Lecture 10: Basic Surface Topology [1]

Manifolds

In three dimensions the topology of surfaces becomes an important factor in modeling. We will mean a manifold when we talk about surfaces in 3D.

A 2-manifold $S \in R^k$ is a topological space such that each point in S is *homeomorphic* to a 2-disk. Homeomorphism is a continuous function defined between two spaces which is bijective and also has a continuous inverse. For example, a square in plane is homeomorphic to a disk, a "surface patch" in 3D as we will call it is also homeomorphic to a 2-disk.

A 2-manifold may be embedded in R^3 meaning that it has no self-intersection. Or, it might be *immersed* in R^3 in which case there may be self-intersection.

Some examples of 2-manifolds are spheres, torus, double torus. We know that some of the surfaces as we know might have boundaries. For example, a "surface patch" has a boundary. Our definition of 2-manifolds does not allow such "surface patch". So, we need another definition. We define 2-manifold *with boundary* as a topological space such that each point has a neighborhood homeomorphic to a half-disk. A sphere with a hole cut out is a 2-manifold with boundary. The points that have only half-disk neighborhood constitute the boundary. In general, the boundary of a 2-manifold is a 1-manifold that is a closed curve.

Classification of surfaces

We will calll 2-manifolds as *surfaces* and 2-manifolds with boundary as *surfaces with boundary.*

Surfaces in 3D have a nice charaterization up to topology. A 2-manifold is either a sphere or a join of one or more torus. A join of two tori form the double torus, and in general join k tori form a surface called k-tori.

The *genus* of an orientable surface without boundary is equal to half the minimum number of cuts along simple curves required to make it flat or a disk. For example, a torus needs two cuts one along the equator, and one along meridian to make it flat. Only one cut makes it a cylinder which again can be cut to make the rectangle. Conversely, a rectangle can be made into a torus by identifying the opposite edges of a rectangle. Actually, this process can be generalized for arbitrary 2-manifold surfaces. A genus-g surface can be obtained by identifying appropriate edges of a $4g$-gon. The sphere is a special case whose genus is 0.

A similar charaterization of surfaces with boundary is also possible. The characterization takes into account the genus of the surface and the number of boundaries.

Euler Characteristic

There is a combinatorial characterization of surfaces that is also sometimes useful in modeling. We assume only two types of surface patches that a surface in decomposed into. They are either triangular or rectangular. In each case we assume the surface patches join nicely to form a *complex.*, i.e., any two of the surface patches either do not meet, or meet in an edge or a vertex of both.

Let v, e and f denote the number of vertices, edges and faces (patches) of a surface complex. The quantity $v - e + f$ is called the Euler characteristic of the surface which is a topological invariant. It

[1] Note by Tamal K. Dey

means that any two surfaces that are homeomorphic must have the same Euler characteristic. For example, the boundary of a tetrahedron is homeomorphic to a sphere, and its Euler characteristic is $4-6+4=2$ which is same as the Euler characteristic of a cube boundary $8-12+6=2$. In general, the genus-g surface without boundary has an Euler characteristic of $2-2g$. Thus, the torus has Euler characteristic 0. If there are b boundaries, the Euler charactersitic becomes $2-2g-b$.

We will often want a rectangular net on a surface with each vertex having degree four. Such a net is essential for generating Bézier or B-spline surfaces. Using Euler characteristic we can show that there is no rectangular net that can span a sphere and has degree four at each vertex. If this were possible we would have:

$$\begin{aligned} 2e &= 4v \; from\ degree\ consideration \\ 2e &= 4f \; each\ face\ rectangular \\ v-e+f &= 2 \; Eulercharacteristic \\ or,\ \frac{1}{2}e-e+\frac{1}{2}e &= 2,\ an\ impossibility \end{aligned}$$

Thus, it is always difficult to fit a Bézier or B-spline surfaces on sphere.

