"Like buildings made of bricks, complex shapes can be built by gluing a large number of simple ones."

6 Simplicial complexes

The faces of a graph embedded in the plane are a consequence of the edges decomposing $\mathbf{R}^2$. Alternatively, faces can be specified as elements of the abstract structure, and an explicit embedding may be required. This leads to the notion of a complex, which is a fundamental concept in combinatorial and algebraic topology [1, 2]. Although the next few sections are predominantly two-dimensional, it is convenient to present the definitions for arbitrary dimensions, d.

Simplices. A set $T \subseteq \mathbf{R}^d$ is ***affinely independent*** (*a.i.*) if no point $x \in T$ is an affine combination of $T - \{x\}$. A necessary condition for T being a.i. is that $\text{card}\, T \leq d + 1$. A ***simplex*** is the convex hull of an a.i. point set, T. If $\text{card}\, T = k + 1$ then $\sigma = \text{conv}\, T$ is a ***k-simplex*** and its ***dimension*** is $\dim \sigma = k$, see figure 6.1. Consistent with these definitions, there is a unique (-1)-simplex, namely the empty set, $\emptyset$. The ***faces*** of σ are the simplices defined by subsets $U \subseteq T$, and if $\text{card}\, U = \ell + 1$ then $\tau = \text{conv}\, U$ is an ***ℓ-face*** of σ. The ***improper*** faces of σ are $\emptyset$ and σ itself, and all other faces are ***proper***. Since every subset of T defines a face of σ, we know there are $\binom{k+1}{\ell+1}$ ℓ-faces, for $-1 \leq \ell \leq k$, and $2^{k+1} = \sum_{\ell=-1}^{k} \binom{k+1}{\ell+1}$ faces altogether.

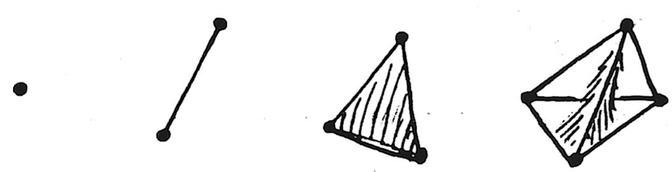

Figure 6.1: A 0-simplex is a point or vertex, a 1-simplex is an edge, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Simplicial complexes. We consider sets of simplices that are nicely arranged. Intuitively, this means that simplices do not overlap, and if two simplices touch each other then they do this along a common face. This is now formalized. A finite collection $\mathcal{K}$ of simplices is a ***simplicial complex*** if the following two conditions are satisfied:

(i) if $\sigma \in \mathcal{K}$ and τ is a face of σ then $\tau \in \mathcal{K}$, and

(ii) if $\sigma, \sigma' \in \mathcal{K}$ then $\sigma \cap \sigma'$ is a face of both.

Note that $\emptyset$ is a face of every simplex, so condition (ii) allows for the possibility that σ and σ' are disjoint. By condition (i), $\emptyset$ belongs to every non-empty simplicial complex. A ***subcomplex*** of $\mathcal{K}$ is a simplicial complex $\mathcal{L} \subseteq \mathcal{K}$.

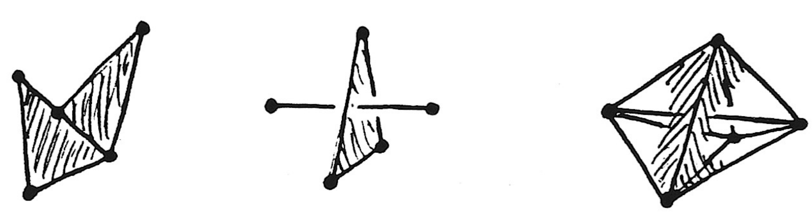

Figure 6.2: The two triangles on the left do not define a simplicial complex because an edge of one triangle overlaps with a portion of an edge of the other. Neither do the edge and the pierced triangle in the middle. The three tetrahedra on the right, together with their faces, form a simplicial complex. Its underlying space is a triangular double-pyramid.

Note that every subset inherits property (ii) from $\mathcal{K}$, so $L \subseteq \mathcal{K}$ is a subcomplex if it satisfies (i). For any integer i, the *i-skeleton* of $\mathcal{K}$ is $\mathcal{K}^{(i)} = \{\sigma \in \mathcal{K} \mid \dim \sigma \leq i\}$. The *dimension* of $\mathcal{K}$ is $\dim \mathcal{K} = \max_{\sigma \in \mathcal{K}}\{\dim \sigma\}$. The *underlying space* of $\mathcal{K}$ is $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$, and a subset of space is called a *polytope* or *polyhedron* if it is the underlying space of a simplicial complex.

Abstract simplicial complexes. It is sometimes convenient to study a simplicial complex without worrying about the embedding in space. This can be done by taking the vertices as abstract elements, and the simplices as sets of vertices. A finite collection $\mathcal{A}$ of finite sets is an *abstract simplicial complex* if $\alpha \in \mathcal{A}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathcal{A}$. The sets in $\mathcal{A}$ are *(abstract) simplices*, and the *dimension* of $\alpha \in \mathcal{A}$ is $\dim \alpha = \operatorname{card} \alpha - 1$. The *dimension* of $\mathcal{A}$ is $\dim \mathcal{A} = \max_{\alpha \in \mathcal{A}}\{\dim \alpha\}$. The *vertex set* of $\mathcal{A}$ is $\operatorname{vert} \mathcal{A} = \bigcup_{\alpha \in \mathcal{A}} \alpha$. For example, a graph is a one-dimensional abstract simplicial complex.

