CS 565

Programming Languages (graduate)
Spring 2024

Week 11

Loop Invariants and Introduction to Dafny

Loop Invariants

Hoare Logic is a structural model-theoretic proof system

- Rules characterize a set of states consistent with the requirements imposed by the pre- and post-conditions
- Highly mechanical: intermediate states can almost always be automatically constructed
- One major exception:

$$\frac{\vdash \{I \wedge b\}\ c\ \{I\}}{\vdash \{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \textbf{end}\ \{I \wedge \neg b\}}\ \text{HLWHILE}$$

The invariant must:

- be weak enough to be implied by the precondition
- hold across each iteration
- be strong enough to imply the postcondition

Example


```
{{ True }}
  if X <= Y then
                {{                          }}
    Z := Y - X
                {{                          }}
  else
                {{                          }}
    Y := X + Z
                {{                          }}
  end
{{ Y = X + Z }}
```

Our proof rules provide a systematic way of generating intermediate assertions. The fully decorated program constitutes a proof that the program when executed in a state that satisfies the precondition, will produce a state satisfying the postcondition.

Example


```
{{ True }}
  if X <= Y then
              {{                                  }}
    Z := Y - X
              {{                                  }}
  else
              {{                                  }}
    Y := X + Z
              {{Y = X + Z}}
  end
{{ Y = X + Z }}
```

follows from the postcondition

Example


```
{{ True }}
  if X <= Y then
                {{                                  }}
    Z := Y - X
                {{Y = X + Z}}
  else
                {{                                  }}
    Y := X + Z
                {{ Y = X + Z}}
  end
{{ Y = X + Z }}
```

Example


```
{{ True }}
  if X <= Y then
                {{X <= Y}}
    Z := Y - X
                {{Y = X + Z}}
  else
                {{X > Y}}
    Y := X + Z
                {{Y = X + Z}}
  end
{{ Y = X + Z }}
```

- *By If Rule*
- *But, shape of precondition in assignments does not match the shape demanded by the Assgn rule*

Example


```
{{ True }}
  if X <= Y then
                {{X <= Y}} ->
                {{Y = X + (Y - X)}}
    Z := Y - X
                {{Y = X + Z}}
  else
                {{X > Y}} ->
                {{X + Z = X + Z}}
    Y := X + Z
                {{Y = X + Z}}
  end
{{ Y = X + Z }}
```

Strengthen precondition using rule of consequence and Assgn rule

Non-trivial implication

Proof can be constructed automatically, reasoning backwards from the postcondition

Loops


```
{{ True }} -->
      {{                          }}
   X := a;
      {{                          }}
   Y := b;
      {{                          }}
   Z := 0;
      {{                          }}
   while X <> 0 && Y <> 0 do
      {{                          }}
    X := X - 1;
      {{                          }}
    Y := Y - 1;
      {{                          }}
    Z := Z + 1;
      {{                          }}
   end
{{ Z = min a b }}
```

Loops


```
{{ True }}
      {{                          }}
   X := a;
      {{                          }}
   Y := b;
      {{                          }}
   Z := 0;
      {{                          }}
   while X <> 0 && Y <> 0 do
      {{                          }}
    X := X - 1;
      {{                          }}
    Y := Y - 1;
      {{                          }}
    Z := Z + 1;
      {{                          }}
   end
{{ Z = min a b }}
```

postcondition given in terms of inputs a and b

loop invariant expresses constraints on local variables X and Y

candidate invariant:

```
Z + min X Y = min a b
```

Loops


```
{{ True }}
      {{  min a b = min a b  }}
   X := a;
      {{  min X b = min a b  }}
   Y := b;
      {{  min X Y = min a b  }}
   Z := 0;
      {{  Inv                }}
   while X <> 0 && Y <> 0 do
      {{ Z + 1 + min (X - 1) (Y - 1) = min a b }}
    X := X - 1;
      {{ Z + 1 + min X (Y - 1) = min a b }}
    Y := Y - 1;
      {{ Z + 1 + min X Y = min a b }}
    Z := Z + 1;
      {{ Inv  }}
   end
{{ ~(X <> 0 /\ Y <> 0) /\ Inv) }} ->
{{ Z = min a b }}
```

Inv == (Z + min X Y = min a b)

