"Everyday life is full with decisions based on vicinity,
such as where to get coffee and where to drink it."

7 Voronoi diagrams

A region equipped with a metric and a collection of sites is naturally decomposed into cells, each representing the neighborhood of one site. In computational geometry, a decomposition obtained following this general idea is called a *Voronoi diagram*, honoring the pioneering work of the Russian mathematician Georges Voronoi [5, 6]. This idea has emerged independently in different disciplines of science and thought, and the resulting decompositions are known under a variety of names, including Dirichlet tesselations [3], Thiessen polygons [4], and Blum's medial axis transform [2]. The survey article by Aurenhammer [1] is a good recent and general reference.

Voronoi cells for points. The *Euclidean distance* between two points, $x = (\xi_1, \xi_2)$ and $y = (v_1, v_2)$ in the plane is $|xy| = ((\xi_1 - v_1)^2 + (\xi_2 - v_2)^2)^{\frac{1}{2}}$. Given a finite set $S \subseteq \mathbf{R}^2$, the *Voronoi cell* of $p \in S$ is

$$V_p = \{x \in \mathbf{R}^2 \mid |xp| \leq |xq|, q \in S\}.$$

Note that $V_p = \bigcap_{q \in S-\{p\}} H_{pq}$, where $H_{pq} = \{x \in \mathbf{R}^2 \mid |xp| \leq |xq|\}$ is the half-plane bounded by the perpendicular bisector of p and q that contains p. The following properties are immediate consequences of the definition and are illustrated in figure 7.1.

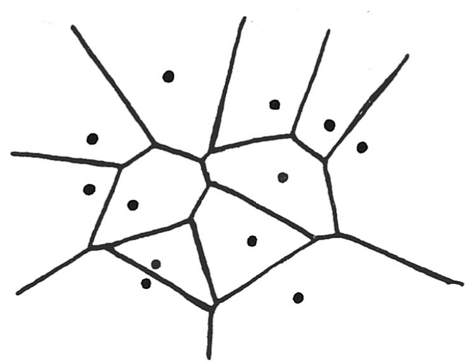

Figure 7.1: The Voronoi cells of 12 points. There are 25 Voronoi edges and 14 Voronoi vertices.

(i) V_p is a convex polygonal region bounded by at most $n-1$ edges, $n = \operatorname{card} S$.

(ii) $p \in V_p$, so $V_p \neq \emptyset$ for all $p \in S$.

(iii) V_p is unbounded iff $p \in \operatorname{bd} \operatorname{conv} S$.

(iv) $\operatorname{int} V_p \cap \operatorname{int} V_q = \emptyset$ if $p \neq q$.

(v) $\bigcup_{p \in S} V_p = \mathbf{R}^2$.

Assume general position of the points in S, which in this case means no 4 points lie on a common circle. Then no point $x \in \mathbf{R}^2$ belongs to more than three Voronoi cells. The intersection of any two Voronoi cells is then either empty or a line segment, called a *Voronoi edge* of S. The intersection of any three Voronoi cells is either empty or a point, called a *Voronoi vertex* of S. By assumption of general position, every Voronoi vertex belongs to three Voronoi edges and three Voronoi cells.

LEMMA. A set of n point in $\mathbf{R}^2$ defines at most $3n-6$ Voronoi edges and at most $2n-5$ Voronoi vertices.

PROOF. We can assume general position because a slight perturbation can only increase the number of Voronoi edges and vertices. To use Euler's relation for planar graphs (Thm. 3.1), we bend all unbounded Voronoi edges so they join at a common endpoint, z. Let e_∞ be the number of such edges. Now we have a connected plane graph with n faces, e edges, and $v+1$ vertices. Every vertex has degree 3, except for z, which has degree e_∞. Hence, $3v + e_\infty = 2e$. From Euler's relation we get $e = 3n - 3 - e_\infty \leq 3n - 6$ and $v = e - n + 1 \leq 2n - 5$. ☐

Cones and paraboloids. For a fixed point $p \in \mathbb{R}^2$, the Euclidean distance from p is a map $\delta_p : \mathbb{R}^2 \to \mathbb{R}$. Consider its graph, $C_p = \{(x, \xi) \in \mathbb{R}^2 \times \mathbb{R} \mid \xi = |xp|\}$, which is the surface of a cone in $\mathbb{R}^3$, see figure 7.2. Its apex is $(p, 0)$, its

