1.2 Sampling

We will mainly concentrate on curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ as the sampled space. The notation Σ will be used to denote this generic sampled space throughout this course. Obviously it is not possible to extract any meaningful information about Σ if it is not sufficiently sampled. This means features of Σ should be represented with sufficiently many samples. But, this brings up the question of defining what are features. We aim for a measure that can tell us how complicated Σ is around each point $x \in \Sigma$. A geometric structure called *medial axis* turns out to be useful to define such a measure.

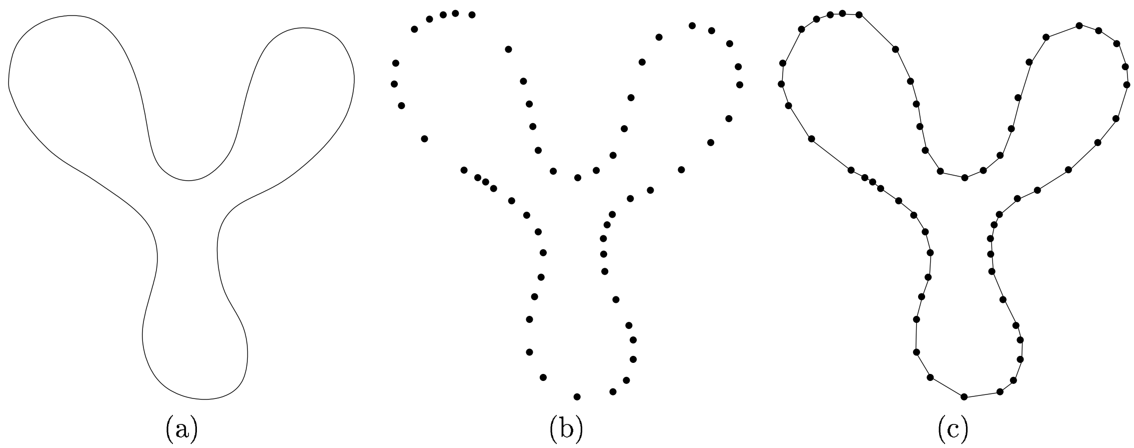

Figure 1.4: (a) A curve in the plane. (b) A sample. (c) Reconstructed curve.

Local feature size. Consider Σ to be a curve in $\mathbb{R}^2$. The closure of the centers of the balls that meet Σ only tangentially in two or more points constitutes the *medial axis* of Σ. Figure 1.5 (a) shows a subset of the medial axis of a curve. Notice that the medial axis may have a branching point such as v and boundary points such as u and w. Also, the medial axis need not be connected. For example, the interior of the curve shown in Figure 1.5 (a) has a medial axis component disjoint from the ones in the outside.

It follows from the definition that, if one grows a ball around an interior point on the medial axis, it will meet Σ for the first time tangentially in two or more points, see Figure 1.5 (b). These maximal empty balls with centers on the medial axis are called *medial balls*. Conversely, one can start at a point $x \in \Sigma$ and start growing a ball keeping it tangent to Σ at x untill it hits another point $y \in \Sigma$. At this moment the ball is medial and the segments joining the center m to x and y are normal to Σ at x and y respectively.

If we move along the medial axis and consider the medial balls as we move, the radius of the medial balls increases or decreases accrodingly to maintain the touch with Σ, and at the boundaries coincides with the *curvature ball* where all touching points merge into a single one. At that point the radius becomes equal to the radius of curvature of Σ. See Figure 1.5 (b).