Loops


```
{{ True }} ->
      {{  min a b = min a b  }}
   X := a;
      {{  min X b = min a b  }}
   Y := b;
      {{  min X Y = min a b  }}
   Z := 0;
      {{  Inv                }}
   while X <> 0 && Y <> 0 do
      {{ Inv /\ (X <> 0) /\ Y <> 0) }} ->
      {{ Z + 1 + min (X - 1) (Y - 1) = min a b }}
     X := X - 1;
       {{ Z + 1 + min X (Y - 1) = min a b }}
     Y := Y - 1;
       {{ Z + 1 + min X Y = min a b }}
     Z := Z + 1;
       {{ Inv  }}
   end
{{ ~(X <> 0 /\ Y <> 0) /\ Inv) }} ->
{{ Z = min a b }}
```

This style of proof construction is known as weakest precondition inference

Identify a precondition that satisfies the largest set of states that still enable verification of the postcondition

Can automate this inference once we know the loop invariant

Concept Check


```
{{ ? }}  skip  {{ X = 5 }}                    X = 5

{{ ? }}  X := Y + Z {{ X = 5 }}               Y + Z = 5

{{ ? }}  X := Y  {{ X = Y }}                  True
```

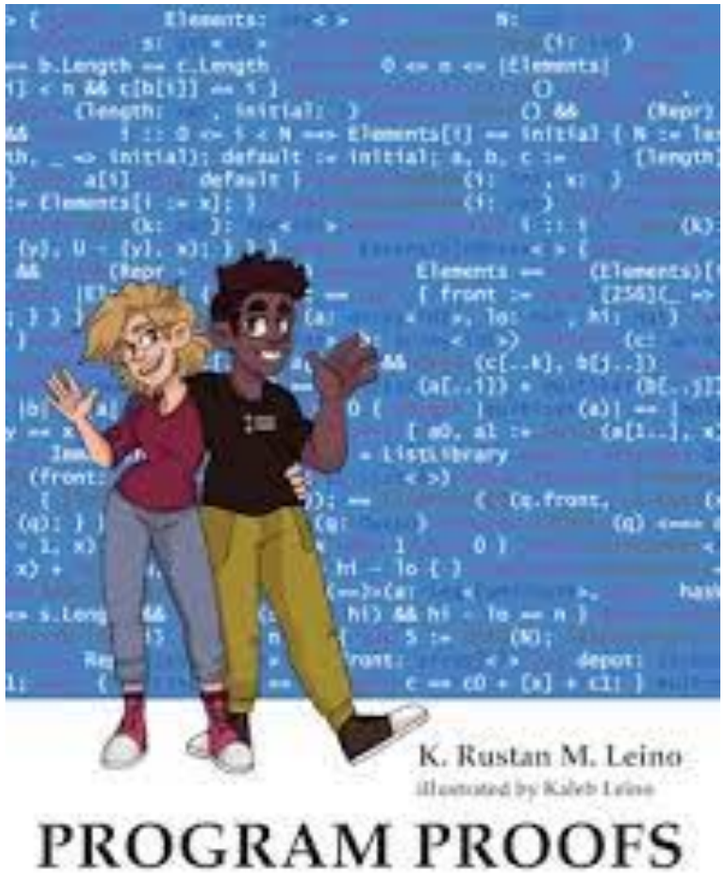
```
{{ ? }}                                       (X = 0 /\ Z = 4) \/ (X <> 0 /\ W = 3)
  if X = 0 then
       Y := Z + 1
  else Y := W + 2
{{ Y = 5 }}
```

```
{{ ? }} X := 5 {{ X = 0 }}                    False
```

```
{{ ? }} while true do                         True
          X := 0
        end
{{ X = 0 }}
```

Dafny

- Solver-aided language and verifier
- Language is statically-typed
- Imperative (with lots of functional language features)
- Compiles to C#, Java, Go, Python, …



Reference manual:

