"If $ab$ is an edge and $abc, abd$ are triangles, then the angle at $c$ exceeds the angle at $d$ if $d$ lies outside the circle through $a, b, c$."

# 8 Delaunay triangulations

The convex hull of a finite point set gives a specific meaning to the region spanned or occupied by the set. A decomposition into triangles lends structure to this region, which is expressed by how the triangles are connected via shared edges. Even if only edges connecting points of the set are considered, the number of possible decompositions is in general exponential in the number of points. Among these decompositions there is an arguably most natural one introduced in 1934 by Boris Delaunay, also Delone [2].

**Triangulating a point set.** Given a finite set $S \subseteq \mathbf{R}^2$, we consider decompositions of $\text{conv}\, S$ into triangles using the points in $S$ as vertices. Formally, a *(geometric) triangulation* of $S$ is a simplicial complex $\mathcal{K}$ so that $|\mathcal{K}| = \text{conv}\, S$ and $\mathcal{K}^{(0)} = S$. Later, we will relax the second condition and admit simplicial complexes with vertex sets different from $S$. Note that $\mathcal{K}^{(1)}$ is a plane graph with $n = \text{card}\, S$ vertices. If $h$ of the vertices lie on $\text{bd}\,\text{conv}\, S$, we need $h - 3$ additional edges to obtain a maximally connected planar graph. Hence, $\mathcal{K}$ has $e = 3n - h - 3$ edges and $e - n + 1 = 2n - h - 2$ triangles.

Instead of a detailed discussion of algorithms that construct a triangulation of $S$, we refer back to section 5 where convex hull algorithms are studied. Indeed, both the incremental algorithm and the divide-and-conquer algorithm effectively triangulate $S$. Whenever common tangents are sought, the intermediate edges constructed in the process need to be retained, see figure 5.3. Both algorithms take time $O(n)$ after sorting, so we have two $O(n \log n)$ time algorithms for constructing some arbitrary triangulation of $S$.

**Nerves.** We use Voronoi diagrams and certain set systems to introduce the triangulation of $S$ that we consider most natural. The systems can be defined for any arbitrary collection of sets. Let $A$ be a collection that is finite and otherwise arbitrary. The *nerve* of $A$ is

$$\mathcal{N} = \mathcal{N}(A) = \{X \subseteq A \mid \bigcap X \neq \emptyset\}.$$

For example, $\mathcal{N} = 2^A$ if the sets in $A$ have a non-empty common intersection. Clearly $\mathcal{N}$ is finite because $A$ is finite and $\mathcal{N} \subseteq 2^A$. By convention, $\bigcap \emptyset \neq \emptyset$ so $\emptyset \in \mathcal{N}$. Furthermore, if $Y \subseteq X$ then $\bigcap X \subseteq \bigcap Y$, so if $X \in \mathcal{N}$ then $Y \in \mathcal{N}$. It follows that $\mathcal{N}$ is an abstract simplicial complex. Nerves have been introduced as a tool in topology by Alexandrov, also Alexandroff [1]. By Thm. 6.1, $\mathcal{N}$ has a geometric realization if the dimension of the containing space is sufficiently high. We will consider nerves that can be realized in $\mathbf{R}^2$.

**Delaunay triangulations.** Let $S$ be a finite set of points in general position in $\mathbf{R}^2$, and let $V_S = \{V_p \mid p \in S\}$ be the collection of Voronoi cells. By assumption of general position, at most three cells have a non-empty common

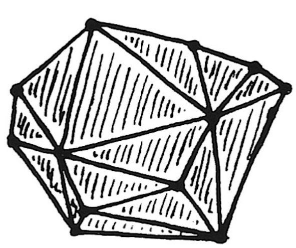

Figure 8.1: The Delaunay triangulation of the 12 points with Voronoi diagram shown in figure 7.1.