Figure 7.2: For each $p \in S$ take a cone with apex p, vertical axis, and right opening angle. The lower envelope of the cones projects vertically to the Voronoi diagram of S. Similarly, we can take a paraboloid per point and project the lower envelope of the paraboloids.

axis of rotation is the vertical line (p, ξ), $\xi \in \mathbb{R}$, and its opening angle is $\frac{\pi}{2}$, or a quarter of a full angle. The graph of the point-wise minimum, $\delta(x) = \min_{p \in S}\{\delta_p(x)\}$, consists of patches of cone surfaces. The vertical projection into $\mathbb{R}^2$ of the patch in C_p is the Voronoi cell of p.

Similarly, let $C_p^2 = \{(x, \xi^2) \mid \xi = |xp|\}$ be the graph of the Euclidean distance square from p. C_p^2 is a paraboloid of revolution with the same axis and the same bottommost point as C_p. The graph of the point-wise minimum, $\min_{p \in S}\{\delta_p^2(x)\}$, consists of paraboloid patches, and the vertical projection of the patch in C_p^2 is V_p.

Power distance. There are many generalizations of Voronoi diagrams, some using weights to modify the distance function. A particular such generalization is achieved by replacing Euclidean distance with power distance. Assign to each point $p \in S$ a real weight w_p. The ***power distance*** of a point $x \in \mathbb{R}^2$ from $p \in S$ is $\pi_p(x) = |xp|^2 - w_p$, see figure 7.3. Notice the power distance give rise to paraboloids, similar to those on the right side of figure 7.2 except that their bottommost points do not lie in the same plane if the weights are different. Consider the disk

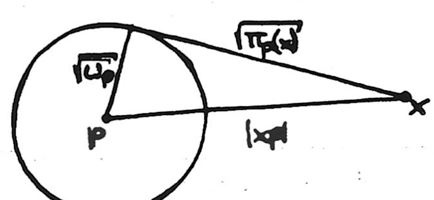

Figure 7.3: For a point x outside the disk with center p and radius $(w_p)^{\frac{1}{2}}$, the power distance is the length square of the line segment connecting x to a tangent point on the bounding circle.

$D_p = \{x \in \mathbb{R}^2 \mid |xp|^2 \leq w_p\}$, which is a point if $w_p = 0$ and empty if $w_p < 0$. Obviously,

$$\pi_p(x) \begin{cases} > 0 & \text{if } x \notin D_p, \\ = 0 & \text{if } x \in \text{bd } D_p, \text{ and} \\ < 0 & \text{if } x \in \text{int } D_p. \end{cases}$$

Given points $p = (\phi_1, \phi_2)$ and $q = (\psi_1, \psi_2)$ with weights w_p and w_q, the points $x = (\xi_1, \xi_2)$ with equal power distance from both satisfy $\pi_p(x) = \pi_q(x)$, and therefore

$$(\xi_1 - \phi_1)^2 + (\xi_2 - \phi_2)^2 - w_p = (\xi_1 - \psi_1)^2 + (\xi_2 - \psi_2)^2 - w_q,$$

which is equivalent to

$$2\langle x, p - q\rangle = \langle p, p\rangle - \langle q, q\rangle - w_p + w_q.$$

The set of such points x is therefore a line perpendicular to $p - q$, called the ***radical axis*** of p and q. It is a generalization of the perpendicular bisector, and a few examples covering the cases of incomparable, nested, and disjoint disks are shown in figure 7.4. Notice that adding the same amount to the weights of p and q does not change the radical axis.