`https://dafny.org/dafny/DafnyRef/DafnyRef.html`

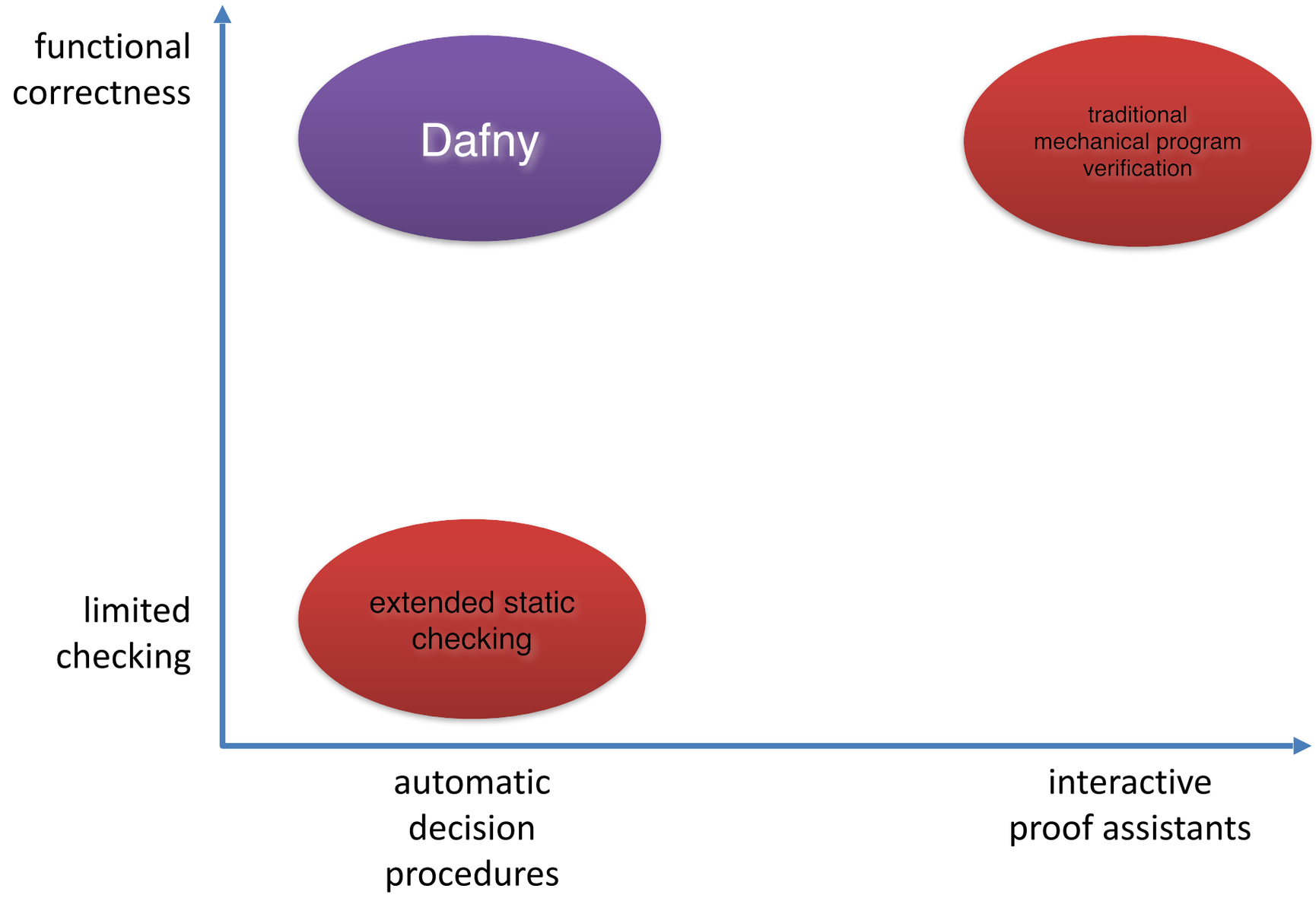
Dafny

- Applies Hoare reasoning to programs
- User provides specifications in the form of pre- and postconditions, along with other assertions
- Dafny verifies that the program meets the specification
  - When successful, Dafny guarantees (total) functional correctness of the program

Correctness:

