> "In drawing 7 or more lines on a piece of paper
> there are always 3 or more through a common point."

15 Line arrangements

One of the oldest topics in discrete geometry is the study of decompositions of the plane obtained by drawing finitely many lines, see Steiner [5] for one of the earliest references and Grünbaum [3] for a comprehensive survey. Such decompositions are fundamental in geometry, which will become apparent when we see multiple connections to other geometry problems, including those discussed in earlier sections. To avoid unnecessary confusion, we note that a line in $\mathbb{R}^2$ has no endpoints and decomposes $\mathbb{R}^2$ into two half-planes. In other words, an edge connecting two points is a line segment and not a line at all.

Definitions. Let H be a finite collection of lines in the plane. The decomposition of $\mathbb{R}^2$ defined by the lines can be formalized as a plane graph, with some edges going to infinity, or as a complex with convex cells of dimension $0, 1, 2$. A *vertex* is a point contained in two or more lines. An *edge* is a segment connecting two contiguous endpoint on a line (one or both endpoints can be at infinity in which case the edge is a half-line or a line). A *chamber* is a maximal convex subset of $\mathbb{R}^2$ whose interior is non-empty and avoids all lines in H, see figure 15.1. The *arrangement* of H,

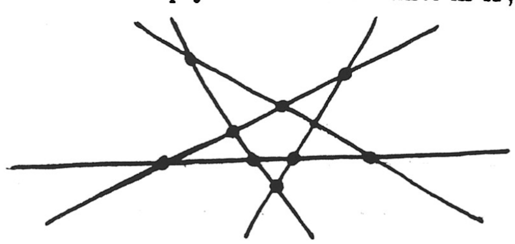

Figure 15.1: A simple arrangement of 5 lines consists of 10 vertices, 25 edges (10 of which are half-lines), and 16 chambers (10 of which are unbounded).

$\mathcal{A} = \mathcal{A}(H)$, is the collection of vertices, edges, and chambers defined by the lines. The elements of $\mathcal{A}$ are generically referred to as *faces*. $\mathcal{A}$ is *simple* if every 2 and no 3 lines meet in a common point.

Counting faces. We will see shortly that every simple arrangement of n lines has the same number of vertices, edges, and chambers. The numbers for non-simple arrangements can only be less. Indeed, if two lines are parallel we can turn one slightly and gain 1 vertex, 2 edges, and 1 chamber. If three lines are concurrent we can move one slightly and gain 2 vertices, 3 edges, and 1 chamber. Similar gains result from slight perturbations involving one or several groups of parallel or concurrent lines. We present two proofs for the following face counting formulas for line arrangements.

THM. 15.1 A simple arrangement of n lines has $\binom{n}{2}$ vertices, $2\binom{n}{2} + n$ edges, and $\binom{n}{2} + n + 1$ chambers.

PROOF (plane sweep argument). Every 2 lines meet in a unique point, so there are $\binom{n}{2}$ vertices. Assume no line in H is horizontal and sweep $\mathcal{A}(H)$ with a horizontal line, ℓ, from bottom to top. Initially, ℓ meets n edges (one per line) and $n+1$ chambers. Each time ℓ passes through a vertex, it encounters 2 new edges and 1 new chamber. ☐

PROOF (induction). Write f_0, f_1, f_2 for the number of vertices, edges, chambers of a simple arrangement. For one line we have $f_0(1) = 0$, $f_1(1) = 1$, and $f_2(1) = 2$. The nth line contributes $n-1$ new vertices and n new edges, and it cuts $n-1$ old edges and n old chambers in two each. Hence,

$$f_0(n) \;=\; f_0(n-1) + n - 1 = \sum_{i=1}^{n} i - 1 = \binom{n}{2},$$

$$f_1(n) = f_1(n-1) + 2n - 1 = \sum_{i=1}^{n}(2i-1) = 2\binom{n}{2} + n, \text{ and}$$

$$f_2(n) = f_2(n-1) + n = 1 + \sum_{i=1}^{n} i = \binom{n}{2} + n + 1.$$

□

REMARK. Although any two simple arrangements have the same number of vertices, edges, and chambers, they need not be isomorphic as complexes. For example, all simple arrangements of 4 lines are isomorphic but there are two non-isomorphic arrangements of 5 lines. The number of non-isomorphic arrangements grows rapidly with n and the enumeration of all non-isomorphic arrangements quickly becomes cumbersome and difficult, for simple and for general line arrangements, see [4] for $n \leq 7$.

Incremental construction. Possibly the most straightforward algorithm for constructing $\mathcal{A}(H)$ adds a line at a time to the growing arrangement. It follows the idea of the inductive proof of thm. 15.1. Let $H = \{h_1, h_2, \ldots, h_n\}$, and for $1 \leq i \leq n$ define $H_i = \{h_1, h_2, \ldots, h_i\}$. For simplicity assume $\mathcal{A}(H)$ is simple, see also section 13.

Construct $\mathcal{A}(H_2)$;
for $i := 3$ **to** n **do**
    find an unbounded chamber $\chi \in \mathcal{A}(H_{i-1})$ whose intersection with h_i is a half-line; $\epsilon := \emptyset$;
    loop add $h_i \cap \chi$ as a new edge that splits χ;
        find edge $\epsilon' \neq \epsilon$ of χ that intersects h_i;
        if ϵ' exists **then** let $\chi' \neq \chi$ be the chamber that shares ϵ' with χ;
                $\epsilon := \epsilon'$; $\chi := \chi'$; add $h_i \cap \epsilon'$ as a new vertex that splits ϵ'
            else exit
        endif
    forever
endfor.