Being a set system, $\mathcal{A}$ defines the poset $(\mathcal{A}, \subseteq)$, where $\subseteq$ denotes set inclusion, see figure 6.3. If $\gamma \subseteq \alpha$ then γ is said to be *smaller* than α, and α is *larger* than γ. For every two simplices, $\alpha, \beta \in \mathcal{A}$, there is a unique largest

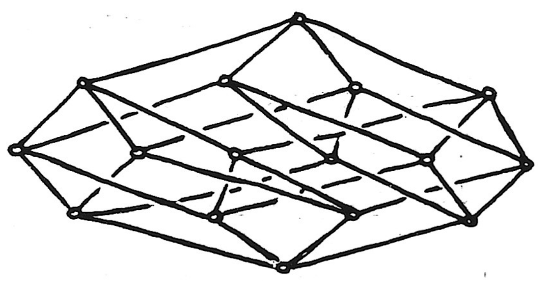

Figure 6.3: The poset corresponds to the abstract simplicial complex consisting of a tetrahedron and its faces.

simplex smaller than both, namely $\alpha \cap \beta$. This also holds in the other direction if we add $\operatorname{vert} \mathcal{A}$ to $\mathcal{A}$, that is, there is a unique smallest simplex larger than both. This smallest simplex is either $\alpha \cup \beta$ or $\operatorname{vert} \mathcal{A}$ if $\alpha \cup \beta \notin \mathcal{A}$. A poset with this property is a *lattice*.

A general geometric realization result. A simplicial complex, $\mathcal{K}$, is a *geometric realization* of $\mathcal{A}$ if there exists a bijection $\varphi : \operatorname{vert} \mathcal{A} \to \mathcal{K}^{(0)}$ so that $\alpha \in \mathcal{A}$ iff $\operatorname{conv} \varphi(\alpha) \in \mathcal{K}$. Note that a geometric realization is similar to an embedding, except it insists on embedding each set as a "straight" simplex. Fáry's theorem can now be reformulated: every graph that has a plane embedding also has a geometric realization in $\mathbf{R}^2$. This result does not generalize to higher dimensions, that is, there are two-dimensional simplicial complexes that can be embedded in $\mathbf{R}^3$ without improper intersections, but they have no geometric realization in $\mathbf{R}^3$. Nevertheless, geometric realizations always exist if the dimension is high enough. Such a realization can be constructed using the moment curve: $M_d = \{x(t) = (t, t^2, \ldots, t^d) \mid t \in \mathbf{R}\}$.

LEMMA. A hyperplane in $\mathbf{R}^d$ intersects M_d in at most d points.

Indeed, let $h : \sum_{i=1}^{d} a_i x_i - a_{d+1} = 0$ be a hyperplane, and observe that $x(t) \in M_d \cap h$ iff $\sum_{i=1}^{d} a_i t^i - a_{d+1} = 0$. Thus, t is root of a degree-d polynomial, and by the fundamental theorem of algebra, such a polynomial has at most d roots.

THM. 6.1 Every k-dimensional abstract simplicial complex has a geometric realization in $\mathbf{R}^{2k+1}$.

PROOF. The idea is to map the vertices of $\mathcal{A}$ to points on M_d, for $d = 2k+1$. Let $\varphi : \operatorname{vert} \mathcal{A} \to M_d$ be injective. We claim that $\mathcal{K} = \{\operatorname{conv} \varphi(\alpha) \mid \alpha \in \mathcal{A}\}$ is a simplicial complex. Condition (i) is automatically satisfied. To see that condition (ii) holds observe that any $d+1 = 2k+2$ points of M_d are a.i. Two simplices $\sigma, \sigma' \in \mathcal{K}$ together have $m+1 \leq 2k+2$ vertices, which are therefore a.i. The $m+1$ points thus span an m-simplex, and σ and σ' are faces of this m-simplex. If follows that $\sigma \cap \sigma'$ is another face of the m-simplex and thus a face of both, σ and σ'. ☐

Homework exercises

6.1 Let $\mathcal{K}$ contain a d-simplex together with all its faces.

(i) Prove

$$\chi(\mathcal{K}) = \sum_{\tau \in \mathcal{K}} (-1)^{\dim \tau} = 0.$$

(ii) Consider the poset of $\mathcal{K}$ and count the number of pairs $\upsilon \subseteq \tau$ with $\dim \upsilon = \dim \tau - 1$. Argue that the number of such pairs is $(d+1)2^d$.

6.2 Let $\mathcal{K}$ be a simplicial complex in $\mathbf{R}^2$ so that $|\mathcal{K}|$ is a polygon with inside. Show that $\chi(\mathcal{K}) = 0$. *(Hint. Observe that removing a triangle together with one of its edges, or an edge with one of its vertices leaves χ invariant.)*

References

[1] P. J. Giblin. *Graphs, Surfaces, and Homology.* 2nd edition, Chapman and Hall, London, 1981.

[2] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.