useful property is

(i) $V_p \cap \bigcup D_S(\alpha) = V_p \cap D_p(\alpha)$.

This is because a point $x \in V_p$ is closest to p and therefore belongs to $D_p(\alpha)$ if it belongs to any disk in $D_S(\alpha)$. Use Voronoi cells to decompose the union into smaller cells $E_p(\alpha) = V_p \cap \bigcup D_S(\alpha)$. By property (i), $E_p(\alpha)$ is the intersection of V_p with a single disk, $D_p(\alpha)$, and is therefore convex. Define $E_S(\alpha) = \{E_p(\alpha) \mid p \in S\}$, see figure 11.2. Together, the Voronoi cells cover the entire plane, and any two Voronoi cells have disjoint interiors. This implies

(ii) $\bigcup E_S(\alpha) = \bigcup D_S(\alpha)$, and

(iii) $\operatorname{int} E_p(\alpha) \cap \operatorname{int} E_q(\alpha) = \emptyset$ if $p \neq q$.

Clearly, $E_p(\alpha) \subseteq V_p$ for all $p \in S$. For bounded Voronoi cells, $E_p(\alpha) = V_p$ provided α is sufficiently large, namely at least the maximum distance from p to any $x \in V_p$. For unbounded cells, $E_p(\alpha)$ covers all of V_p in the limit.

Alpha shapes and complexes. The nerve of the collection of cells in the decomposition, $\mathcal{N}(\alpha) = \mathcal{N}(E_S(\alpha))$, is an abstract simplicial complex. Strictly speaking, $\mathcal{N}(\alpha)$ is not a subset of $\mathcal{N}(V_S)$ because it contains collections of cells E_p rather than collections of Voronoi cells. This is only a technicality, and we have

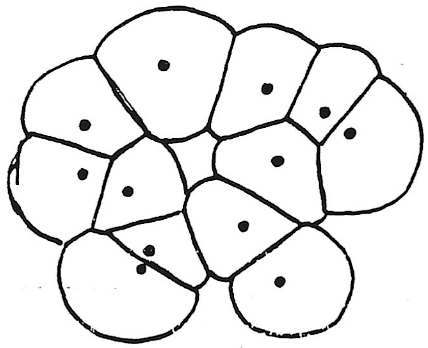

Figure 11.2: The Voronoi cells decompose the disk union into convex cells without shared interior points.

(iv) $\mathcal{N}(\alpha)$ is isomorphic to a subcomplex of $\mathcal{N}(V_S)$.

The subcomplex of $\mathcal{N}(V_S)$ that is naturally isomorphic to $\mathcal{N}(\alpha)$ is obtained by substituting V_p for every occurrence of $E_p(\alpha)$ in $\mathcal{N}(\alpha)$. The *α-complex* of S is the geometric realization, $\mathcal{K}(\alpha) = \mathcal{K}_S(\alpha)$, of $\mathcal{N}(\alpha)$ defined by the injective map $\varphi : E_S(\alpha) \rightarrow \mathbf{R}^2$ with $\varphi(E_p(\alpha)) = p$, see figure 11.3. Note that the geometric realization uses the same points in $\mathbf{R}^2$ for every α, and these points are the same as used to derive the Delaunay triangulation from the collection of Voronoi cells. Therefore,

(v) $\mathcal{K}(\alpha_1) \subseteq \mathcal{K}(\alpha_2) \subseteq \mathcal{D}$ whenever $\alpha_1 \leq \alpha_2$.

The *α-shape* of S is the underlying space, $|\mathcal{K}(\alpha)|$, of $\mathcal{K}(\alpha)$, see figure 11.3.

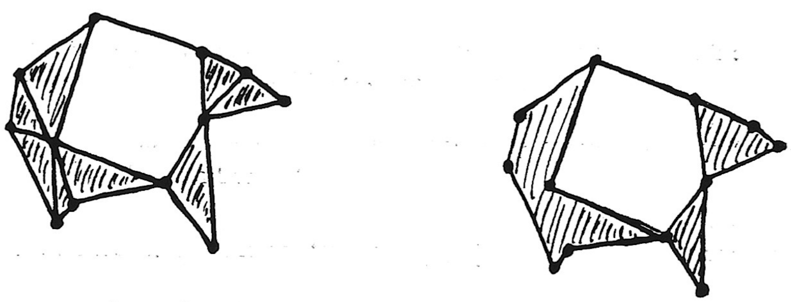

Figure 11.3: The α-complex and α-shape for the points and value of α as shown in figure 11.2.

Filtrations. As α grows continuously from 0 to $+\infty$, $\bigcup D_S(\alpha)$ also grows, and we get a continuous family of disk unions. The corresponding family of alpha shapes is discrete because $\mathcal{K}$ changes only when a new subset of E_S is added to its nerve. This happens when a pair of cells touches for the first time, or a triplet of cells meet at a common Voronoi vertex. Sequences of complexes similar to the resulting sequence of alpha complexes have been studied in algebraic topology. A *filtration* is a sequence of simplices,

$$\sigma_0, \sigma_1, \ldots, \sigma_m,$$

so that for each $0 \leq j \leq m$, $\mathcal{K}_j = \{\sigma_i \mid 0 \leq i \leq j\}$ is a simplicial complex, see e.g. [3, chapter 9]. The sequence of alpha complexes is almost a filtration; it falls short of being one because two contiguous alpha complexes may differ by more than just one simplex, even if general position is assumed. It is possible, however, to add complexes between such contiguous alpha complexes to get a genuine filtration. Below, we study in detail how to decide for two simplices $\sigma, \sigma' \in \mathcal{D}$ whether σ joins the growing alpha complex before or after or simultaneously with σ'.