The medial axis with the distance r_m to Σ at each point m captures the shape of

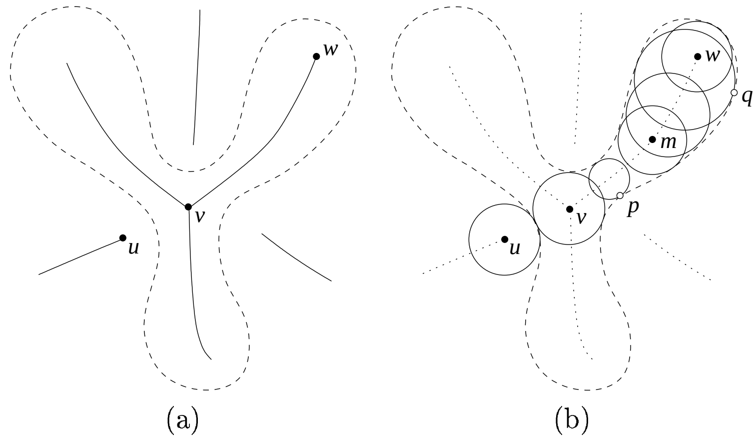

Figure 1.5: (a) Subset of the medial axis of the curve in Figure 1.4. (b) Medial ball centered at v touch the curve in three points, whereas the ones with centers u and w touch it in only one point and coincide with the curvature ball.

Σ. In fact, one can reconstruct Σ completely from its medial axis and the distances r_m. This means if we measure the distance of a point $x \in \Sigma$ from the medial axis we can approximate size of features around x. Keeping this in mind we define a distance function called *local feature size*:

$$f : \Sigma \to \mathbb{R} \text{ where } f(x) \text{ is the least distance to the medial axis of any } x \in \Sigma.$$

As can be observed from Figure 1.5 (b), the local feature size at p is small compared to the local feature size at q in coherence with our intuitive notion of features.

ε-sampling. Once we have quantized the feature size, we would require the sample respect the features, i.e., we require more sample points where the local feature size is small compared to the regions where it is not.

Definition 1 *A sample P of Σ is an ε-sample if each point $x \in \Sigma$ has a sample point $p \in P$ so that $\|x - p\| \leq \varepsilon f(x)$.*

The value of ε has to be smaller than 1 to have a dense sample. In fact, it seems from practical evidences that $\varepsilon < 0.4$ constitutes a dense sample useful for reconstructing Σ from P. Notice that an ε-sample is also an ε'-sample for any $\varepsilon' > \varepsilon$. The definition of ε-sample allows a sample arbitrarily dense anywhere on Σ. It only puts a lower bound on the density. Figure 1.6 illustrates a sample of a circle which is a 0.2-sample and also a 0.3-sample of the same.

Useful properties. One useful property of the local feature size function $f(\cdot)$ is that it is Lipschitz continuous.

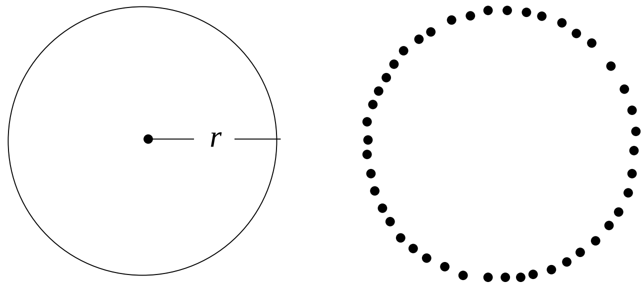

Figure 1.6: Local feature size at any point on the circle is equal to the radius r. Each point on the circle has a sample point within $0.2r$ distance.

Lemma 1 (Lipschitz Continuity.)

$$f(x) \leq f(y) + \|x - y\|$$

for any two points x and y in Σ.

PROOF. Let m be a point on the medial axis so that $f(y) = \|y - m\|$. By triangular inequality,

$$\|x - m\| \leq \|y - m\| + \|x - y\|, \text{ and}$$
$$f(x) \leq \|x - m\| \leq f(y) + \|x - y\|$$

□

An useful application of the Lipschitz Continuity Lemma is that the distance between two points expressed in terms of the local feature size of one can be expressed in terms of that of the other.

Lemma 2 (Feature Translation.) *For any two points x, y in Σ with $\|x - y\| \leq \varepsilon f(x)$ we have $\|x - y\| \leq \frac{\varepsilon}{1-\varepsilon} f(y)$.*

PROOF. We have

$$\begin{aligned} f(x) &\leq f(y) + \|x - y\| \\ f(x) &\leq f(y) + \varepsilon f(x) \\ f(x) &\leq \frac{1}{1 - \varepsilon} f(y). \end{aligned}$$

Plug the above inequality in $\|x - y\| \leq \varepsilon f(x)$ to obtain the result. □

It will be useful for our proofs later to know the following property of balls intersecting the sampled space Σ.