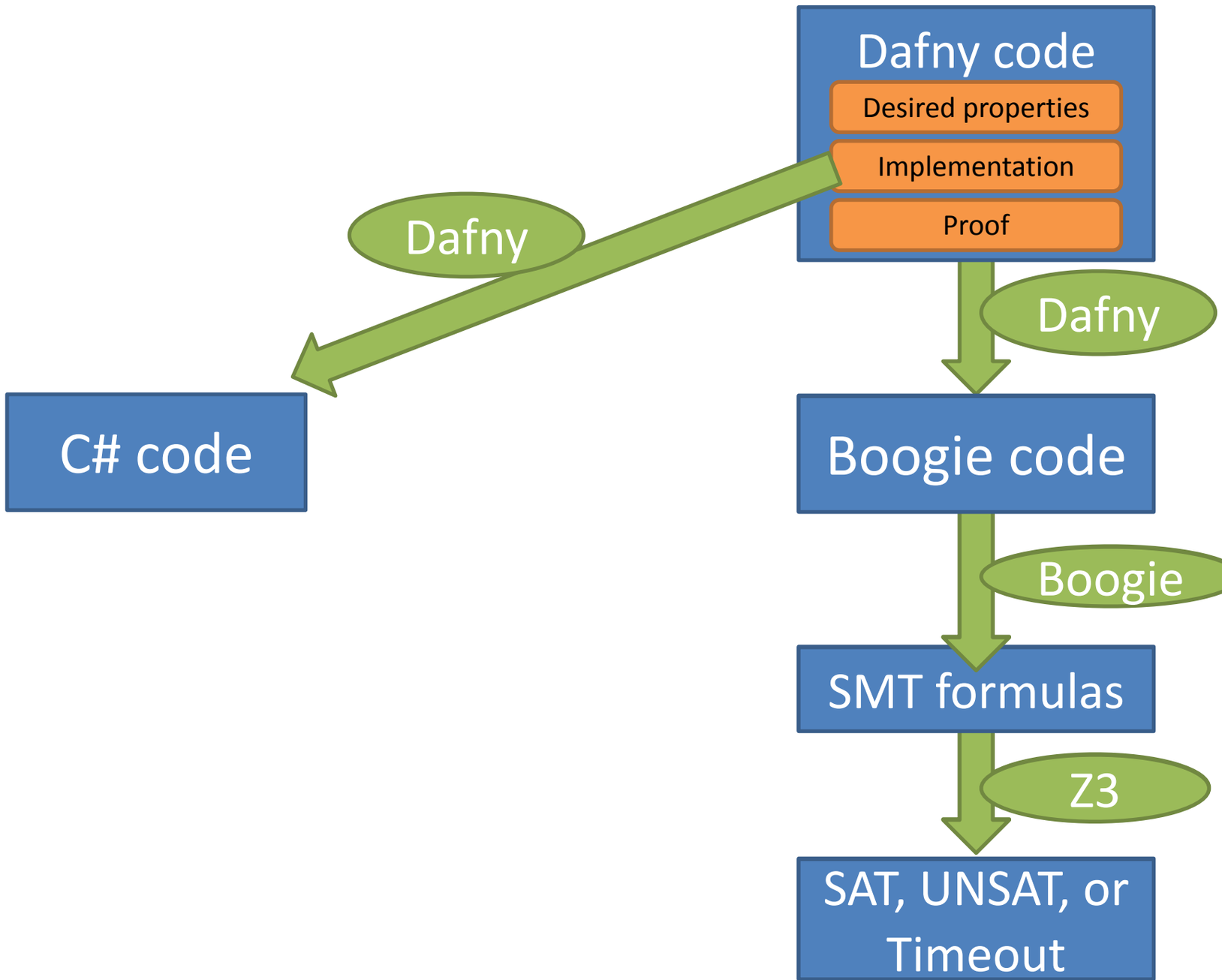
- Reflects base-level semantic properties (no runtime errors (e.g., divide-by-zero, null pointer dereferences, etc.)
- But, also justifies higher-level application-specific properties (e.g., correctness of distributed systems, …)