Vertices. Each vertex belongs to $\mathcal{K}(0)$, and for $\alpha < 0$ we define $\mathcal{K}(\alpha) = \{\emptyset\}$. So all vertices are added simultaneously, at time $\alpha = 0$, before all edges and triangles. We may order them arbitrarily as $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Edges. For an edge $ab \in \mathcal{D}$ define $\varrho(ab) = \frac{|ab|}{2}$; it is the radius of the smallest enclosing disk. If $\alpha \geq \varrho(ab)$ then $D_a(\alpha) \cap D_b(\alpha) \neq \emptyset$, but this does not necessarily imply that $E_a(\alpha) \cap E_b(\alpha)$ have a non-empty intersection, see the three points at the upper right in figure 11.2. If ab intersects the dual Voronoi edge, $V_a \cap V_b$, then $\alpha \geq \varrho(ab)$ suffices for $ab \in \mathcal{K}(\alpha)$. This is the case when no points of S lie inside the circle with center $\frac{a+b}{2}$ and radius $\varrho(ab)$. If ab does not intersect $V_a \cap V_b$ then $\alpha \geq \varrho(ab)$ is not sufficient. We will come back to this case shortly.

Triangles. For $abc \in \mathcal{D}$, let $\varrho(abc)$ be the radius of the circumcircle; it is also the distance of a, b, and c from the Voronoi vertex $x = V_a \cap V_b \cap V_c$. At the moment when the cells $E_a(\alpha), E_b(\alpha), E_c(\alpha)$ reach x, which happens simultaneously, they have a common intersection. They reach x at time $\alpha = \varrho(abc)$, so $\alpha \geq \varrho(abc)$ is necessary and sufficient for $abc \in \mathcal{K}(\alpha)$.

Attached edges. An edge $ab \in \mathcal{D}$ is *attached* if it does not intersect the dual Voronoi edge, $V_a \cap V_b$. It belongs to $\mathcal{K}(\alpha)$ only if α is large enough for $E_a(\alpha)$ and $E_b(\alpha)$ to reach a common Voronoi vertex. This vertex also belongs to a third cell, $E_c(\alpha)$. So ab is added to the alpha complex simultaneously with abc at time $\alpha = \varrho(abc)$. In order to get a genuine filtration, we let ab precede abc in the sequence; otherwise, one of the prefixes would correspond to a collection of simplices that do not form a complex.

It should be clear that a filtration that contains the sequence of alpha complexes as a subsequence is a convenient data structure representing the entire family of alpha shapes or complexes. The size of this filtration is just the number of simplices in $\mathcal{D}$, which is less than $6n$. For a given $\alpha \in \mathbf{R}$, we can retrieve $\mathcal{K}(\alpha)$ by searching for the index i so that $\varrho(\sigma_i) \leq \alpha < \varrho(\sigma_{i+1})$. Then $\mathcal{K}(\alpha) = \mathcal{K}_i$. This stipulates that each attached edge $ab \in \mathcal{D}$ has its original ϱ-value replaced by the smallest ϱ-value of any triangle (one or two) that contains ab.

Homework exercises

11.1 Let $\mathcal{K}$ be the α-complex of a finite point set in $\mathbf{R}^2$. The *closure* of $L \subseteq \mathcal{K}$ is the smallest subcomplex $\mathrm{Cl}\, L \subseteq \mathcal{K}$ that contains L. The *star* of a simplex $\tau \in \mathcal{K}$ is $\mathrm{St}\,\tau = \{\sigma \in \mathcal{K}_\alpha \mid \tau \subseteq \sigma\}$, and the *link* of τ is $\mathrm{Lk}\,\tau = \{\tau' \in \mathrm{Cl}\,\mathrm{St}\,\tau \mid \tau' \cap \tau = \emptyset\}$.

  (i) What is the link of a triangle in $\mathcal{K}$?

  (ii) What are the three types of links an edge $\tau \in \mathcal{K}$ can have?

  (iii) Consider $A = \mathrm{bd}\, D_a(\alpha) \cap \mathrm{bd} \bigcup D_\alpha$. Show that $A = \emptyset$ iff $\mathrm{Lk}\, a$ is a cycle of edges.

11.2 Let D be a set of n disks in general position in $\mathbf{R}^2$. The boundary of the union, $\mathrm{bd} \bigcup D$, consists of at least one and possibly several components, called *boundary cycles*, each a cyclic sequence of circular arcs.

  (i) Show $\mathrm{bd} \bigcup D$ has fewer than $2n$ boundary cycles.

  (ii) Show each boundary cycle consists of fewer than $2n$ arcs.

  (iii) Show the total number of arcs in the collection of boundary cycles is less than $6n$.

11.3 Let $\mathcal{D} = \mathcal{D}_S$ be the Delaunay triangulation of a finite set $S \subseteq \mathbf{R}^2$. Let $\varrho'(\sigma)$ be the radius of the smallest disk that covers σ. Define $\mathcal{L}(\alpha) = \{\sigma \in \mathcal{D} \mid \varrho'(\sigma) \leq \alpha\}$. Show $\mathcal{L}(\alpha)$ is a simplicial complex and $\mathcal{K}(\alpha) \subseteq \mathcal{L}(\alpha)$.

References

[1] H. EDELSBRUNNER. The union of balls and its dual shape. *In* "Proc. 9th Ann. Sympos. Comput. Geom., 1993", 218–231.

[2] H. EDELSBRUNNER, D. G. KIRKPATRICK AND R. SEIDEL. On the shape of a set of points in the plane. *IEEE Trans. Inform. Theory* **IT-29** (1983), 551–559.

[3] E. H. SPANIER *Algebraic Topology.* Springer-Verlag, New York, 1966.