"No matter into how many segments we cut an interval,
the vertices will always outnumber the edges by one."

26 Euler's relation for convex polyhedra

The *Euler characteristic* of a simplicial complex, $\mathcal{K}$, is the alternating sum of simplex numbers,

$$\chi(\mathcal{K}) = \sum_{i=-1}^{d} (-1)^i s_i,$$

where $s_i = s_i(\mathcal{K})$ is the number of i-dimensional simplices and d is the dimension of $\mathcal{K}$. Alternatively, we can write $\chi(\mathcal{K}) = \sum_{\sigma \in \mathcal{K}} (-1)^{\dim \sigma}$. A nice introduction to the Euler characteristic and some of its applications can be found in [6]. Interestingly, the Euler characteristic does not depend on $\mathcal{K}$, but rather on $|\mathcal{K}|$, the underlying space of $\mathcal{K}$. For example, it can be shown that $\chi(\mathcal{K}) = \chi(\mathcal{L})$ if $|\mathcal{K}|$ and $|\mathcal{L}|$ are homeomorphic.

This section considers the Euler characteristic for boundary complexes of convex polyhedra in $\mathbf{R}^d$. In this special case, $\chi(\mathcal{K}) = 1$ provided the polyhedron is bounded. Early proofs of this relation in arbitrary finite dimensions date back as early as 1901 [5]. This century witnessed a variety of different proofs based on line shelling [1], sweeping a hyperplane [3], decomposing with an arrangement [4], etc. We present an elementary inductive proof of the relation taken from [2].

Inclusion-exclusion. Consider a finite set H of closed half-spaces in $\mathbf{R}^d$. The intersection of the half-spaces is a convex polyhedron, $\bigcap H = \bigcap_{h \in H} h$. For convenience, assume the hyperplanes bounding the half-spaces are in general position and $\bigcap H \neq \emptyset$. The faces of this convex polyhedron do not, in general, form a simplicial complex, but the nerve of the collection of facets is a simplicial complex. Faces of dimension less than $d-1$ correspond to simplices of dimension larger than 0. The alternating sum of face numbers for $\bigcap H$ can thus be interpreted as the Euler characteristic of this nerve, see also section 12. For each point $x \in \mathbf{R}^d$ and each subset $I \subseteq H$, define the indicator function

$$\gamma_I(x) = \begin{cases} 1 & \text{if } x \notin h \text{ for all } h \in I, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To get an intuitive feeling of what γ indicates consider a fixed subset $I \subseteq H$. Clearly, $\bigcap H \subseteq \bigcap I$. The hyperplanes bounding half-spaces in I define a simple arrangement of which $\bigcap I$ is one chamber. If $\bigcap I$ is unbounded there is a unique opposite chamber, and $\gamma_I(x) = 1$ iff x lies in the interior of the opposite chamber, see figure 26.1. For a set

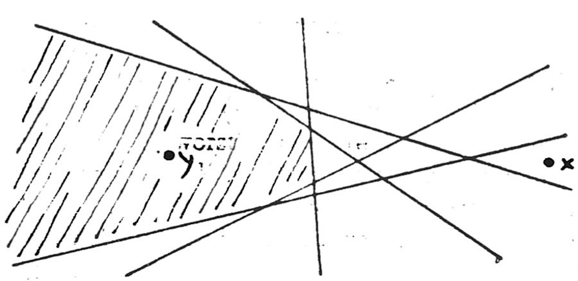

Figure 26.1: $\gamma_\emptyset(y) = 1$ and $\gamma_I(y) = 0$ for all non-empty subsets I of H. $\gamma_I(x) = 1$ for all $I \subseteq H$.

system $S \subseteq 2^H$ define

$$\Gamma_S(x) = \sum_{I \in S} (-1)^{\operatorname{card} I} \gamma_I(x).$$

Note the effect of γ is that it restricts the sum to half-spaces that do not contain x. The inclusion-exclusion principle can be expressed by setting $S = 2^H$.