intersection, so the nerve of $V_S$, $\mathcal{N} = \mathcal{N}(V_S)$, is a 2-dimensional abstract simplicial complex. The *Delaunay triangulation* of $S$ is the geometric realization, $\mathcal{D} = \mathcal{D}_S$ of $\mathcal{N}$ defined by the injective map $\varphi : V_S \to \mathbf{R}^2$ with $\varphi(V_p) = p$. In other words, whenever two Voronoi cells share an edge, $\mathcal{D}$ contains the edge connecting the points generating the two cells, and whenever three Voronoi cells share a vertex, $\mathcal{D}$ contains the triangle spanned by the three corresponding points, see figure 8.1. We should be careful and make sure $\varphi$ really defines a geometric realization of $\mathcal{N}$. We will see in lemma B that a triangle in $\mathcal{D}$ contains no vertices, other than its own. So it suffices to show that no two vertex-disjoint edges intersect.

LEMMA A. If $pq, rs \in \mathcal{D}$ and $\{p,q\} \cap \{r,s\} = \emptyset$ then $pq \cap rs = \emptyset$.

PROOF. Observe that $pq \in \mathcal{D}$ implies there is a circle, $C_{pq}$, through $p$ and $q$ so that all other points lie outside $C_{pq}$. Similarly, there is a circle $C_{rs}$ through $r$ and $s$ so that all other points lie outside $C_{rs}$. $pq$ is an edge whose endpoints lie on $C_{pq}$ and outside $C_{rs}$, and $rs$ is an edge whose endpoints lie on $C_{rs}$ and outside $C_{pq}$. Because $pq \cap rs$ is a point inside both, $C_{pq}$ and $C_{rs}$ must cross in at least four points, see figure 8.2. This contradicts the fact that

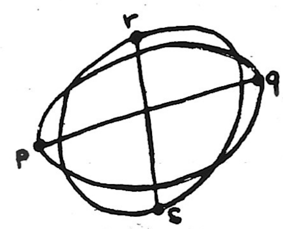


Figure 8.2: Two vertex-disjoint edges of the Delaunay triangulation have a non-empty intersection only if two circles cross in 4 or more points, which is impossible.

two different circles intersect in at most two points. □

**Empty circle criterion.** The Delaunay triangulation of $S$ is not necessarily defined if $S$ contains 4 or more cocircular points, so we assume throughout that this is not the case. A most important property of Delaunay triangulations is the following.

LEMMA B. $\sigma = \operatorname{conv} T$ is a simplex in $\mathcal{D}$ iff there is a disk $D$ with $S \cap \operatorname{bd} D = T$ and $S \cap \operatorname{int} D = \emptyset$.

PROOF. $\sigma$ belongs to $\mathcal{D}$ iff $V_T = \{V_p \mid p \in T\}$ belongs to $\mathcal{N}(V_S)$. Take a point $x$ in the interior of $\bigcap V_T$. All points in $T$ are equally far from $x$, say at distance $\delta$, and all points in $S - T$ are further from $x$ than $\delta$. Define $D = \{y \in \mathbf{R}^2 \mid |yx| \leq \delta\}$ and observe that $S \cap \operatorname{bd} D = T$ and $S \cap \operatorname{int} D = \emptyset$. To see the implication in the other direction, notice that a disk $D$ with the above properties implies that its center, $x$, belongs to the interior of $\bigcap V_T$. □

Lemma B is often used to define the Delaunay triangulation of a set $S$. It certainly implies that $\mathcal{D} = \mathcal{D}_S$ is unique if $S$ is in general position. For point sets not in general position, there are several ways to define a complex similar to the Delaunay triangulation, and the choice will depend on the application. If, for example, $k > 3$ Voronoi cells share a common vertex, then the nerve of $V_S$ contains all subsets of the $k$ corresponding points as abstract simplices, and the ones of dimension higher than two cannot be realized in $\mathbf{R}^2$. One possibility is to give up on simpliciality and to replace these simplices by the $k$-gon spanned by the $k$ cocircular points generating the cells. Another is to give up on uniqueness and to retain $k-2$ triangles that decompose the $k$-gon. Such triangles are automatically generated if the general position assumption is simulated by an infinitesimal perturbation [3, 4].