Lemma 3 (Feature Ball.) *If a d-ball B intersects a k-manifold $\Sigma \subset \mathbb{R}^d$ at more than two points with either (a.) $B \cap \Sigma$ is not a k-ball, or (b.) $(\mathrm{bd}\, B \cap \Sigma)$ is not a $(k-1)$-sphere, then a medial axis point must be in B.*

PROOF. First we show that if $(B \cap \Sigma)$ has more than one component or B touches Σ tangentially at least at one point, B must contain a medial axis point. See Figure 1.7 for an illustration on curves. If a point of tangency does not exist, shrink B centrally until it meets a component only tangentially. Let x be the point of this tangency. Shrink B further keeping the tangential touch at x. This means the center of B moves towards x along a normal direction at x. We stop when B meets Σ only tangentially. Observe that this will happen as the intersection of B with other components only shrinks as B does so. At this moment B becomes a medial ball and its center is a medial axis point which must lie in the original ball B.

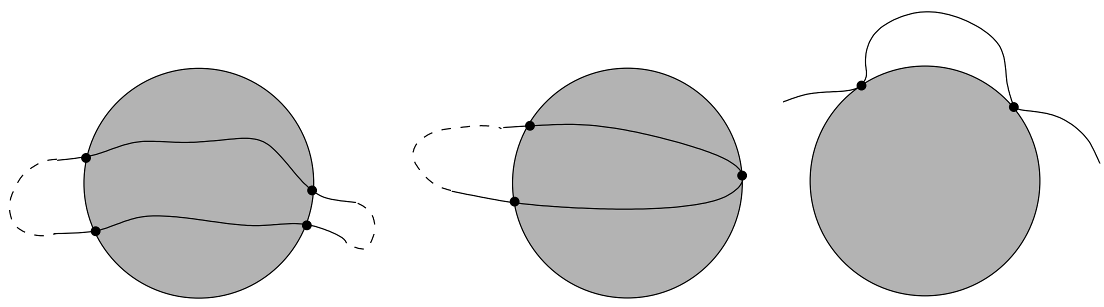

Figure 1.7: (a) $B \cap \Sigma$ is not a 1-ball, (b) $B \cap \Sigma$ is a 1-ball, but $\mathrm{bd}\, B \cap \Sigma$ is not a 0-sphere, (c) $\mathrm{bd}\, B \cap \Sigma$ is a 0-sphere, but $B \cap \Sigma$ is not a 1-ball.

So, assume B does not meet Σ tangentially at any point. Then $\mathrm{bd}\, B \cap \Sigma$ must be empty or a set of 1-spheres, or cycles. If condition (b) holds, shrink B untill two cycles meet, or a new cycle appears, or an existing cycle vanishes. At this point, B must touch Σ tangentially at a point and our previous argument applies to claim that B has a medial axis point.

So, suppose that condition (b) does not hold and condition (a) holds. In that case we must have two points whose normals meet at a point in B when extended and this point is a medial axis point by definition. Otherwise, project each point of $B \cap \Sigma$ onto $\mathrm{bd}\, B$ with the function $h : (B \cap \Sigma) \to \mathrm{bd}\, B$ where $h(x)$ is the point where the line of oriented normal at x hits $\mathrm{bd}\, B$. We claim that h is a homeomorphism between $B \cap \Sigma$ and its image $h(B \cap \Sigma)$. It is one-to-one, otherwise we have a point in B where lines along the normals at two points of Σ meet in B. It is onto on $h(B \cap \Sigma)$. Since $B \cap \Sigma$ is continuous and normals on Σ varies continuously, h is continuous. The patch $h(B \cap \Sigma)$ has the same boundary as $\mathrm{bd}\, B \cap \Sigma$ which is a 1-sphere. This means $h(B \cap \Sigma)$ is a 2-ball and so is $B \cap \Sigma$ violating the assumption that condition (a) holds.