Types of Program Verification



Adapted from slides of Bryan Parno

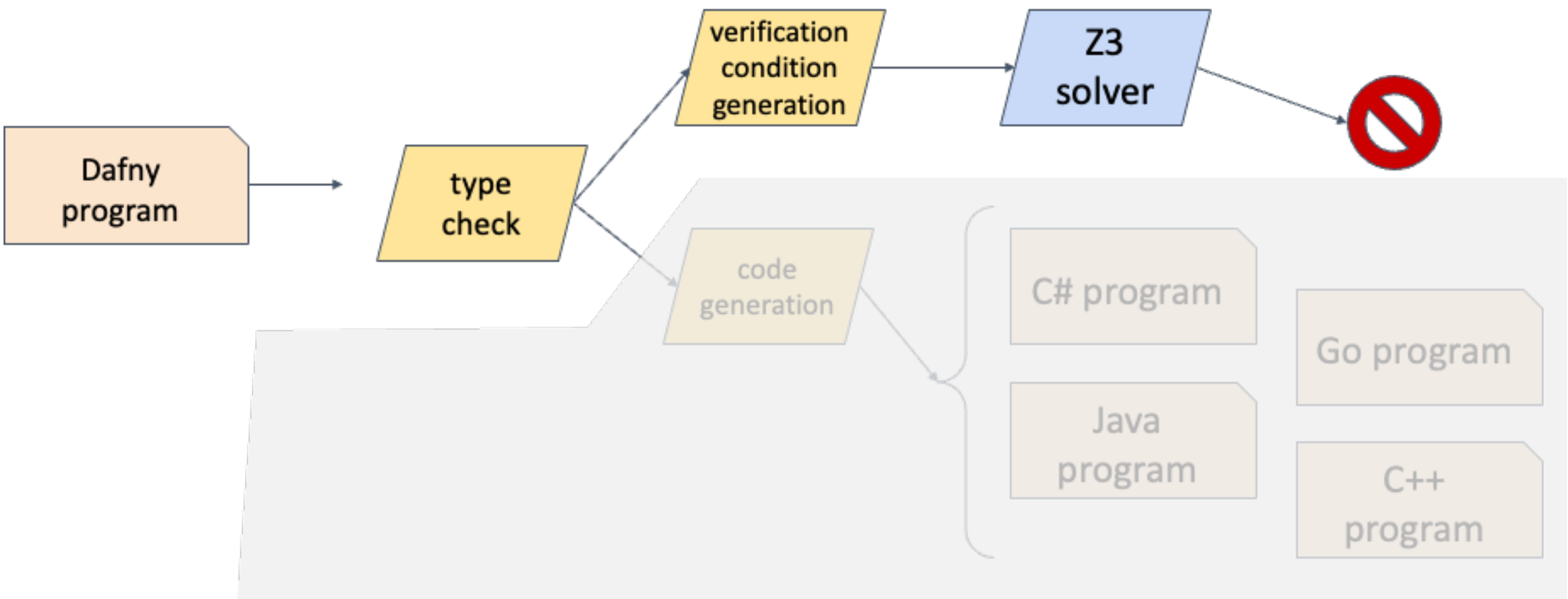
Architecture



Dafny's architecture

Adapted from slides of Bryan Parno

Pipeline



Adapted from slides of Jon Howell and Manos Kapritsos

Specifications

- Specifications are meant to capture salient behavior of an application, eliding issues of efficiency and low-level representation.

```
forall k:int :: 0 <= k < a.Length ==> 0 < a[k]
```

- Specifications in Dafny can be arbitrarily sophisticated.

- We can think of Dafny as being two smaller languages rolled into one:

  - An imperative core that has methods, loops, arrays, if statements... and other features found in realistic programming languages. This core can be compiled and executed.

  - A pure (functional) specification language that supports functions, sets, predicates, algebraic datatypes, etc. This language is used by the prover but is not compiled.

Examples


```
method Triple (x: int) returns (r : int) {
  var y := 2 * x;
  r := x + y;
  assert r == 3 * x;
}

method Caller () {
  var t := Triple(9);
  assert t == 27;      // assert fails: why?
}

method TripleSpec (x: int) returns (r : int)
   ensures r == 3 * x
{
      var y := 2 * x;
      r := x + y;
}

method CallerSpec () {
  var t := TripleSpec(9);
   assert t == 27; // assert succeeds
}
```

Examples

// Valid method

```
method Index (n: int) returns (i : int)
    requires 1 <= n
    ensures 0 <= i <= n
{
    i := n /2;
}
```

// Invalid assert - how would you fix this?

```
method CallIndex() {
    var t1 := Index(50);
    var t2 := Index(50);
    assert t1 == t2;
}
```

Examples


```
method Min (x : int, y : int) returns (m : int)
  ensures m <=x && m <= y
{
   m := if x <= y then x else y;
}
```

The implementation satisfies the spec but does not capture the intended behavior!

```
method Min (x : int, y : int) returns (m : int)
  ensures m <=x && m <= y
  ensures m == x || m == y
{
   m := if x <= y then x else y;
}
```

Functions vs. Methods

- Functions in Dafny have no computational effect
  - Deterministic
  - Can be used in specifications!

```
function average(a: int, b: int): int {
  (a + b) / 2
}

method Triple (x: int) returns (r: int)
   ensures average(r, 3 * x) == 3 * x
{
  if (x < 0) { return -x; } else { return x; }
}
```