However, a torus will admit such a net because its Euler characteristic is indeed 0 which is required by the first two equations above. In fact, it is only the torus among all surfaces without boundary that admits a rectangular net. There are many surfaces with boundary that admits a rectangular net where the degree of a vertex on the boundary is 3. For eaxmple, a disk admits such a rectangular net. All surfaces admit triangular net with no restriction on degree.

Lecture 11: Introduction to Surface Equations [1]

Implicit surfaces

An equation of the form $f(x,y,z)=0$ is an implicit equation of a surface. In addition if $f(x,y,z)$ is a polynomial in x,y,z we have a polynomial surface with implicit equation

$$\Sigma_{i,j,k} a_{i,j,k} x^i y^j z^k = 0$$

The degree of the surface is the maximum degree of a term, $i+j+k$. If we solve one variable in terms of the other, say, z, we get explicit equation $z=f(x,y)$ of the same surface. A rational parametric form of a surface is expressed as $z=f(u,v), x=g(u,v), y=h(u,v)$ where f,g,h are polynomials. Converting a parametric form to an implicit form is called *implicitization* and it is possible to implicitize every parametric surface.

Quadric surfaces

A quadric polynomial surface is expressed with an implicit equation of degree 2. A general form of a quadric surface is:

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fzx + 2Gx + 2Hy + 2Jz + K = 0$$

In matrix form one can write: $\mathbf{P}\mathbf{Q}\mathbf{P}^T$ where $\mathbf{P} = [x\ y\ z\ 1]$ and

$$\mathbf{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{D} & \mathrm{F} & \mathrm{G} \\ \mathrm{D} & \mathrm{B} & \mathrm{E} & \mathrm{H} \\ \mathrm{F} & \mathrm{E} & \mathrm{C} & \mathrm{J} \\ \mathrm{G} & \mathrm{H} & \mathrm{J} & \mathrm{K} \end{bmatrix}$$

Except for some degenerate cases, all quadric surfaces that has explicit eqaution $z=f(x,y)$ of degree 2 must be one of the following three types:

$$\begin{aligned} z &= x^2/a^2 + y^2/b^2 \ \textit{elliptic paraboloid} \\ z &= x^2/a^2 - y^2/b^2 \ \textit{hyperbolic paraboloid} \\ y^2 &= 4ax \ \textit{parabolic cylinder} \end{aligned}$$

It should be noted that a parametric surface of degree 2 may correspond to an explicit surface of degree more than 2. For example,

$$\begin{aligned} x &= u \\ y &= u^2 + v \\ z &= v^2 \end{aligned}$$

An explicit equation of this surface is $z=(y-x^2)^2$ which is of degree 4 and its degree cannot be lower.

[1] Note by Tamal K. Dey, Ohio State U.

Curvatures

For each point on a surface we can define its curvatures that tells us how fast the surface is bending locally. Consider the tangent plane at a point p on the surface. The surface normal at that point is defined as the normal to the tangent plane. Consider the planes passing through the line supporting the normal. These planes intersect the surface in open curves in an open neighborhood of p. Among these curves, the two curves with maximum and minimum curvatures are of most importance to us. These two curvatures with signs are called *principal curvatures* at p. The larger (absolute value) of the two, say κ_1 is called the maximum curvature and the smaller of the two, say κ_2 is called the minimum curvature at p. The *Gaussian curvature* at p is defined as $\kappa = \kappa_1\kappa_2$. The arithmetic mean of the two curvatures is called the *mean curvature* at p.

Now one can observe that hyperbolic paraboloid has a Gaussian curvature negative at the origin. Elliptic paraboloid has Gaussian curvature positive at the origin. The cylinderical paraboloid has Gaussian curvature 0 at the origin.

Any surface locally can be approximated with a degree 2 explicit surface and thus each point is either elliptical, or hyperbolic or parabolic.

Parametric surfaces

We will talk about it when we cover Bézier and B-spline surfaces.