**Locality lemma.** Let $\mathcal{K}$ be an arbitrary geometric triangulation of $S$. We call an edge $pq \in \mathcal{K}$ *locally delone* if it belongs to only one triangle or it belongs to two triangles, $pqr, pqs \in \mathcal{K}$, and there is a circle through $p$ and $q$ so that $r$ and $s$ lie outside the circle. If $pq \in \mathcal{D} = \mathcal{D}_S$ then $pq$ is locally delone (lemma B, but it is possible that an edge of $\mathcal{K}$ is locally delone and does not belong to $\mathcal{D}$, see figure 8 3. A useful property is that if *all* edges of $\mathcal{K}$ are locally delone, then $\mathcal{K}$ is necessarily the Delaunay triangulation.

LEMMA C. (Delaunay, 1934). If every edge of a triangulation $\mathcal{K}$ of $S$ is locally delone then $\mathcal{K} = \mathcal{D}_S$.

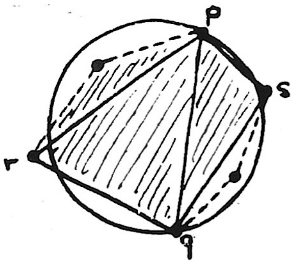


Figure 8.3: $pq$ is locally delone because $r$ and $s$ lie outside a circle through $p$ and $q$. It is still possible to have two points inside the circle that prevent $pq$ from being an edge of the Delaunay triangulation.

PROOF. We show that if $pqr$ is a triangle and $s \in S - \{p, q, r\}$ is a vertex of $\mathcal{K}$ then $s$ lies outside the circle $C_{pqr}$ through $p, q, r$. It follow that all triangles of $\mathcal{K}$ satisfy the condition of lemma B and thus belong to the Delaunay triangulation of $S$.

Take a point $x \in pqr$ and consider the triangles $pqr = \tau_1, \tau_2, \ldots, \tau_k$ that intersect the open line segment from $x$ to $s$ in this sequence. Since $x$ can be chosen freely within $pqr$, we can assume that $xs$ contains no vertex, so the sequence of $\tau_i$ is unambiguously defined. By construction, $s$ and $\tau_{i+1}$ lie on the same side of the line separating $\tau_i$ and $\tau_{i+1}$, for $1 \leq i \leq k-1$. This line is the radical axis of the disks $D_i$ and $D_{i+1}$ bounded by the circles through the vertices of $\tau_i$ and $\tau_{i+1}$. Because the common edge of $\tau_i$ and $\tau_{i+1}$ is locally delone, we have

$$\pi_{D_i}(s) > \pi_{D_{i+1}}(s).$$

Transitivity implies $\pi_{D_1}(s) > \pi_{D_k}(s)$. Since $s$ is a vertex of $\tau_k$, we also have $\pi_{D_k}(s) = 0$. So $\pi_{D_1}(s) > 0$, which implies $s \notin D_1$ and therefore $s$ lies outside $C_{pqr}$. ☐

## Homework exercises

8.1 Let $A$ and $B$ be two disjoint finite sets in $\mathbf{R}^2$. Prove that if $a \in A$ and $b \in B$ are such that $|ab| \leq |xy|$ for all $x \in A$ and $y \in B$ then $ab$ is an edge of the Delaunay triangulation of $A \cup B$.

8.2 Let $S$ be a finite set in $\mathbf{R}^2$, let $G = (S, \binom{S}{2})$ be the complete graph with vertex set $S$, and define the *length* of an edge $xy \in \binom{S}{2}$ equal to the Euclidean distance between $x$ and $y$. Prove that the minimum spanning tree of $G$ is a subgraph of $\mathcal{D}^{(1)}$, the 1-skeleton of the Delaunay triangulation of $S$.

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