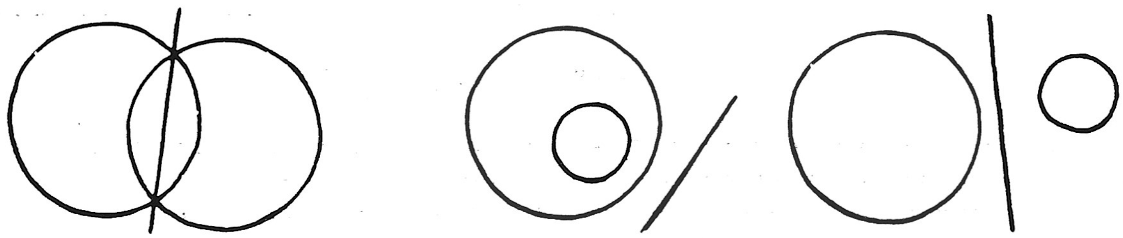

Figure 7.4: The radical axis passes through the intersection of the bounding circles, if non-empty. If the disks are nested, the radical axis misses both and recedes to infinity when the centers of the disks get close to coincidence. Two disjoint disks are separated by their radical axis.

Power cells. Given a finite set S of points in $\mathbf{R}^2$, each with a weight in $\mathbf{R}$, we can now generalize the Voronoi diagram to the ***power diagram*** (also ***weighted*** Voronoi diagram) of S. The ***power cell*** of $p \in S$ is

$$P_p = \{x \in \mathbf{R}^2 \mid \pi_p(x) \leq \pi_q(x), q \in S\}.$$

$P_p = V_p$ if the weights of all points are the same. Similar to V_p, we have $P_p = \bigcap_{q \in S - \{p\}} H_{pq}$, where $H_{pq} = \{x \in \mathbf{R}^2 \mid \pi_p(x) \leq \pi_q(x)\}$, see figure 7.5. The following properties are straightforward and correspond to properties (i)

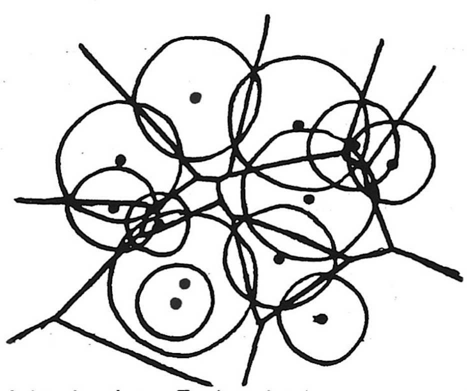

Figure 7.5: The power cells of 12 weighted points. Each point has a non-empty cell, so the number of edges and vertices are the same as in the Voronoi diagram of the same but unweighted points.

through (v) of Voronoi diagrams. Only property (ii) does not extend from Voronoi to power diagrams.

(i) P_p is a convex polygonal region bounded by at most $n - 1$ edges, $n = \text{card}\, S$.

(ii) p is not necessarily a point of P_p and $P_p = \emptyset$ is possible

(iii) P_p is unbounded if p is a vertex of $\text{conv}\, S$.

(iv) $\operatorname{int} P_p \cap \operatorname{int} P_q = \emptyset$ if $p \neq q$.

(v) $\bigcup_{p \in S} P_p = \mathbb{R}^2$.

If the points and weights are in general position, we can again define *power edges* as non-empty intersections of two, and *power vertices* as non-empty intersections of three cells. In this case, general position means there is no $x \in \mathbb{R}^2$ with equal power distance from 4 or more weighted points. Intuitively, this means there is no circle (with center x) that orthogonally intersects 4 or more bouding circles of disks D_p.

Homework exercises

7.1 Let S be a finite set of points in $\mathbb{R}^3$, and for each $p \in S$, let $V_p = \{x \in \mathbb{R}^3 \mid |xp| \leq |xq|, q \in S\}$ be the Voronoi cell of p. Show that for every plane h there is a set of weighted points, $S_h = \{p_h \in h \mid p \in S\}$, so that $V_p \cap h$ is the power cell of p_h within h.

7.2 Let S be a set of n weighted points, and define the distance of $x \in \mathbb{R}^2$ from $p \in S$ equal to $|xp| - w_p$. The *cell* of p is the set of points x whose distance from p is less than or equal to the distance to any other point in S.

(i) Show that the cell of any $p \in S$ is connected.

(ii) Argue that the number of edges and vertices bounding the cells are less than $3n$ and $2n$, respectively.

(Hint. Notice that the edges are pieces of hyperbolas, and that two cells can share more than just one edge.)

References

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