Alternative definition:

```
function average(a: int, b: int): int
  requires 0 <= a && 0 <= b
{
  (a + b) / 2
}
```

Functions


```
function fib(n: nat): nat
{
   if n == 0 then 0 else
   if n == 1 then 1 else
            fib(n - 1) + fib(n - 2)
}

method Fib (n: nat) returns (x: nat)
   ensures x == fib(n);
{
  var i := 0;
  x := 0;
  var y := 1;
  while (i < n) {
    x, y := y, x+y;
    i := i + 1;
   }
}
```

Dafny fails to verify this program. Why?

Invariants

- Follows the same principle as Hoare logic

```
method ComputeFib (n: nat) returns (y: nat)
    ensures y == fib(n);
{
    if (n == 0) { return 0; }

  var i := 1;
  var x := 0;
  y := 1;

   while (i < n)
      invariant 0 < i <= n
      invariant x == fib (i - 1)
      invariant y == fib (i)
   {
      x, y := y, x+y;
      i := i + 1;
   }
}
```

Invariants


```
method loopEx (n : nat)
{
    var i : int := 0;
    while (i < n)
        invariant 0 <= i
        {
            i := i + 1;
    }
    assert i == n;
}
```

Dafny will not verify this program. Why?

Need invariants to be inductive!

- hold in the initial state
- hold in every state reachable from the initial state
- strong enough to imply the postcondition

```
method loopExCheckFixed (n : nat)
{
    var i : int := 0;
    while (i < n)
        invariant 0 <= i <= n
        {
            i := i + 1;
    }
    assert i == n;
}
```

Ghost vs. Compiled

- Ghost constructs are syntactic forms used only in specifications
  - Pre- (requires) and post- (ensures) conditions are ghost constructs
  - As are assert and invariant

  - Some constructs such as functions exist in both ghost and compiled form
  - Can explicitly declare variables, parameters, methods, etc. as ghost; such objects are not compiled into executables
    - ★ Cannot assign ghost entities to complied ones

```
method Triple(x : int) returns (r: int)
  ensures r = 3 * x
{
   var y := 2 * x;
   r := x + y;
   ghost var a, b := DoubleQuadruple(x);
   assert a <= r <= b || b <= r <= a;
}

ghost method DoubleQuadruple (x : int) returns (a: int, b: int)
   ensures a = 2 * x && b = 4 * x
{
   a := 2 * x;
   b := 2 * a;
}
```

Assert

- `assert E` is a no-op if `E` holds, otherwise program faults.
- To show postcondition `Q` holds, i.e.,

  `WP(assert E, Q)`

  we must prove `E && Q`
- Backward reasoning

Alternative interpretation:

- `assert E` evaluates `E` and if `E` does not crash, continues
- No proof obligations introduced
- Forward reasoning

Concept Check


```
method MultipleReturnsSpec(x: int, y: int) returns (more: int, less: int)
{
  more := x + y;
  less := x - y;
}
```

What is a meaningful spec for this method?

```
method MultipleReturnsSpec(x: int, y: int) returns (more: int, less: int)
  ensures less < x
  ensures x < more
{
  more := x + y;
  less := x - y;
}
```

Concept Check


```
method Max (a : int, b : int) returns (c : int)
{
    if (a < b) {
        c := b;
    }
    else { c := a; }
}
```

What is a meaningful spec for this method?

```
method Max (a : int, b : int) returns (c : int)
    ensures (a <= c && b <= c) && (b == c || a == c)
{
    if (a < b) {
        c := b;
    }
    else { c := a; }
}
```

Concept Check


```
method Abs(x: int) returns (r: int)
  ensures r >= 0
   {
    if (x < 0)
      { return -x; }
    else
      { return x; }
  }
```

What's wrong with this spec? How would you fix it?

```
method AbsFixed(x: int)    returns (y: int)
  ensures 0 <= x ==> y == x
  ensures x < 0 ==> y == -x
{
  if (x < 0) { return -x; }
     else { return x; }
}
```

```
method AbsFixedA(x: int)   returns (y: int)
    ensures 0 <= y && ( y == x || y == -x)
{
  if (x < 0) { return -x; }
     else { return x; }
}
```