We assume a data structure so that the edges of a chamber can be enumerated in constant time per edge. Similarly, the vertices of an edge, the edges that contain a vertex, and the chambers that contain an edge are assumed to be available in constant time. There are several different ways to find the first chamber whose intersection with h_i is a half-line in time $O(i)$. Each time the inner loop is executed, a chamber is cut in two, which happens less than n^2 time, see thm. 15.1. However, it is not clear how much time is required to find edges ϵ' and split chambers χ. The time needed for a single chamber χ is proportional to its *degree*, $\deg \chi$, defined as the number of edges bounding χ.

Zones. Let H be a set of n lines in $\mathbf{R}^2$ and $b \notin H$ another line so that $\mathcal{A}(H \cup \{b\})$ is simple. The *zone* of b in $\mathcal{A}(H)$ is $Z_b(H) = \{\chi \in \mathcal{A}(H) \mid \chi \text{ is a chamber}, \chi \cap b \neq \emptyset\}$, see figure 15.2. For the purpose of analyzing the incremental

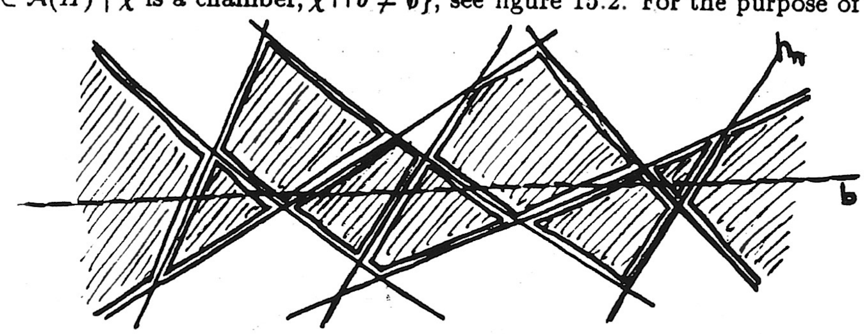

Figure 15.2: The shaded chambers belong to the zone of b.

algorithm, we are interested in $z_b(H) = \sum_{\chi \in Z_b(H)} \deg \chi$ and

$$z(n) = \max_{\text{card } H = n,\, b \notin H} \{z_b(H)\}.$$

Suppose b were to be added to $\mathcal{A}(H)$ using the above method. Then $Z_b(H)$ is the set of chambers visited, and $z_b(H)$ is proportional to the time required to add b.

THM. 15.2 $z(n) < 6n$.

PROOF. Assume b is horizontal and label the lines h_1 through h_n so h_i intersect b to the left of h_j if $i < j$. By assumption of simplicity, all h_i are non-vertical, so we can unambiguously distinguish between lines to the left and lines to the right of a chamber. An edge of a chamber $\chi \in \mathcal{A}(H)$ is a *left* edge if its line is to the left of χ. Otherwise, it is a *right* edge of χ. We use induction to show the chambers in $Z_b(H)$ have fewer than $3n$ left edges.

For $n = 1$ there is only one left edge. Let $H_{n-1} = H - \{h_n\}$ and assume inductively that the chambers in $Z_b(H_{n-1})$ have fewer than $3(n-1)$ left edges. h_n splits the rightmost chamber in two. At the same time, it splits up to two left edges, one above and one below b. In addition, h_n contributes one new left edge, namely to the new rightmost chamber. No other left edges are added, although some are possibly removed. The argument for right edges is symmetric (lines are added from right to left). It follows the sum of degrees of chambers in the zone of b, which is the total number of left and right edges, is less than $3n + 3n = 6n$. ◻

Analysis. The time required to add the ith line to $\mathcal{A}(H_{i-1})$ is $O(i)$, and the total time to construct $\mathcal{A}(H)$ is $O(\sum_{i=1}^{n} i) = O(n^2)$. This is asymptotically optimal because $\mathcal{A}(H)$ can consist of more than n^2 faces.

REMARK. An extension of thm. 15.2 to three and higher dimensions has been proved in [2]. Algorithmic questions related to arrangements of lines, planes, and hyperplanes are studied extensively in [1].

Homework exercises

15.1 A car race is organized on an infinitely long straight road. The cars start out at different positions and travel with constant speed (different speed for different cars) in the same direction. The car immediately in front a given car is its *leader*. The leader of a fixed car, C, changes when C overtakes its leader, C is overtaken by another car, or the leader of C overtakes its own leader. Show that the number of times the leader of C changes is less than $3n$.

15.2 Consider the race as explained in exercise 15.1 and assume the number of participating cars is $2n-1$. Show that the number of times the car in the middle, or nth position changes is $O(n\sqrt{n})$.

15.3 Let H be a set of $n \geq 4$ lines in $\mathbf{R}^2$ defining a simple arrangement. For $h \in H$ let $a_h \leq b_h$ be the numbers of vertices in the two open half-planes defined by h. Show there is is a constant c, independent of n, so that H contains a line h with $b_h \leq 3a_h + \frac{c}{n}$.

References

[1] H. EDELSBRUNNER. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Heidelberg, 1987.

[2] H. EDELSBRUNNER, R. SEIDEL AND M. SHARIR. On the zone theorem for hyperplane arrangements. *SIAM J. Comput.* **22** (1993), 418–429.

[3] B. GRÜNBAUM. *Arrangements and Spreads.* Regional Conf. Ser. Math., Amer. Math. Soc., Providence, 1972.

[4] R. KLEE. *Über die einfachen Konfigurationen der euklidischen und der projektiven Ebene.* Focken und Oltmanns, Dresden, 1938.

[5] J. STEINER. Einige Gesetze über die Theilung der Ebene und des Raumes. *J. Reine Angew. Math.* **1** (1826), 349–364.